Unification of integrability in supersymmetric gauge theories

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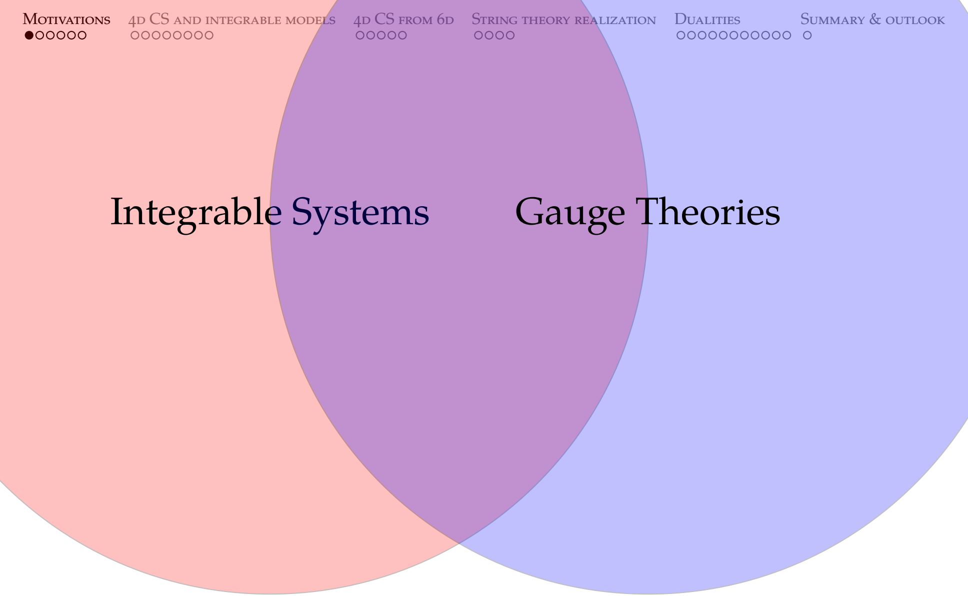
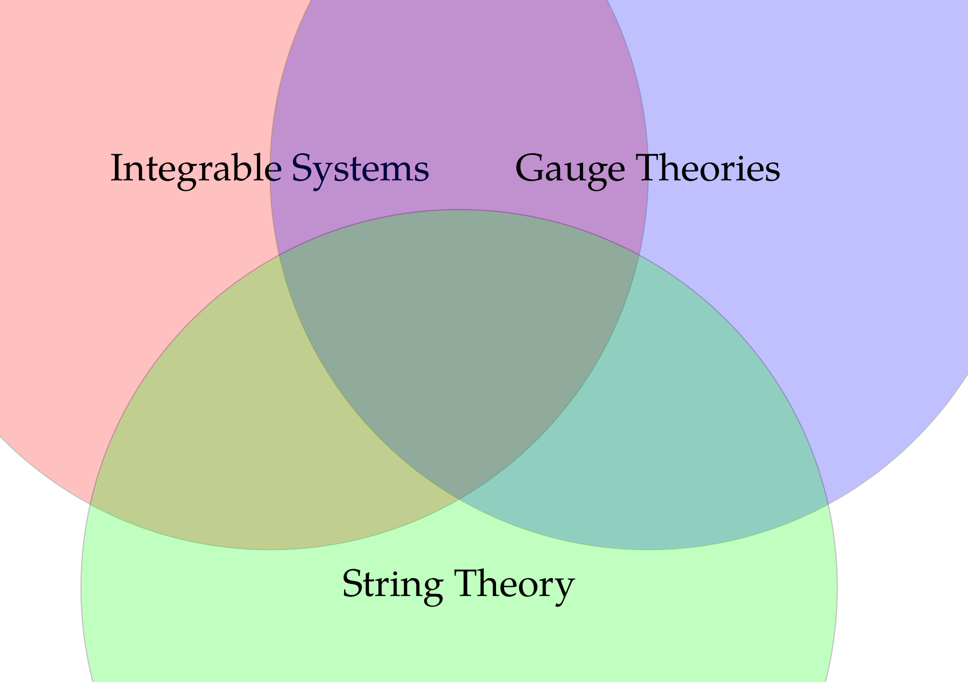
December 18, 2018

