A probabilistic approach to Liouville field theory

Vincent Vargas [1]

ENS Paris

[1] based on a series of works with: David, Kupiainen, Rhodes

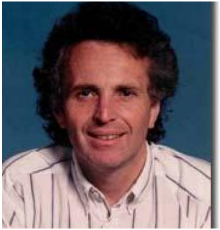
Outline

1. LCFT in the conformal bootstrap
2. Path integral formulation of LCFT and probabilistic statement of the DOZZ formula
3. Ideas of the probabilistic proof of the DOZZ formula

Plan of the talk

1. LCFT in the conformal bootstrap
2. Path integral formulation of LCFT and probabilistic statement of the DOZZ formula
3. Ideas of the probabilistic proof of the DOZZ formula

Conformal Field Theory: the legacy of Polyakov



A. Polyakov

Polyakov introduces LCFT:

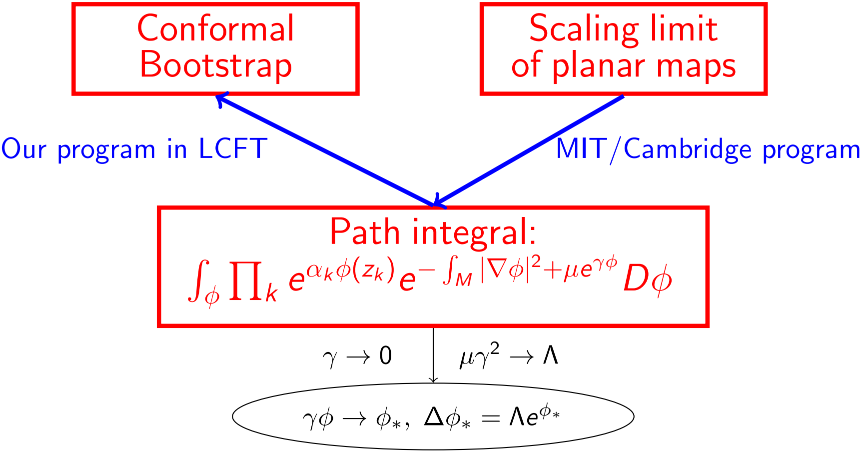
Polyakov (1981): *Quantum geometry of bosonic strings*.

Conformal Field Theory to solve LCFT:

Belavin, Polyakov, Zamolodchikov (1984): *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*.

The **DOZZ** proposal for LCFT in the bootstrap: Dorn-Otto (1994), Zamolodchikov-Zamolodchikov (1996).

Unifying LCFT



Motivation from discrete planar maps: courtesy of F. David

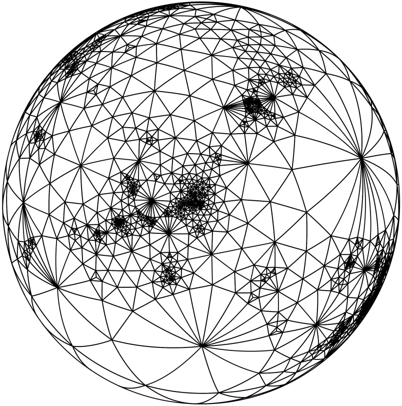


Figure: The scaling limit of large circle packed triangulations should be described by LCFT

Motivation from discrete planar maps

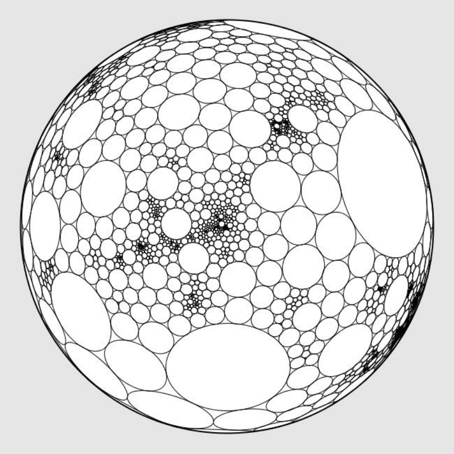


Figure: Circles of the circle packed triangulation

Motivation from discrete planar maps

The KPZ relation reads:

$$\langle \prod_{i=4}^{n} \Phi(x_i) \rangle = \frac{\langle \prod_{i=4}^{n} \sigma(x_i) \rangle \langle e^{\gamma\phi(x_1)} e^{\gamma\phi(x_2)} e^{\gamma\phi(x_3)} \prod_{i=4}^{n} e^{\alpha\phi(x_i)} \rangle_{\gamma,\mu}}{\langle e^{\gamma\phi(x_1)} e^{\gamma\phi(x_2)} e^{\gamma\phi(x_3)} \rangle_{\gamma,\mu}}$$

where:

- $x_1, x_2, x_3 \in \mathbb{S}^2$ fixed points in the embedding of the map in the sphere
- σ scaling limit of primary field on **regular lattice** with conformal weight Δ_σ
- Φ scaling limit of same primary field on **planar map**
- Conformal weight condition: $\Delta_\sigma + \Delta_\alpha = 1$

Introduction to LCFT in the conformal bootstrap

Main ingredients of LCFT in the conformal bootstrap:

- Shift equations to determine 3 point structure constant $C_{\gamma}^{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$
- Base of vertex operators $e^{\alpha\phi(z)}$ where α in $\mathbf{S} = Q + i\mathbb{R}$ (Reminder: $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$).
- Recursive procedure called the Operator Product Expansion (OPE):

$$e^{\alpha_1\phi(z_1)}e^{\alpha_2\phi(z_2)} = \int_{-\infty}^{\infty} \sum_{L,\bar{L}} C_{\alpha_1,\alpha_2}^{Q+iP}(z_1, z_2) L\bar{L}(e^{(Q+iP)\phi(z_2)}) dP$$

where L and $\bar{L}$ are differential operators acting on z_2 and $\bar{z}_2$.

The 4 point correlation function

Using the OPE, one can then get an expression for the 4 point correlation function for $(\alpha_i)_i \in \mathbf{S}$:

$$< e^{\alpha_1 \phi(z)} e^{\alpha_2 \phi(0)} e^{\alpha_3 \phi(1)} e^{\alpha_4 \phi(\infty)} >$$
$$= \int_{-\infty}^{\infty} C_\gamma^{DOZZ}(\alpha_1, \alpha_2, Q - iP) C_\gamma^{DOZZ}(Q + iP, \alpha_3, \alpha_4) |\mathcal{F}_P(z)|^2 dP$$

where $\mathcal{F}_P := \mathcal{F}_{(\Delta_{\alpha_i})_i, P}$ are the (universal) conformal blocks of CFT.

Similarly for general n point correlations, etc...

Zamolodchikov's $\Upsilon_{\frac{\gamma}{2}}$ function for $\gamma \in \mathbb{C} \setminus i\mathbb{R}$

The $\Upsilon_{\frac{\gamma}{2}}$ function defined as analytic continuation of

$$\ln \Upsilon_{\frac{\gamma}{2}}(z) = \int_0^\infty \left(\left(\frac{Q}{2} - z \right)^2 e^{-t} - \frac{(\sinh((\frac{Q}{2} - z)\frac{t}{2}))^2}{\sinh(\frac{t\gamma}{4}) \sinh(\frac{t}{\gamma})} \right) \frac{dt}{t}$$

for $0 < \Re(z) < \Re(Q)$.

Remarkable functional relation

$$\Upsilon_{\frac{\gamma}{2}}(z+\frac{\gamma}{2}) = \ell(\frac{\gamma z}{2})(\frac{\gamma}{2})^{1-\gamma z}\Upsilon_{\frac{\gamma}{2}}(z), \; \Upsilon_{\frac{\gamma}{2}}(z+\frac{2}{\gamma}) = \ell(\frac{2z}{\gamma})(\frac{\gamma}{2})^{\frac{4z}{\gamma}-1}\Upsilon_{\frac{\gamma}{2}}(z)$$

with $\ell(x) = \Gamma(x)/\Gamma(1-x)$.

The **DOZZ** formula for $\gamma \in \mathbb{C} \setminus i\mathbb{R}$.

For all $\mu > 0$ and $\gamma \in \mathbb{C} \setminus i\mathbb{R}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$,

$$C_{\gamma,\mu}^{DOZZ}(\alpha_1, \alpha_2, \alpha_3) = \left(\pi\,\mu\,\ell(\frac{\gamma^2}{4})\,(\frac{\gamma}{2})^{2-\gamma^2/2}\right)^{\frac{2Q-\bar{\alpha}}{\gamma}} \times \frac{\Upsilon'_{\frac{\gamma}{2}}(0)\Upsilon_{\frac{\gamma}{2}}(\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2Q}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_1}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_2}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_3}{2})}$$

with $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$.

Reminder: $\ell(x) = \Gamma(x)/\Gamma(1-x)$.

