Singularity theorems and the stability of compact extra dimensions

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In 2002, as a present to S.W. Hawking, Penrose argued that spatial compact extra-dimensions are likely to be unstable [2003 On the instability of extra space dimensions, *The Future of Theoretical Physics and Cosmology*, ed G W Gibbons et al]
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To reach such conclusion he used the celebrated **singularity theorems**.
The classical Hawking-Penrose theorem

**Theorem (Hawking and Penrose 1970)**

If the convergence, causality and generic conditions hold and if there is one of the following:

- a closed achronal set without edge,
- a closed trapped surface,
- a point with re-converging light cone

then the space-time is causal geodesically incomplete.
Penrose’s argument

To use the singularity theorems, Penrose starts with a 
$(4 + n)$-dimensional direct product $M_4 \times \mathcal{Y} = \mathbb{R} \times \mathbb{R}^3 \times \mathcal{Y}$ 
with metric as in e.g.

$$g = -dt^2 + dx^2 + dy^2 + dz^2 + g_Y$$

and perturbs initial data given on a slice $\mathbb{R}^3 \times \mathcal{Y}$ (say $t = 0$) 
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He then forgets about the 3-dimensional typical space (in red) and considers a \((1 + n)\)-dimensional “reduced spacetime” \((\mathcal{Z}, g_{\text{red}})\) whose metric \(g_{\text{red}}\) is the evolution (e.g. Ricci-flat solution) of the initial data specified at \(\mathcal{Y}\) \((t = 0)\).
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- the entire spacetime would be given by $\mathbb{R}^3 \times \mathcal{Z}$ with direct product metric

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- But then, the H-P singularity theorem applies to $({\mathcal{Z}},g_{\text{red}})$ as it contains a compact slice and satisfies the convergence condition (because $R_{\mu\nu} = 0$).
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- He concluded that “if we wish to have a chance of perturbing $\mathcal{Y}$ in a finite generic way so that we obtain a non-singular perturbation of the full $(4+n)$- spacetimes $M_4 \times \mathcal{Y}$, then we must turn to consideration of disturbances that significantly spill over into the $M_4$ part of the spacetime”.

- However, he claimed that such general disturbances are even more dangerous (due to the large approaching Planck-scale curvatures that are likely to be present in $\mathcal{Y}$).

- He defended that there is good reason to believe that these general perturbations will also result in spacetime singularities using again the H-P singularity theorem, but now using the point with reconverging light cone condition.
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"I believe that it is possible to show that with a generic but small perturbation (...) this saving property will be destroyed, so that the (...) singularity theorem will indeed apply, but a fully rigorous demonstration (...) is lacking at the moment. Details of this argument will be presented elsewhere in the event that it can be succinctly completed"
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Their conclusions were criticized by Koster and Postma, where one can find references to many other no-go and instability theorems. [R. Koster and M. Postma, A no-go for no-go theorems prohibiting cosmic acceleration in extra dimensional models, JCAP 12 (2011) 015]
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- I want to concentrate here on the arguments based on the existence of singularities.
The classical Hawking-Penrose theorem (again)

**Theorem (Hawking and Penrose 1970)**

If the convergence, causality and generic conditions hold and if there is one of the following:

- a closed achronal set without edge, *(co-dimension 1)*
- a closed trapped surface, *(co-dimension 2)*
- a point with re-converging light cone *(co-dimension D)*

then the space-time is causal geodesically incomplete.
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What about co-dimensions 3, . . . , $D - 1$ — for instance, closed spacelike curves?
The Penrose singularity theorem

**Theorem (The 1965 Penrose singularity theorem)**

*If the spacetime contains a non-compact Cauchy hypersurface and a closed trapped surface, and if the null convergence condition holds, then there exist incomplete null geodesics.*

Here, the germinal and very fruitful notion of closed trapped surface was introduced.
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If the spacetime contains a non-compact Cauchy hypersurface and a closed trapped surface, and if the null convergence condition holds, then there exist incomplete null geodesics.

Here, the germinal and very fruitful notion of closed trapped surface was introduced.

These are closed surfaces (that is, compact without boundary) such that their area tends to decrease locally along any possible future direction. (There is a dual definition to the past).
“Normal situation”

\[ t = t_0 + dt \]

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Possible trapping in contracting worlds

$t = t_0 + dt$

$(x, y, t)$
Trapped submanifolds of arbitrary dimension?