Kavli IPMU Based on 1810.01970 with Kevin Costello

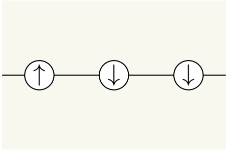
Integrable Systems


Integrable Systems
Gauge Theories


Integrable Systems
Gauge Theories
String Theory

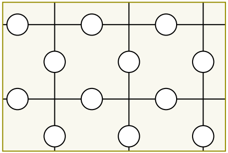
Two of the most famous quantum integrable systems

XYZ spin chain



$\Longleftrightarrow$

8-vertex model



arise from various QFTs:

1. 2d/3d/4d gauge theories, 4 SUSY [NS '09, Maulik–Okounkov '12, . . .]
2. 4d/5d/6d gauge theories, 8 SUSY [NS '09, Chen–Dorey–Hollowood–Lee '11]
3. 3d gauge theories, 8 SUSY [Bullimore–Dimofte–Gaiotto '15, Braverman–Finkelberg–Kamnitzer–Kodera–Nakajima–Webster–Weekes '16]
4. 4d "brane tiling" & "class-$\mathcal{S}_k$" theories [Spiridonov '10, Yamazaki '13, Y '15, Gaiotto–Rastelli–Razamat '12, Gadde–Gukov '13, GR '15, Maruyoshi–Y '16, Y '17]
5. 4d Chern–Simons theory [Costello '13, C–Witten–Yamazaki '17, '18, CY '18]

Question:

Why does a single quantum integrable system appear in multiple QFT setups?

Answer:

They are different descriptions of the same physical system, related by dualities in string theory.

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Another motivation:

Understand 4d Chern–Simons theory.

4d CS has the most direct relation to integrable lattice models, but it is a strange theory:

- It can only be defined on $\Sigma \times C$ with
$$C = \mathbb{C}\,, \quad \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \quad \text{or} \quad E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})\,.$$
- It has a complex gauge group $G_\mathbb{C}$.
- It has a complex action functional
$$S = \frac{\mathrm{i}}{\pi\hbar} \int_{\Sigma \times C} \mathrm{d}z \wedge \mathrm{CS}(\mathcal{A})\,,$$
$$\mathrm{CS}(\mathcal{A}) = \mathcal{A} \wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\,, \quad \hbar \in \mathbb{C}\,,$$
constructed from a partial connection
$$\mathcal{A} = \mathcal{A}_x \mathrm{d}x + \mathcal{A}_y \mathrm{d}y + A_{\bar{z}} \mathrm{d}\bar{z}\,, \quad \mathcal{A}_{x,y} = A_{x,y} + \mathrm{i}\phi_{x,y}\,.$$

Reason:

4d CS arises from 6d maximally SUSY Yang–Mills.

It turns out

$$\text{4d CS} = \text{topologically twisted, } \Omega\text{-deformed 6d MSYM}\,.$$

In turn, this allows us to realize 4d CS in string theory.

Outline

1. Motivations
2. 4d CS and integrable lattice models
3. 4d CS from 6d
4. String theory realization
5. Dualities
6. Summary & outlook

4d CS and integrable lattice models

Though 4d CS is strange, we can study it perturbatively. [C, CWY]

Basic observables are Wilson lines

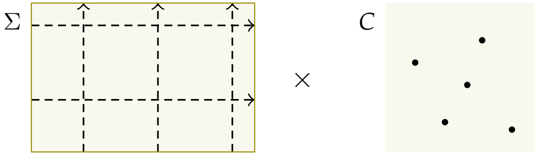
$$W_\alpha = \mathrm{Tr}_{V_\alpha} P \exp\left(\oint_{K_\alpha \times \{z_\alpha\}} \mathcal{A}\right)$$

in reps $G \to \mathrm{GL}(V_\alpha)$, around 1-cycles $K_\alpha \times \{z_\alpha\} \subset \Sigma \times C$.

