A probabilistic approach to Liouville field theory

Vincent Vargas\(^1\)

ENS Paris

\(^1\)based on a series of works with: David, Kupiainen, Rhodes
Outline

1. LCFT in the conformal bootstrap

2. Path integral formulation of LCFT and probabilistic statement of the DOZZ formula

3. Ideas of the probabilistic proof of the DOZZ formula
Plan of the talk

1. LCFT in the conformal bootstrap
2. Path integral formulation of LCFT and probabilistic statement of the DOZZ formula
3. Ideas of the probabilistic proof of the DOZZ formula
Polyakov introduces LCFT:

**Polyakov** (1981): *Quantum geometry of bosonic strings*.

Conformal Field Theory to solve LCFT:


Unifying LCFT

Conformal Bootstrap

Scaling limit of planar maps

Our program in LCFT

MIT/Cambridge program

Path integral:
\[ \int_\phi \prod_k e^{\alpha_k \phi(z_k)} e^{-\int_M |\nabla \phi|^2 + \mu e^{\gamma \phi}} D\phi \]

\[ \gamma \rightarrow 0 \quad \mu \gamma^2 \rightarrow \Lambda \]

\[ \gamma \phi \rightarrow \phi_* \quad \Delta \phi_* = \Lambda e^{\phi_*} \]
Figure: The scaling limit of large circle packed triangulations should be described by LCFT
Motivation from discrete planar maps

Figure: Circles of the circle packed triangulation
The KPZ relation reads:

$$\langle \prod_{i=4}^{n} \Phi(x_i) \rangle = \frac{\langle \prod_{i=4}^{n} \sigma(x_i) \rangle \langle e^{\gamma \phi(x_1)} e^{\gamma \phi(x_2)} e^{\gamma \phi(x_3)} \prod_{i=4}^{n} e^{\alpha \phi(x_i)} \rangle_{\gamma,\mu}}{\langle e^{\gamma \phi(x_1)} e^{\gamma \phi(x_2)} e^{\gamma \phi(x_3)} \rangle_{\gamma,\mu}}$$

where:

- $x_1, x_2, x_3 \in S^2$ fixed points in the embedding of the map in the sphere
- $\sigma$ scaling limit of primary field on **regular lattice** with conformal weight $\Delta_{\sigma}$
- $\Phi$ scaling limit of same primary field on **planar map**
- Conformal weight condition: $\Delta_{\sigma} + \Delta_{\alpha} = 1$
Main ingredients of LCFT in the conformal bootstrap:

- Shift equations to determine 3 point structure constant $C_{\gamma}^{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$

- Base of vertex operators $e^{\alpha \phi(z)}$ where $\alpha$ in $S = Q + i\mathbb{R}$
  (Reminder: $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$).

- Recursive procedure called the Operator Product Expansion (OPE):

$$e^{\alpha_1 \phi(z_1)} e^{\alpha_2 \phi(z_2)} = \int_{-\infty}^{\infty} \sum_{L, \bar{L}} C_{\alpha_1, \alpha_2}^{Q+iP} (z_1, z_2) L \bar{L}(e^{(Q+iP)\phi(z_2)}) dP$$

where $L$ and $\bar{L}$ are differential operators acting on $z_2$ and $\bar{z}_2$. 
Using the OPE, one can then get an expression for the 4 point correlation function for $(\alpha_i)_i \in S$:

\[
< e^{\alpha_1 \phi(z)} e^{\alpha_2 \phi(0)} e^{\alpha_3 \phi(1)} e^{\alpha_4 \phi(\infty)} >
= \int_{-\infty}^{\infty} C_{\gamma}^{DOZZ}(\alpha_1, \alpha_2, Q - iP) C_{\gamma}^{DOZZ}(Q + iP, \alpha_3, \alpha_4) |F_P(z)|^2 dP
\]

where $F_P := F(\Delta_{\alpha_i}, P)$ are the (universal) conformal blocks of CFT.