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Some time ago, Galloway and I started to analyze the reasons behind the absence of other co-dimensions in the H-P singularity theorem, and we realized that the three conditions (on the point with reconverging light cone, on the closed trapped surface, and on the spacelike compact slice) can be unified into one single criterion of geometrical basis.

And that this criterion is valid for any other co-dimension!
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The unification concept of trapping for arbitrary co-dimension:

⇒ The mean curvature vector $\vec{H}$!
Mathematical interlude: trapped submanifolds

Consider an embedded **spacelike submanifold** $\zeta$ of any co-dimension $m$ with first fundamental form $\gamma_{AB}$. 

- Decomposing the derivatives of tangent vector fields $\{\vec{e}_A\}$ into its tangent and normal parts we have
  
  $e^\rho_A \nabla^\rho e^\mu_\beta = \Gamma^C_{AB} e^\mu_C - K^\mu_{AB}$

  - The mean curvature vector (Notice that $H^\mu$ is normal to $\zeta$)
    
    $H^\mu \equiv \gamma_{AB} K^\mu_{AB}$

  - An expansion of $\zeta$ relative to any normal vector $\vec{n}$ is:
    
    $\theta(\vec{n}) \equiv n^\mu H^\mu$

    - There are $m$ independent expansions.

    - If they correspond to (future) null normals, they are called (future) null expansions.
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Definition (Trapped submanifold)

A spacelike submanifold $\zeta$ is said to be **future trapped** (f-trapped from now on) if $\vec{H}$ is timelike and future-pointing everywhere on $\zeta$, and similarly for past trapped.
Future-trapped submanifolds: $\vec{H}$ is future on $\zeta$

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Now that we have trapped submanifolds of any dimension, can we still get singularity theorems based on them?
Closed trapped submanifolds at work:

XXI-century singularity theorems
The parallel propagated projector $P^{\mu\nu}$

**Notation**

- $n_\mu$: *future-pointing* normal to the spacelike submanifold $\zeta$,
- $\gamma$: geodesic curve tangent to $n^\mu$ at $\zeta$,
- $u$: affine parameter along $\gamma$ ($u = 0$ at $\zeta$),
- $N^\mu$: geodesic vector field tangent to $\gamma$ ($N^\mu|_{u=0} = n^\mu$).
- $\vec{E}_A$: vector fields defined by parallel propagating $\vec{e}_A$ along $\gamma$ ($\vec{E}_A|_{u=0} = \vec{e}_A$)
- By construction $g_{\mu\nu}E^\mu_A E^\nu_B$ is independent of $u$, so that $g_{\mu\nu}E^\mu_A E^\nu_B = g_{\mu\nu}e^\mu_A e^\nu_B = \gamma_{AB}$
- $P^{\nu\sigma} \equiv \gamma^{AB} E^\nu_A E^\sigma_B$ (at $u = 0$ this is the projector to $\zeta$).
Notation on a picture
Generalized Hawking-Penrose singularity theorem

**Theorem (Generalized Hawking-Penrose singularity theorem)**

If the chronology, generic and convergence conditions hold and there is a closed $f$-trapped submanifold $ζ$ of arbitrary co-dimension such that

$$R_{μνρσ}N^μN^ρP_{νσ} \geq 0$$

along every null geodesic emanating orthogonally from $ζ$ then the spacetime is causal geodesically incomplete.

Remarks:

\[ R_{\mu\nu\rho\sigma} N^{\mu} N^{\rho} P^{\nu\sigma} \geq 0 \]  \hspace{1cm} (1)

1. **Spacelike hypersurfaces:** \( m = 1 \), there is a unique timelike orthogonal direction \( n_{\mu} \). Then \( P^{\mu\nu} = g^{\mu\nu} - (N_{\rho} N^{\rho})^{-1} N^{\mu} N^{\nu} \) and (1) reduces to

\[ R_{\mu\nu} N^{\mu} N^{\nu} \geq 0 \]