Take

$$\Sigma = T^2$$

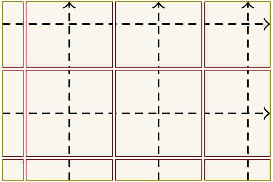
and make a lattice of Wilson lines:



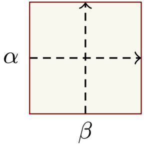
4d CS is topological on Σ and holomorphic on C:

$$\int_{\Sigma\times C} \mathrm{d}z \wedge \mathrm{CS}(\mathcal{A})\,, \qquad \mathcal{A} = \mathcal{A}_x \mathrm{d}x + \mathcal{A}_y \mathrm{d}y + A_{\bar{z}} \mathrm{d}\bar{z}\,.$$

Cut Σ into square pieces:

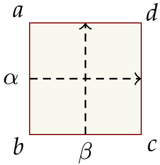


A single piece has two intersecting segments of line operators:

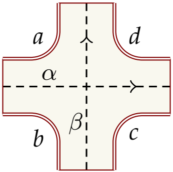


By topological invariance, precise shapes are irrelevant.

Pick boundary conditions a, b, c, d at the corners:



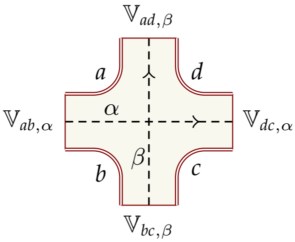
We can view this picture as scattering of two open strings:



Each string carries a particle (of type α or β) and ends on branes (labeled a, b, etc.).

We are dealing with 2d open–closed TQFT with line defects.

Let $\mathbb{V}_{ab,\alpha}$, $\mathbb{V}_{bc,\beta}$, $\mathbb{V}_{ad,\beta}$, $\mathbb{V}_{dc,\alpha}$ be spaces of states for these strings:

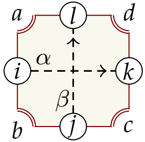


Path integral on this picture produces the R-matrix

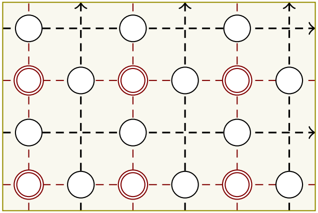
$$\check{R}_{\alpha\beta}\begin{pmatrix} a & d \\ b & c \end{pmatrix} : \mathbb{V}_{ab,\alpha} \otimes \mathbb{V}_{bc,\beta} \to \mathbb{V}_{ad,\beta} \otimes \mathbb{V}_{dc,\alpha} \, .$$

We represent its matrix elements as

$$\check{R}_{\alpha\beta}\begin{pmatrix} a & d \\ b & c \end{pmatrix}^{lk}_{ij} =$$



To reconstruct the whole Σ, we glue the pieces back:



To do this:

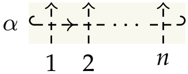
1. Pick branes for all ◎ and states for all ○. We get a bunch of R-matrix elements.
2. Take the product of all R-matrix elements.
3. Sum over all possible configurations of branes and states.

This is the partition function of a lattice model:

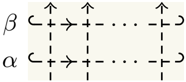
spin sites = ○ , ◎ ,   Boltzmann weight = R-matrix .