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Path integral of LCFT on the Riemann sphere $\mathbb{S}^2$

Formal definition of correlations:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu} := \int \left(\prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \right) e^{-S_L(X)} DX,$$

where

- DX "Lebesgue measure" on functional space
- S_L Liouville action:

$$S_L(X) := \frac{1}{4\pi} \int_{\mathbb{S}^2} \left(|\nabla_g X(z)|^2 + 2QX(z) + 4\pi\mu e^{\gamma X(z)} \right) g(z)\, d^2 z$$

  with $g(z) = \frac{4}{(1+|z|^2)^2}$ round metric, $\gamma \in]0,2]$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and $\mu > 0$.
- **Liouville field**: $\phi(z) = X(z) + \frac{Q}{2} \ln g(z)$.

Existence of the correlation functions

Theorem (DKRV, 2014)

One can define the correlations $\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu}$ *by a regularization procedure. The correlations are non trivial if and only if:*

$$\forall i, \; \alpha_i < Q \quad \text{and} \quad Q - \frac{\sum_{i=1}^{n} \alpha_i}{2} < \frac{2}{\gamma} \wedge \inf_{1 \le i \le n} (Q - \alpha_i) \quad (*)$$

In particular, existence implies $n \geq 3$*!*

Remark: see region I and II in Harlow, Maltz, Witten (2011).

Idea of proof: interpret the gradient term in Liouville action as Gaussian Free Field with average distributed as Lebesgue (zero mode).

An explicit expression for the correlation functions

The existence is in fact based on the following explicit expression:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu} = A \left(\prod_{1 \leq j < k \leq n} \frac{1}{|z_j - z_k|^{\alpha_j \alpha_k}} \right) \mu^{-s} \Gamma(s) \mathbb{E}[Z_1^{-s}]$$

where $s = \frac{\sum_{i=1}^{n} \alpha_i - 2Q}{\gamma}$, A some constant (depending on the α_i and γ) and

$$Z_1 = \int_{\mathbb{C}} e^{\gamma X_g(z) - \frac{\gamma^2}{2} \mathbb{E}[X_g(z)^2]} \left(\prod_{i=1}^{n} \frac{1}{|z - z_i|^{\gamma \alpha_i}} \right) g(z)^{1 - \frac{\gamma}{4} \sum_{i=1}^{n} \alpha_i} d^2 z$$

with X_g GFF with vanishing mean on the sphere.

The KPZ formula

Theorem (DKRV, 2014)

Let $(\alpha_i)_i$ satisfy $(*)$. If ψ is a Möbius transform, we have

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(\psi(z_i))} \rangle_{\gamma,\mu} = \prod_{i=1}^{n} |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu}$$

where $\Delta_{\alpha_i} = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ is the conformal weight of $e^{\alpha_i \phi(z)}$.

Reminder: $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

Central charge: $c_L = 1 + 6Q^2 \geq 25$.

The 3 point correlation function

By conformal covariance:

$$\langle \prod_{i=1}^{3} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu}$$
$$= |z_1 - z_2|^{2\Delta_{12}} |z_2 - z_3|^{2\Delta_{23}} |z_1 - z_3|^{2\Delta_{13}} \langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu}$$

where:

- $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$, etc...
- $\langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu}$ is the 3 point structure constant

Exact expression for $\langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu}$?

The 3 point correlation function

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where:

- $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$, etc...
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Exact expression for $\langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu}$?
The DOZZ formula $C_{\gamma,\mu}^{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$...

Exact expression the 3 point structure constants

We have the following expression with $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$:

$$\langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu} = A \mu^{\frac{2Q-\bar{\alpha}}{\gamma}} \Gamma\Big(\frac{\bar{\alpha} - 2Q}{\gamma}\Big) \mathbb{E}[Z_1^{\frac{2Q-\bar{\alpha}}{\gamma}}]$$

where A is some constant (depending on the α_i and γ) and

$$Z_1 = \int_{\mathbb{C}} e^{\gamma X_g(z) - \frac{\gamma^2}{2}\mathbb{E}[X_g(z)^2]} \frac{g(z)^{1-\frac{\gamma}{4}\bar{\alpha}}}{|z|^{\gamma\alpha_1}|z-1|^{\gamma\alpha_2}} d^2 z$$

with X_g GFF with vanishing mean on the sphere (and $g(z) = \frac{4}{(1+|z|^2)^2}$).