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2. **Spacelike ‘surfaces’**: \( m = 2 \), there are two independent null normals on \( \zeta \), say \( n_\mu \) and \( \ell_\mu \). (Define \( L_\mu \) parallelly propagating \( \ell_\mu \) on \( \gamma \)). Then, \( P^{\mu\nu} = g^{\mu\nu} - (N_\rho L^\rho)^{-1}(N^\mu L^\nu + N^\nu L^\mu) \) and again (1) reduces to

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Points: $m = D$, (1) could be rewritten as a ‘generic’ condition

$$R_{\mu\nu\rho\sigma} N^\mu N^\rho > 0.$$
The generalized Penrose singularity theorem

**Theorem (Generalized Penrose singularity theorem)**

If $(M, g)$ contains a non-compact Cauchy hypersurface $\Sigma$ and a closed $f$-trapped submanifold $\zeta$ of arbitrary co-dimension, and if

$$R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} \geq 0$$

holds along every future-directed null geodesic emanating orthogonally from $\zeta$, then $(M, g)$ is future null geodesically incomplete.

(G.J. Galloway and J.M.M. Senovilla, *ibid.*)
The conclusion of the generalized Penrose theorem remains valid if the curvature condition and the trapping condition assumed there are jointly replaced by

$$\int_0^a R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} du > \theta(\vec{n}) ,$$

along each future inextendible null geodesic $\gamma : [0, a) \to M$ emanating orthogonally from $\zeta$ with initial tangent $n^\mu$. 

No need for trapped submanifold!
Theorem

If \((M, g)\) contains a non-compact Cauchy hypersurface \(\Sigma\) and is null geodesically complete, then for every closed spacelike submanifold \(\zeta\) there exists at least one null geodesic \(\gamma\) with initial tangent \(n^\mu\) orthogonal to \(\zeta\) along which

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2nd generalized Penrose singularity theorem

**Theorem**

If $(M, g)$ contains a non-compact Cauchy hypersurface $\Sigma$ and is null geodesically complete, then for every closed spacelike submanifold $\zeta$ there exists at least one null geodesic $\gamma$ with initial tangent $n^\mu$ orthogonal to $\zeta$ along which

$$\int_0^\infty R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} du \leq \theta(\vec{n}).$$

Observe that there is no restriction on the sign of $\theta(\vec{n})$. 
Higher-dimensional spacetimes:

(warped) products
Consider a spacetime $M = M_1 \times M_2$, $x^\mu = (x^a, x^i)$, with direct product metric

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Direct product: “it just fails”

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- Let $\varsigma \subset M_2$ be compact and $\dim \varsigma = k$, and let $e_A^i$ be tangent vectors to $\varsigma$. Then, their parallel transports along geodesics are such that \( \vec{E}_A = (0, \vec{E}_A^i) \)
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Geodesics decompose too, tangent vectors are $\vec{N} = (\hat{N}^a, \bar{N}^i)$, with $\hat{N}^a$ and $\bar{N}^i$ geodesic in $(M_1, \hat{g})$ and $(M_2, \bar{g})$, respectively.
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- Then $R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} = \bar{R}_{ijkl} \bar{N}^i \bar{N}^k P^{jl}$, $P^{jl} = \gamma^{AB} E^j_A E^l_B$. 
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g_{\mu\nu} dx^\mu dx^\nu = \hat{g}_{ab}(x^c) dx^a dx^b + \bar{g}_{ij}(x^k) dx^i dx^j
\]

\( R^{\alpha}{}_{\beta\mu\nu} = \left( \hat{R}^{a}{}_{bcd}, \bar{R}^{i}{}_{jkl} \right) \)

Let \( \varsigma \subset M_2 \) be compact and \( \dim \varsigma = k \), and let \( e^i_A \) be tangent vectors to \( \varsigma \). Then, their parallel transports along geodesics are such that \( \vec{E}_A = (0, E^i_A) \)

Geodesics decompose too, tangent vectors are \( \vec{N} = (\hat{N}^a, \bar{N}^i) \), with \( \hat{N}^a \) and \( \bar{N}^i \) geodesic in \( (M_1, \hat{g}) \) and \( (M_2, \bar{g}) \), respectively.

Then \( R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} = \bar{R}_{ijkl} \bar{N}^i \bar{N}^k P^{jl}, P^{jl} = \gamma^{AB} E_A^j E_B^l \).