"Extra dimensions" C $\Longrightarrow$

1. Wilson line W_α carries a spectral parameter $z_\alpha \in C$.
2. Transfer matrices

$t_\alpha(z_\alpha) =$  $\in \mathrm{End}\left(\bigotimes_{\beta=1}^{n} V_\beta\right)$

commute:



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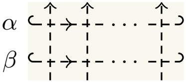


$\Longleftrightarrow \quad [t_\alpha(z_\alpha), t_\beta(z_\beta)] = 0\,.$

$\Longrightarrow$ series of commuting conserved charges:

$$t(z) = \sum_{n=-\infty}^{\infty} t_n z^n\,; \qquad [t_m, t_n] = 0\,.$$

The lattice model is integrable.

The R-matrix solves the Yang–Baxter equation.

For Wilson lines in the vector rep, we get the R-matrix for

- rational 6-vertex model for $C = \mathbb{C}$; [C]
- trigonometric 6-vertex model for $C = \mathbb{C}^\times$. [CWY]

Their conserved charges include Hamiltonians for

- XXX spin chain;
- XXZ spin chain.

For $C = E$, we get Felder's elliptic dynamical R-matrix, carrying an extra parameter $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ associated with d.o.f. on faces. [CY]

This is related by conjugation to Baxter's elliptic R-matrix for

$$\text{8-vertex model} \longleftrightarrow \text{XYZ spin chain}\,.$$

4d CS from 6d

6d MSYM is dimensional reduction of 10d SYM:

- 4 scalars $\phi_\mu = A_{6+\mu}$, $\mu = 1, \ldots, 4$
- R-symmetry $\mathrm{Spin}(4)_R$

Put 6d MSYM on

$$M \times C$$

and topologically twist along M:

- $\mathrm{Spin}(4)_M \to \mathrm{Spin}(4)'_M = \mathrm{diag}(\mathrm{Spin}(4)_M \times \mathrm{Spin}(4)_R)$
- Scalars become a 1-form $\phi = \phi_\mu \mathrm{d}x^\mu$ on M.
- $\exists$ supercharge Q that is scalar on M and

$$Q^2 = 0\,, \quad \partial_\mu = \{Q, \ldots\}\,, \quad \partial_{\bar{z}} = \{Q, \ldots\}\,.$$

Q-cohomology is topological on M and holomorphic on C.

Take

$$M = \mathbb{R}^2 \times \Sigma$$

and describe the 6d theory as a 2d theory on $\mathbb{R}^2$.

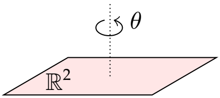
We get a B-twisted $\mathcal{N} = (2,2)$ SUSY gauge theory with

- gauge group $\mathcal{G} = \mathrm{Map}(\Sigma \times C, G)$;
- vector multiplet A_1, A_2, ϕ_1, ϕ_2;
- 3 adjoint chiral multiplets $\mathcal{A}_3 = A_3 + \mathrm{i}\phi_3$, $\mathcal{A}_4 = A_4 + \mathrm{i}\phi_4$, $A_{\bar{z}}$;
- superpotential

$$W = -\frac{\mathrm{i}}{e^2} \int_{\Sigma \times C} \mathrm{d}z \wedge \mathrm{CS}(\mathcal{A}) \,.$$

Quite generally, a B-twisted gauge theory on $\mathbb{R}^2$ can be subjected to Ω-deformation: [Nekrasov '02, Y '14, Luo–Tan–Y–Zhao '14]

$$Q^2 = \epsilon \partial_\theta \,, \quad \epsilon \in \mathbb{C} \,.$$



This reduces the path integral to

$$\int_{\gamma/\mathcal{G}_{\mathbb{C}}} \mathrm{d}\varphi_0 \exp\left(\frac{2\pi}{\epsilon} W(\varphi_0)\right) ,$$

where

- φ_0 is the constant mode of the chiral multiplet φ;
- γ is constructed from gradient flows ("Lefschetz thimble").

This is a 0d gauged sigma model on γ, with gauge group $\mathcal{G}_{\mathbb{C}}$.

Apply this mechanism to the 6d theory on $\mathbb{R}^2 \times \Sigma \times C$.