The DOZZ formula

Recall the $(*)$ condition

$$\forall i,\ \alpha_i < Q \quad \text{and} \quad Q - \frac{\sum_{i=1}^3 \alpha_i}{2} < \frac{2}{\gamma} \wedge \inf_{1 \leq i \leq 3} (Q - \alpha_i) \quad (*)$$

Theorem (Kupiainen, Rhodes, V., 2017)

For all $\gamma \in (0,2)$ and (α_i) satisfying $(*)$ the following identity holds

$$\langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle_{\gamma,\mu} = C_{\gamma,\mu}^{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$$

Some comments on the previous theorem

- Previous result provides analytic continuation for $\gamma \in \mathbb{C} \setminus i\mathbb{R}$ of the path integral approach

- Observation: $C_{\gamma,\mu}^{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$ invariant under **duality**:

$$\frac{\gamma}{2} \leftrightarrow \frac{2}{\gamma}, \quad \mu \leftrightarrow \tilde{\mu} = \frac{(\mu\pi\ell(\frac{\gamma^2}{4}))^{\frac{4}{\gamma^2}}}{\pi\ell(\frac{4}{\gamma^2})}$$

Path integral interpretation?

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The **quantum** Fuchsian equation: the BPZ differential equation of order 2

The fields $e^{-\frac{\gamma}{2}\phi}$ and $e^{-\frac{2}{\gamma}\phi}$ satisfy BPZ of order 2:

Theorem (Kupiainen, Rhodes, V., 2015)

For real $(\alpha_i)_i$ *staisfying* $(*)$*, one has for* $\alpha \in \{-\frac{\gamma}{2}, -\frac{2}{\gamma}\}$

$$\frac{1}{\alpha^2}\partial^2_{zz}\langle e^{\alpha\phi(z)}\prod_{i=1}^{3} e^{\alpha_i\phi(z_i)}\rangle_{\gamma,\mu} + \sum_{k=1}^{3}\frac{\Delta_{\alpha_k}}{(z-z_k)^2}\langle e^{\alpha\phi(z)}\prod_{i=1}^{3} e^{\alpha_i\phi(z_i)}\rangle_{\gamma,\mu}$$
$$+\sum_{k=1}^{3}\frac{1}{z-z_k}\partial_{z_k}\langle e^{\alpha\phi(z)}\prod_{i=1}^{3} e^{\alpha_i\phi(z_i)}\rangle_{\gamma,\mu} = 0,$$

Consequences of the BPZ equation

By studying $\langle e^{-\frac{\gamma}{2}\phi(z)} \prod_{i=1}^{3} e^{\alpha_i \phi(z_i)} \rangle_{\gamma,\mu}$ around $z = 0, 1$, one gets by monodromy argument

$$\frac{\langle e^{(\alpha_1+\frac{\gamma}{2})\phi(0)} e^{\alpha_2\phi(1)} e^{\alpha_3\phi(\infty)} \rangle_{\gamma,\mu}}{\langle e^{(\alpha_1-\frac{\gamma}{2})\phi(0)} e^{\alpha_2\phi(1)} e^{\alpha_3\phi(\infty)} \rangle_{\gamma,\mu}}$$

$$= -\frac{1}{\pi\mu} \frac{\ell(-\frac{\gamma^2}{4})\ell(\frac{\gamma\alpha_1}{2})\ell(\frac{\alpha_1\gamma}{2} - \frac{\gamma^2}{4})\ell(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_1 - \frac{\gamma}{2}))}{\ell(\frac{\gamma}{4}(\bar{\alpha} - \frac{\gamma}{2} - 2Q))\ell(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_3 - \frac{\gamma}{2}))\ell(\frac{\gamma}{4}(\bar{\alpha} - 2\alpha_2 - \frac{\gamma}{2}))}.$$

and a dual equation where $\frac{\gamma}{2} \leftrightarrow \frac{2}{\gamma}$.

Reminder: $\ell(x) = \Gamma(x)/\Gamma(1-x)$ and $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$.

Perspectives and open problems

Perspectives:

- Construction of the Virasoro algebra of LCFT based on the two Ward identities: probabilistic construction of $e^{\alpha\phi(z)}$ where α in $\mathbf{S} = Q + i\mathbb{R}$? (Kupiainen, Rhodes, V.).
- DOZZ type formulas for LCFT with a boundary: Teschner, Zamolodchikov brothers (probabilistic statement: Remy).
- Can one make sense of the path integral for $\gamma \in i\mathbb{R}$ using the path integral? Can one relate this path integral to the imaginary DOZZ formula? Related to critical FK model, CLE, etc...