But, there are \( \perp \varsigma \)-null geodesics with \( \bar{n}^i = \bar{N}^i(0) = 0 \), and for these \( \bar{N}^i(u) = 0 \), and \( \theta(\bar{n}) = 0 \), so that any of the two conditions would read

\[
0 > 0 \quad \text{(just fails)}
\]
Consider perturbing the previous spacetime. The simplest way to do it (geometrically) is by breaking the direct product structure and letting one of the two pieces influence the other:

\[ g_{\mu\nu} dx^\mu dx^\nu = \hat{g}_{ab}(x^c) dx^a dx^b + f^2(x^c) \bar{g}_{ij}(x^k) dx^i dx^j \]
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These are called *warped products* \( M_1 \times_f M_2 \), with Base \( M_1 \), Fiber \( M_2 \) and warping function \( f : M_1 \rightarrow \mathbb{R} \).
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There are two possibilities here:
Perturbations: warped products

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- These are called \textit{warped products} $M_1 \times_f M_2$, with Base $M_1$, Fiber $M_2$ and warping function $f : M_1 \rightarrow \mathbb{R}$.

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  - 1. \textbf{Extra-dimension spreading}: the fiber $(M_2, \bar{g})$ is Lorentzian.
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1. **Extra-dimension spreading**: the fiber \((M_2, \bar{g})\) is Lorentzian.
2. **Dynamical**: the base \((M_1, \hat{g})\) is Lorentzian.
Consider perturbing the previous spacetime. The simplest way to do it (geometrically) is by breaking the direct product structure and letting one of the two pieces influence the other:

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There are two possibilities here:

1. **Extra-dimension spreading**: the fiber $(M_2, \bar{g})$ is Lorentzian.
2. **Dynamical**: the base $(M_1, \hat{g})$ is Lorentzian.

They imply very different physical consequences! Actually, case 1 does not lead to singularities.
Extra-dimension spreading: “just fails” too

For case 1, extra-dimension spreading over the Lorentzian part, either the latter is geodesically incomplete by itself or not, the extra dimensions being unable to turn it into null geodesically incomplete.

This follows for instance from a known result that if the Riemannian base of a warped product is complete —which is always the case for compact base— then the spacetime is geodesically complete if and only if the fiber so is.

Extra-dimension spreading: “just fails” too

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Thus, Penrose’s suggestion that “disturbances that significantly spill over into the 4-dimensional part of the spacetime” would be more dangerous and will result in singularities does not seem to sustain—at least in this warped-product situation.
Warped products: Curvature

- Recall \( g_{\mu\nu} dx^\mu dx^\nu = \hat{g}_{ab}(x^c) dx^a dx^b + f^2(x^c) \bar{g}_{ij}(x^k) dx^i dx^j \)

  \( a, b, \ldots, h \) indices on 4-dimensional \( M_1 \); \( i, j, k, l \) indices on \( n \)-dimensional \( M_2 \). Total dimension \( D := 4 + n \)
Warped products: Curvature

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  $a, b, \ldots, h$ indices on 4-dimensional $M_1$; $i, j, k, l$ indices on $n$-dimensional $M_2$. Total dimension $D := 4 + n$

- $R^a_{\ ijk} = 0, \quad R^i_{\ abc} = 0, \quad R^i_{\ jab} = 0$
- $R^a_{\ ibj} = -f\hat{\nabla}_b\hat{\nabla}^a f \bar{g}_{ij}$
- $R^i_{\ jkl} = \bar{R}^i_{\ jkl} - \hat{\nabla}^a f\hat{\nabla}_a f (\delta^i_k\bar{g}_{jl} - \delta^i_l\bar{g}_{jk})$
- $R^a_{\ bcd} = \hat{R}^a_{\ bcd}$
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- \( R^a_{bcd} = \hat{R}^a_{bcd} \)

- \( R_{ab} = \hat{R}_{ab} - n \frac{1}{f} \hat{\nabla}_a \hat{\nabla}_b f \)
- \( R_{ai} = 0 \)
- \( R_{ij} = \bar{R}_{ij} - \bar{g}_{ij} \left( f \hat{\nabla}^b \hat{\nabla}_b f + (n - 1) \hat{\nabla}_b f \hat{\nabla}^b f \right) \)
Warped products: null geodesics