Viewing it as B-twisted gauge theory on $\mathbb{R}^2$, we get

$$\int_{\gamma/\mathcal{G}_\mathbb{C}} \mathcal{D}\mathcal{A} \exp\left(\frac{\mathrm{i}}{\pi\hbar}\int_{\Sigma\times C} \mathrm{d}z \wedge \mathrm{CS}(\mathcal{A})\right), \quad \hbar = -\frac{\epsilon e^2}{2\pi^2}\,.$$

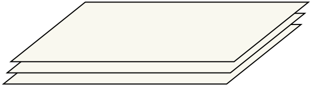
Therefore,

4d CS $=$ 6d MSYM, twisted and Ω-deformed on $\mathbb{R}^2$.

This provides a nonperturbative definition of 4d CS.

[cf. Ashwinkumadr–Tan–Qin]

String theory realization



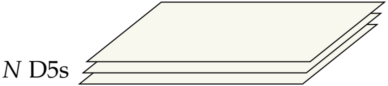
Consider a stack of N D5-branes.

Put them on

$$M \times C \subset T^*M \times C\,.$$

On D5s lives 6d MSYM with $G = \mathrm{SU}(N)$.

Normal directions to D5s are parametrized by "scalars" ϕ_μ.

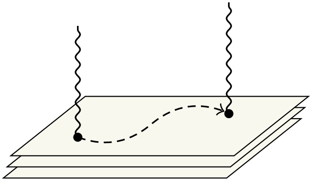


These "scalars" are really a 1-form $\phi = \phi_\mu \mathrm{d}x^\mu$

$\Longrightarrow$ 6d MSYM is twisted.

Take $M = \mathbb{R}^2 \times \Sigma$. Ω-deformation is given by a nontrivial RR 2-form C_2. ("RR fluxtrap" [Hellerman–Orlando–Reffert])

This realizes 4d CS.



Wilson lines are open strings ending on D5s.

N choices of D5s to end on, hence the vector rep of $\mathrm{SU}(N)$.

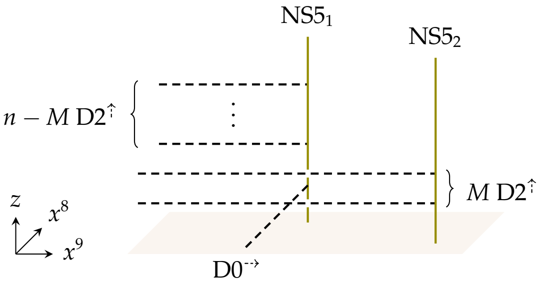
Other reps have more elaborate construction involving additional branes. [Yamaguchi '06, Gomis–Passerini '06]

Dualities

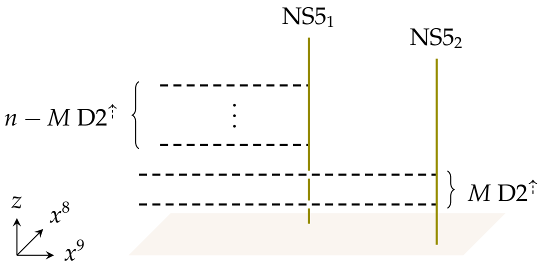
Apply S-duality and T-duality in the horizontal direction of T^2:

$$
\begin{array}{ccccc}
\mathrm{D5} & \xrightarrow{S} & \mathrm{NS5} & \xrightarrow{T} & \mathrm{NS5} \\
\mathrm{F1}^{\dashrightarrow} & \xrightarrow{S} & \mathrm{D1}^{\dashrightarrow} & \xrightarrow{T} & \mathrm{D0}^{\dashrightarrow} \\
\mathrm{F1}^{\uparrow} & \xrightarrow{S} & \mathrm{D1}^{\uparrow} & \xrightarrow{T} & \mathrm{D2}^{\uparrow} \\
C_2 & \xrightarrow{S} & B_2 & \xrightarrow{T} & B_2
\end{array}
$$

For $N = 2$, the brane configuration looks like



The magnon number M counts “up” spins in the spin chain.