Similarly for general n point correlations, etc...
Zamolodchikov’s $\Upsilon_{\frac{\gamma}{2}}$ function for $\gamma \in \mathbb{C} \setminus i\mathbb{R}$

The $\Upsilon_{\frac{\gamma}{2}}$ function defined as analytic continuation of

$$\ln \Upsilon_{\frac{\gamma}{2}}(z) = \int_0^\infty \left( \left( \frac{Q}{2} - z \right)^2 e^{-t} - \frac{(\sinh((\frac{Q}{2} - z)\frac{t}{2}))^2}{\sinh(\frac{\gamma t}{4}) \sinh(\frac{t}{\gamma})} \right) \frac{dt}{t}$$

for $0 < \Re(z) < \Re(Q)$.

Remarkable functional relation

$$\Upsilon_{\frac{\gamma}{2}}(z+\frac{\gamma}{2}) = \ell\left(\frac{\gamma z}{2}\right)\left(\frac{\gamma}{2}\right)^{1-\gamma z} \Upsilon_{\frac{\gamma}{2}}(z), \quad \Upsilon_{\frac{\gamma}{2}}(z+\frac{2}{\gamma}) = \ell\left(\frac{2z}{\gamma}\right)\left(\frac{\gamma}{2}\right)^{\frac{4z}{\gamma}-1} \Upsilon_{\frac{\gamma}{2}}(z)$$

with $\ell(x) = \Gamma(x)/\Gamma(1-x)$. 
The **DOZZ** formula for $\gamma \in \mathbb{C} \setminus i\mathbb{R}$.

For all $\mu > 0$ and $\gamma \in \mathbb{C} \setminus i\mathbb{R}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$,

$$C_{\gamma,\mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3) = (\pi \mu \ell(\frac{\gamma^2}{4}) (\frac{\gamma}{2})^{2-\gamma^2/2})^{\frac{2Q-\bar{\alpha}}{\gamma}}$$

$$\times \frac{\Gamma'_\frac{\gamma}{2}(0) \Gamma_{\frac{\gamma}{2}}(\alpha_1) \Gamma_{\frac{\gamma}{2}}(\alpha_2) \Gamma_{\frac{\gamma}{2}}(\alpha_3)}{\Gamma_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2Q}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_1}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_2}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_3}{2})}$$

with $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$.

Reminder: $\ell(x) = \Gamma(x)/\Gamma(1-x)$. 
Plan of the talk

1. LCFT in the conformal bootstrap

2. Path integral formulation of LCFT and probabilistic statement of the DOZZ formula

3. Ideas of the probabilistic proof of the DOZZ formula
Path integral of LCFT on the Riemann sphere $S^2$

Formal definition of correlations:

$$\langle \prod_{i=1}^{n} e^{\alpha_{i}\phi(z_{i})} \rangle_{\gamma,\mu} := \int \left( \prod_{i=1}^{n} e^{\alpha_{i}\phi(z_{i})} \right) e^{-S_{L}(X)} DX,$$

where

- $DX$ "Lebesgue measure" on functional space
- $S_{L}$ Liouville action:

$$S_{L}(X) := \frac{1}{4\pi} \int_{S^2} \left( |\nabla g X(z)|^{2} + 2QX(z) + 4\pi \mu e^{\gamma X(z)} \right) g(z) d^2 z$$

with $g(z) = \frac{4}{(1+|z|^2)^2}$ round metric, $\gamma \in [0, 2]$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and $\mu > 0$.

- Liouville field: $\phi(z) = X(z) + \frac{Q}{2} \ln g(z)$. 
Existence of the correlation functions

**Theorem (DKRV, 2014)**

One can define the correlations \( \langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu} \) by a regularization procedure. The correlations are non trivial if and only if:

\[
\forall i, \alpha_i < Q \quad \text{and} \quad Q - \frac{\sum_{i=1}^{n} \alpha_i}{2} < \frac{2}{\gamma} \wedge \inf_{1 \leq i \leq n} (Q - \alpha_i) \quad (*)
\]

In particular, existence implies \( n \geq 3 \)!