Let \( \gamma: x^\mu = x^\mu(u) \) be an affinely parametrized null geodesic with tangent vector \( dx^\mu/du := N^\mu = (\hat{N}^a, \bar{N}^i) \).
Warped products: null geodesics

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$$\bar{N}^j \nabla_j \bar{N}^i = -2\hat{N}^a \partial_a (\ln f) \bar{N}^i = -2 \frac{d \ln f}{du} \gamma \bar{N}^i$$
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This states that the curve projected to $M_2$ is itself a geodesic (non-affinely parametrized).
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In particular, if the $M_2$-initial velocity vanishes $\bar{N}^i(0) = n^i = 0$, then $\bar{N}^i(u) = 0$ for all $u$. 

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$$\hat{N}^b \hat{\nabla}_b \hat{N}^a = -\left(\hat{g}_{bc} \hat{N}^b \hat{N}^c\right) \hat{\nabla}^a (\ln f)$$

- This tells us that the acceleration of the $M_1$-projected curve is always parallel to the gradient of $f$. 
Warped products: null geodesics (2)

From the above one knows that

\[ \bar{g}_{ij} \bar{N}^i \bar{N}^j = \frac{C}{f^4}, \quad C = \text{const.} \]
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- For the relevant case \( \geq 2 \), “dynamical” : \( C \geq 0 \).
From the above one knows that

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\[ \hat{g}_{ab} \hat{N}^a \hat{N}^b = -\frac{C}{f^2} \]

For the relevant case 2, “dynamical” : \( C \geq 0 \).

\( C = 0 \) means that the null geodesic lives exclusively in the Lorentzian part \((M_1, \hat{g})\) of the warped product.
As \((M_1, \hat{g})\) is Lorentzian, we choose \(\zeta \subset M_2\) compact with co-dimension \(m\).
As $(M_1, \hat{g})$ is Lorentzian, we choose $\zeta \subset M_2$ compact with co-dimension $m$

$\{\vec{e}_A\}$ ON basis of vector fields tangent to $\zeta$: $e^\mu_A = (0, \vec{e}_A^i)$. 
As \((M_1, \hat{g})\) is Lorentzian, we choose \(\zeta \subset M_2\) compact with co-dimension \(m\).

\(\{\bar{e}_A\}\) ON basis of vector fields tangent to \(\zeta\): \(e_A^\mu = (0, \bar{e}_A^i)\).

One can prove then that

\[
E_A^\mu = (0, \bar{E}_A^i / f)
\]

where \(\bar{E}_A^i\) are the parallel transports of \(\bar{e}_A^i\) along the projected curve \(\bar{\gamma} : x^i(u)\): \(\bar{\nabla}_j \bar{E}_A^i = 0\), \(\bar{E}_A^i(0) = \bar{e}_A^i\).
As \((M_1, \hat{g})\) is Lorentzian, we choose \(\zeta \subset M_2\) compact with co-dimension \(m\)

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\[ g_{\mu\nu} N^\mu E^\nu_A = 0 \implies \bar{g}_{ij} \bar{N}^i \bar{E}^j_A = 0. \]
Parallel transport along null geodesic $\perp \varsigma$, case 2

- As $(M_1, \hat{g})$ is Lorentzian, we choose $\varsigma \subset M_2$ compact with co-dimension $m$
- $\{\bar{e}_A\}$ ON basis of vector fields tangent to $\varsigma$: $e^\mu_A = (0, \bar{e}^i_A)$.
- One can prove then that

$$E^\mu_A = (0, \bar{E}^i_A / f)$$

where $\bar{E}^i_A$ are the parallel transports of $\bar{e}^i_A$ along the projected curve $\bar{\gamma}: x^i(u)$: $\bar{N}^j \nabla_j \bar{E}^i_A = 0$, $\bar{E}^i_A(0) = \bar{e}^i_A$.

- $g_{\mu\nu} N^\mu E^\nu_A = 0 \implies \bar{g}_{ij} \bar{N}^i \bar{E}^j_A = 0$.
- $g_{\mu\nu} E^\mu_B E^\nu_A = \delta_{BA} \implies \bar{g}_{ij} \bar{E}^i_A || \bar{E}^j_B || = \delta_{AB}$. 
As \((M_1, \hat{g})\) is Lorentzian, we choose \(\zeta \subset M_2\) compact with co-dimension \(m\).