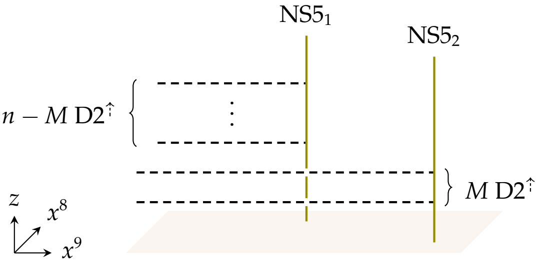


For $C = \mathbb{C}$ and $B_2 = 0$, NS5–D2 realizes 2d $\mathcal{N} = (4, 4)$ gauge theory with $G = \mathrm{U}(M)$ and n hypermultiplets:

$$\boxed{n} — \bigcirc\!\!\!\!\!\!M$$

z_α determine twisted masses associated with $\mathrm{U}(n)$ flavor group.

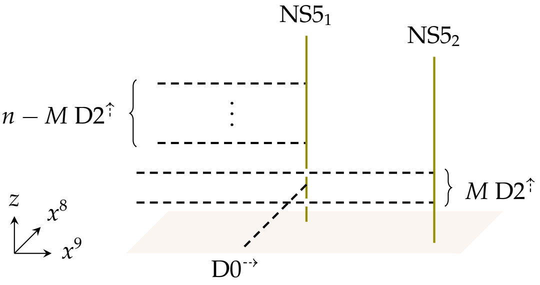
B_2 gives twisted masses breaking $\mathcal{N} = (4, 4)$ SUSY to $(2, 2)$.



Generically, the theory is in the Higgs phase.

For z_α all equal and $B_2 = 0$, the theory flows to A-model on $T^*\mathrm{Gr}(M, n)$.

For $N \geq 2$, the target space is T^*(partial flag manifold).



D0 represents a local operator and generates the chiral ring.

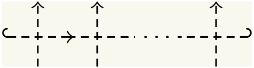
The chiral ring is the quantum cohomology

$$QH(T^*\mathrm{Gr}(M,n))\,.$$

For generic z_α and B_2, it is equivariant quantum cohomology

$$QH_{(\mathbb{C}^\times)^n\times\mathbb{C}^\times}(T^*\mathrm{Gr}(M,n))\,.$$

In the lattice model, D0 represents a transfer matrix:



Therefore,

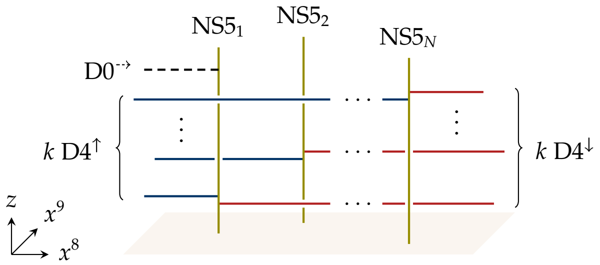
$QH_{(\mathbb{C}^\times)^n \times \mathbb{C}^\times}(T^*\mathrm{Gr}(M,n)) =$ algebra of conserved charges in M-magnon sector of XXX spin chain of length n.

This is 2d Nekrasov–Shatashvili correspondence.

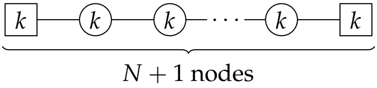
[NS, Maulik–Okounkov, . . .]