Remark: see region I and II in Harlow, Maltz, Witten (2011).

Idea of proof: interpret the gradient term in Liouville action as Gaussian Free Field with average distributed as Lebesgue (zero mode).
An explicit expression for the correlation functions

The existence is in fact based on the following explicit expression:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_{\gamma, \mu} = A \left( \prod_{1 \leq j < k \leq n} \frac{1}{|z_j - z_k|^{\alpha_j \alpha_k}} \right) \mu^{-s} \Gamma(s) \mathbb{E}[Z_1^{-s}]$$

where $s = \frac{\sum_{i=1}^{n} \alpha_i - 2Q}{\gamma}$, $A$ some constant (depending on the $\alpha_i$ and $\gamma$) and

$$Z_1 = \int_{\mathbb{C}} e^{\gamma X_g(z) - \frac{\gamma^2}{2} \mathbb{E}[X_g(z)^2]} \left( \prod_{i=1}^{n} \frac{1}{|z - z_i|^{\gamma \alpha_i}} \right) g(z)^{1-\frac{\gamma}{4}} \sum_{i=1}^{n} \alpha_i d^2 z$$

with $X_g$ GFF with vanishing mean on the sphere.
The KPZ formula

**Theorem (DKRV, 2014)**

Let \((\alpha_i)_i\) satisfy (\(*\)). If \(\psi\) is a Möbius transform, we have

\[
\langle \prod_{i=1}^{n} e^{\alpha_i \phi(\psi(z_i))} \rangle_{\gamma,\mu} = \prod_{i=1}^{n} |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu}
\]

where \(\Delta_{\alpha_i} = \frac{\alpha_i}{2} (Q - \frac{\alpha_i}{2})\) is the conformal weight of \(e^{\alpha_i \phi(z)}\).

Reminder: \(Q = \frac{\gamma}{2} + \frac{2}{\gamma}\).

Central charge: \(c_L = 1 + 6Q^2 \geq 25\).
The 3 point correlation function

By conformal covariance:

\[
\langle \prod_{i=1}^{3} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu} = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} \langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu}
\]

where:

- \( \Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2} \), etc...
- \( \langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu} \) is the 3 point structure constant

Exact expression for \( \langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu} \)?
The 3 point correlation function

By conformal covariance:

\[ \langle \prod_{i=1}^{3} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu} \]

\[ = |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} \langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu} \]

where:

- \( \Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2} \), etc...
- \( \langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu} \) is the 3 point structure constant

Exact expression for \( \langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu} \)?

The DOZZ formula \( C^{DOZZ}_{\gamma,\mu}(\alpha_1, \alpha_2, \alpha_3) \)...
Exact expression the 3 point structure constants

We have the following expression with \( \bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3 \):

\[
\langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma, \mu} = A \mu^{\frac{2Q-\bar{\alpha}}{\gamma}} \Gamma\left( \frac{\bar{\alpha} - 2Q}{\gamma} \right) \mathbb{E}[Z_1^{\frac{2Q-\bar{\alpha}}{\gamma}}]
\]

where \( A \) is some constant (depending on the \( \alpha_i \) and \( \gamma \)) and

\[
Z_1 = \int_{\mathbb{C}} e^{\gamma X_g(z) - \frac{\gamma^2}{2} \mathbb{E}[X_g(z)^2]} \frac{g(z)^{1-\frac{\gamma}{4}\bar{\alpha}}}{|z|^{\gamma \alpha_1} |z-1|^{\gamma \alpha_2}} d^2 z
\]

with \( X_g \) GFF with vanishing mean on the sphere (and \( g(z) = \frac{4}{(1+|z|^2)^2} \)).
Recall the (\(\ast\)) condition

\[
\forall i, \alpha_i < Q \quad \text{and} \quad Q - \frac{\sum_{i=1}^{3} \alpha_i}{2} < \frac{2}{\gamma} \wedge \inf_{1 \leq i \leq 3} (Q - \alpha_i) \quad (\ast)
\]

**Theorem (Kupiainen, Rhodes, V., 2017)**

For all \(\gamma \in (0, 2)\) and \((\alpha_i)\) satisfying (\(\ast\)) the following identity holds

\[
\langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma, \mu} = C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)
\]
Some comments on the previous theorem

• Previous result provides analytic continuation for $\gamma \in \mathbb{C} \setminus i\mathbb{R}$ of the path integral approach.