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\[ E^\mu_A = (0, \bar{E}^i_A \parallel / f) \]

where \(\bar{E}^i_A \parallel\) are the parallel transports of \(\bar{e}_A^i\) along the projected curve \(\bar{\gamma}: x^i(u): \bar{N}^j \nabla_j \bar{E}^i_A \parallel = 0, \quad \bar{E}^i_A \parallel(0) = \bar{e}_A^i\).

\[ g_{\mu\nu} N^\mu E^\nu_A = 0 \quad \Rightarrow \quad \bar{g}_{ij} \bar{N}^i \bar{E}^j_A \parallel = 0. \]

\[ g_{\mu\nu} E^\mu_B E^\nu_A = \delta_{BA} \quad \Rightarrow \quad \bar{g}_{ij} \bar{E}^i_A \parallel \bar{E}^j_B \parallel = \delta_{AB}. \]

In this case the tensor \(P^{\mu\nu} = \gamma^{AB} E^\mu_A E^\nu_B\) reads

\[ P^{ab} = 0, \quad P^{ia} = 0, \quad P^{ij} = \frac{1}{f^2} \delta^{AB} \bar{E}^i_A \parallel \bar{E}^j_B \parallel \]
Expression (1), case

\[ R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} = \delta^{AB} \bar{R}_{ijkl} \bar{N}^i \bar{N}^k \bar{E}^j_A \bar{E}^l_B - (D - m) \frac{1}{f} \frac{d^2 f}{d u^2} \mid_\gamma \]
Expression (1), case 2

\[ R_{\mu\nu\rho\sigma} N^\mu N^\rho P^\nu P^\sigma = \delta^{AB} \bar{R}_{ijkl} \bar{N}^i \bar{N}^k \bar{E}_A^j \bar{E}_B^l - (D - m) \frac{1}{f} \frac{d^2 f}{du^2} |_\gamma \]

- This is written in terms of properties of the Riemannian extra-dimensions in \((M_2, \bar{g})\) and the projected geodesic \(\bar{\gamma}\) plus the second derivative of the warping function along the null geodesic \(\gamma\).
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A simple computation gives, for the initial expansion along \(\bar{n}\):

\[ \theta(\bar{n}) = \bar{\theta}_{\bar{n}} + (D - m) \frac{1}{f_0} \frac{df}{du}(0) \]

where \(\bar{\theta}_{\bar{n}}\) is “expansion of \(\zeta\) as submanifold of \((M_2, \bar{g})\)”. 
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The integrated condition in the singularity theorem reads then

\[
\int_0^\infty \left( \delta^{AB} \bar{R}_{ijkl} \bar{N}^i \bar{N}^k \bar{E}^j_A \bar{E}^l_B - (D - m) \frac{1}{f} \frac{d^2 f}{du^2} |\gamma | \right) du \\
> \bar{\theta}_{\bar{n}} + (D - m) \frac{1}{f_0} \frac{df}{du}(0)
\]
Singularity theorems in warped products

**Theorem**

Let $M = M_1 \times f M_2$ be a null geodesically complete $D$-dimensional warped product spacetime with Riemannian fiber $(M_2, \bar{g})$ and metric

$$g_{\mu\nu} dx^\mu dx^\nu = \hat{g}_{ab}(x^c) dx^a dx^b + f^2(x^c) \bar{g}_{ij}(x^k) dx^i dx^j$$

containing a non-compact Cauchy hypersurface. Then, every compact submanifold $\zeta \subset M_2$, of any possible co-dimension $m$, launches at least one future-directed null geodesic emanating orthogonally to $\zeta$ satisfying the inequality

$$\int_0^\infty \left( \delta^{AB} \bar{R}_{ijkl} \bar{N}^i \bar{N}^k \bar{E}^j_A \bar{E}^l_B - (D - m) \frac{1}{f} \frac{d^2 f}{du^2} |\gamma| \right) du \leq \bar{\theta} + (D - m) \frac{1}{f_0} \frac{df}{du}(0).$$
Analysis of the inequality condition

The negation of the condition:

\[
\int_\gamma \left( \delta^{AB} \bar{R}_{ijkl} \bar{N}^i \bar{N}^k \bar{E}^j_A \bar{E}^l_B - (D - m) \frac{1}{f} \frac{d^2 f}{du^2} \right) du > \bar{\theta}_{\bar{n}} + (D - m) \frac{1}{f_0} \frac{df}{du}(0)
\]

(There is also a version to the past).