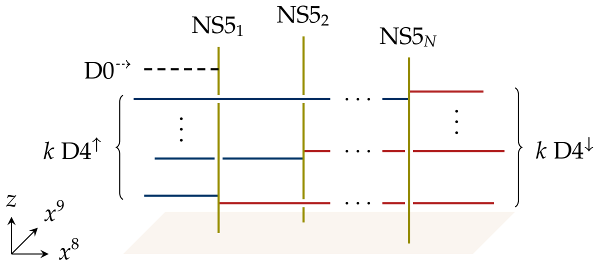
Brane construction provides a concrete realization of the transfer matrix, which has not been understood geometrically.

Replace D2s with D4s:



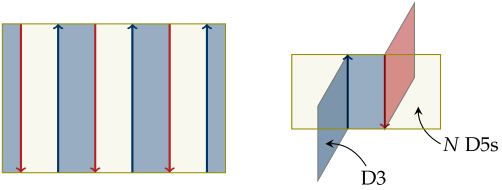
D4–NS5 realizes 4d $\mathcal{N} = 2$ SUSY gauge theory described by a linear quiver:

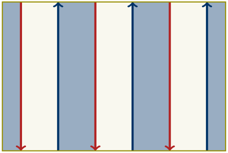


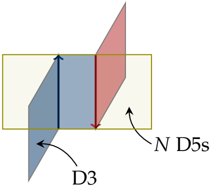


In the original duality frame, D4s are D3s.

D3s combine with D5s and produce strips of surface operators:

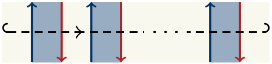






A strip of surface operator is a “thick” line operator, carrying a Verma module of $\mathfrak{sl}_N$. [CY, C–Gaiotto–Y]

The corresponding transfer matrix

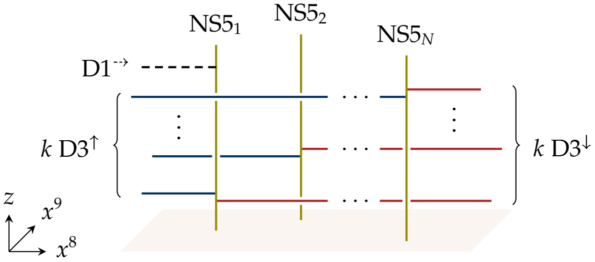


produces a “noncompact” XXX spin chain.

This explains 4d Nekrasov–Shatashvili correspondence.
[NS, Chen–Dorey–Hollowood–Lee, . . .]

What about open spin chains?

If the horizontal direction of the lattice is not S^1, we cannot apply the T-duality. But we can still apply S-duality:



D3–NS5 realizes 3d $\mathcal{N} = 4$ linear quiver theory.

Open XXX spin chain has a large symmetry, Yangian.

This explains why Yangian appears in this 3d theory.
[Bullimore–Dimofte–Gaiotto, BFKKNWW]

Other chains of dualities produce other QFT setups:

- Take $C = E$ and apply T-duality:

$$\text{D1–D3–NS5} \longrightarrow \text{D3–D5–NS5}$$

This produces 4d $\mathcal{N} = 1$ "brane tiling" theories + surface operators. [Maruyoshi–Y]

- Further apply T-duality and lift to M-theory:

$$\text{D3–D5–NS5} \begin{array}{l} \nearrow \text{D2–D4–NS5} \longrightarrow \text{M2–M5} \\ \searrow \text{D4–D4–NS5} \longrightarrow \text{M5–M5} \end{array}$$

We get 4d $\mathcal{N} = 1$ theories of "class $\mathcal{S}_k$" + surface operators
[Gaiotto–Rastelli–Razamat, Gadde–Gukov, Gaiotto–Razamat, …]

Summary:

- 4d CS arises from Ω-deformed twisted 6d MSYM.
- The latter can be embedded into string theory using brane.
- String dualities allow us to connect it to various other QFTs in which integrable systems have been found to arise.

Outlook:

- More chains of dualities.
- T-operators from Wilson lines, Q-operators from 't Hooft lines. [Work in progress with Costello and Gaiotto]
- Relation to spin chains in AdS/CFT integrability?