• Observation: $C_{\gamma,\mu}^{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$ invariant under duality:

\[
\frac{\gamma}{2} \leftrightarrow \frac{2}{\gamma}, \quad \mu \leftrightarrow \tilde{\mu} = \frac{(\mu \pi \ell(\frac{\gamma^2}{4}))^{\frac{4}{\gamma^2}}}{\pi \ell(\frac{4}{\gamma^2})}
\]

Path integral interpretation?
Plan of the talk

1. LCFT in the conformal bootstrap
2. Path integral formulation of LCFT and probabilistic statement of the DOZZ formula
3. Ideas of the probabilistic proof of the DOZZ formula
The quantum Fuchsian equation: the BPZ differential equation of order 2

The fields $e^{-\gamma/2}\phi$ and $e^{-2\gamma}\phi$ satisfy BPZ of order 2:

Theorem (Kupiainen, Rhodes, V., 2015)

For real $(\alpha_i)_i$ satisfying (*), one has for $\alpha \in \{-\gamma/2, -2\gamma\}$

$$\frac{1}{\alpha^2} \partial_{zz}^2 \langle e^{\alpha\phi(z)} \prod_{i=1}^{3} e^{\alpha_i\phi(z_i)} \rangle_{\gamma,\mu} + \sum_{k=1}^{3} \frac{\Delta_{\alpha_k}}{(z - z_k)^2} \langle e^{\alpha\phi(z)} \prod_{i=1}^{3} e^{\alpha_i\phi(z_i)} \rangle_{\gamma,\mu}$$

$$+ \sum_{k=1}^{3} \frac{1}{z - z_k} \partial_{z_k} \langle e^{\alpha\phi(z)} \prod_{i=1}^{3} e^{\alpha_i\phi(z_i)} \rangle_{\gamma,\mu} = 0,$$
Consequences of the BPZ equation

By studying \( \langle e^{-\frac{\gamma}{2}\phi(z)} \prod_{i=1}^{3} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu} \) around \( z = 0, 1 \), one gets by monodromy argument

\[
\frac{\langle e^{\left(\alpha_1 + \frac{\gamma}{2}\right)\phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu}}{\langle e^{\left(\alpha_1 - \frac{\gamma}{2}\right)\phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu}} = -\frac{1}{\pi \mu \ell\left(\frac{\gamma}{4}(\bar{\alpha} - \frac{\gamma}{2} - 2Q)\right)\ell\left(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_3 - \frac{\gamma}{2})\right)\ell\left(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_2 - \frac{\gamma}{2})\right)}
\]

and a dual equation where \( \frac{\gamma}{2} \leftrightarrow \frac{2}{\gamma} \).

Reminder: \( \ell(x) = \Gamma(x)/\Gamma(1-x) \) and \( \bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3 \).
Perspectives and open problems

Perspectives:

- Construction of the Virasoro algebra of LCFT based on the two Ward identities: probabilistic construction of $e^{\alpha \phi(z)}$ where $\alpha$ in $S = Q + i\mathbb{R}$? (Kupiainen, Rhodes, V.).

- DOZZ type formulas for LCFT with a boundary: Teschner, Zamolodchikov brothers (probabilistic statement: Remy).

- Can one make sense of the path integral for $\gamma \in i\mathbb{R}$ using the path integral? Can one relate this path integral to the imaginary DOZZ formula? Related to critical FK model, CLE, etc...