- For any \( \zeta \subset M_2 \), there are always \( \zeta \)-orthogonal null geodesics with \( \bar{n}^i = 0 \) and thus with \( \bar{N}^i(u) = 0 \) (those with \( C = 0 \)).
Analysis of the inequality condition

The negation of the condition:

\[
\int_{\gamma} \left( \delta^{AB} R_{ijkl} \bar{N}^i \bar{N}^k E^j_A \| \bar{E}^l_B \| - (D - m) \frac{1}{f} \frac{d^2 f}{d u^2} \right) d u
\]

\[
> \bar{\theta}_{\bar{n}} + (D - m) \frac{1}{f_0} \frac{d f}{d u}(0)
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(There is also a version to the past).

- For any \( \zeta \subset M_2 \), there are always \( \zeta \)-orthogonal null geodesics with \( \bar{n}^i = 0 \) and thus with \( \bar{N}^i(u) = 0 \) (those with \( C = 0 \)).
- For these geodesics, the above simplifies to

\[
- \int_{\gamma} \frac{1}{f} \frac{d^2 f}{d u^2} d u > \frac{1}{f_0} \frac{d f}{d u}(0)
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Analysis of the inequality condition

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> \bar{\theta}_{\bar{n}} + (D - m) \frac{1}{f_0} \frac{d f}{d u}(0)
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- For any \( \zeta \subset M_2 \), there are always \( \zeta \)-orthogonal null geodesics with \( \bar{n}^i = 0 \) and thus with \( \bar{N}^i(u) = 0 \) (those with \( C = 0 \)).
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\[
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\]

- in more geometrical terms this is

\[
- \int_{\gamma} \frac{1}{f} \hat{N}^a \hat{N}^b \hat{\nabla}_a \hat{\nabla}_b f > \left( \frac{1}{f} \hat{N}^a \hat{\nabla}_a f \right)(0)
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The last is, therefore, a necessary condition (along all null geodesics $\perp$ to $\zeta$ with $C' = 0$) for the singularities to appear.
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If this condition actually holds for all null geodesics starting at a given “\( M_2 \)” (i.e., for a choice of \( x^a = \text{const.} \)), then this \( M_2 \) is itself a compact submanifold leading to null geodesic incompleteness.

Nevertheless, if this does not happen for any choice of “\( M_2 \)”, it can happen for an appropriate subset and one can still have null incompleteness if the corresponding null geodesics are orthogonal to particular submanifolds \( \zeta \subset M_2 \).
The last is, therefore, a necessary condition (along all null geodesics $\perp$ to $\zeta$ with $C = 0$) for the singularities to appear.

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Nevertheless, if this does not happen for any choice of “$M_2$”, it can happen for an appropriate subset and one can still have null incompleteness if the corresponding null geodesics are orthogonal to particular submanifolds $\zeta \subset M_2$.

In this case, one still needs to check that the found inequality condition holds for the remaining null geodesics orthogonal to $\zeta$, those with $C > 0$. 
Recall: $- \int_\gamma \frac{1}{f} \hat{N}^a \hat{N}^b \hat{\nabla}_a \hat{\nabla}_b f > \left( \frac{1}{f} \hat{N}^a \hat{\nabla}_af \right)(0)$
Analysis of the necessary condition for $C = 0$

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- Observe that, from the expression of the Ricci tensor and as $N^\mu = (\hat{N}^a, 0)$ for these null geodesics, the null energy condition (NEC) on them reads

$$R_{\mu\nu} N^\mu N^\nu = \hat{R}_{ab} \hat{N}^a \hat{N}^b - n \frac{1}{f} \hat{N}^a \hat{N}^b \hat{\nabla}_a \hat{\nabla}_b f \geq 0$$

This immediately implies $- \int_{\gamma} \frac{1}{f} \hat{N}^a \hat{N}^b \hat{\nabla}_a \hat{\nabla}_b f \geq - \frac{1}{n} \int_{\gamma} \hat{R}_{ab} \hat{N}^a \hat{N}^b \leq 0$ where the last inequality follows if the NEC holds on average in the noticeable, observed, 4-dimensional spacetime.
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Back to the inequality condition

- This allows us to analyze in greater detail when the necessary condition will hold (easy for instance if $\hat{R}_{ab} = \Lambda \hat{g}_{ab}$ on $(M_1, \hat{g})$).

Even if some of the extra dimensions stay stationary, or expand while the others contract, there may be many situations where it also holds. Still, as mentioned before, we must consider the rest of null geodesics emanating orthogonal to $\zeta$, those with $C > 0$, and thus with $\bar{N}_i(u) \neq 0$.

Recall:

$$\int_{\gamma} (\delta_{AB} R_{ijkl} \bar{N}_i \bar{N}_k \bar{E}_j A \parallel \bar{E}_l B \parallel - (D - m)^1 f d^2 f du > \bar{\theta} \bar{n} + (D - m)^1 f_0 df du (0))$$

Hence, we need an analysis of the behaviour of $d^2 f/du^2$ along these null geodesics.
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\[
\int_{\gamma} (\delta_{AB} R_{ijkl} \bar{N}_i \bar{N}_k \bar{E}_j \bar{E}_l - (D - m) f) \, d\bar{S}_f \, du_2 > \bar{\theta} \bar{n} + (D - m)^{-1} \int_0^1 df \, du_2(0)
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The general expression for this second derivative along the given geodesics is

\[ \frac{d^2 f}{du^2} \bigg|_\gamma = \left( \frac{C}{f^3} \right) \hat{\nabla}^b f \hat{\nabla}_b f + \hat{N}^a \hat{N}^b \hat{\nabla}_a \hat{\nabla}_b f \]

The last summand is analyzed as before, but taking into account that \( \hat{N}^a \) are now timelike. The first summand on the right-hand side favors the singularity if the gradient of \( f \) is non-spacelike: this is the case if the perturbation is truly dynamical (i.e., the dynamical part dominates over other possible accompanying perturbations). Actually, keeping the values of the coupling constants (and the Planck mass) independent of position in space implies \( f \) should depend only on time and thus \( \hat{\nabla}_b f \hat{\nabla}_b f < 0 \). In consequence, \( \frac{d^2 f}{du^2} \bigg|_\gamma \) will become negative in a large class of reasonable situations.
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A useful form of the inequality

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- Furthermore, one actually needs this only on average along $\gamma$. 

Assume then that $X^2 := (D - m)(1 f_0 df_0 du + \int_\gamma 1 f_2 d\gamma_2 du) > 0$ can be proven to be strictly positive for the family of null geodesics orthogonal to a given compact $\zeta$. It follows that the condition such that singularities arise according to the theorem becomes $\int \overline{\gamma} \delta_{AB} \overline{R}_{ijkl} \overline{N}_i \overline{N}_k \overline{E}_j A \| \overline{E}_l B \| du > \overline{\theta} \overline{n} - X^2$. The importance of this form is that the left-hand side is a quantity relative to the extra-dimensional space $(M_2, \bar{g})$ exclusively.
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If the co-dimension is 5, that is dimension $n - 1$, then

$$\delta^{AB} \bar{R}_{ijkl} \bar{N}^i \bar{N}^k \bar{E}^j_A \parallel \bar{E}^l_B = \bar{R}_{ij} \bar{N}^i \bar{N}^j$$ (Ricci-flat $M_2$ OK!)
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Therefore, one can find many (physical) situations where this incompleteness arises.
Concluding remarks

- Allowing for arbitrary dynamical perturbations the function $f$ can satisfy the “destroying” conditions in physically interesting situations. How to avoid the destroying power of generic dynamical $f$ should thus be analyzed.
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- In essence, **dynamical** perturbations can sometimes lead to the appearance of singularities, destroying the stationary **classical** stability of the extra-dimensional space.

- On a positive side, the condition as given involving quantities of only the extra-dimensional space may help in finding the stable possibilities, providing information on which classes of compact extra-dimensions may be viable and why —and for which warping functions $f(t)$. 
Thank you for your attention!

あなたの注意のために大変ありがとう