$2d$ extremal correlators in $\mathcal{N} = (2,2)$ SCFTs

Jin Chen

Institute of Theoretical Physics,
Chinese Academy of Science

November 8, 2018

at Kavli IPMU, University of Tokyo

the talk is based on 1712.01164 and current work in progress

Overview

1. Introduction and motivation
   - Preliminaries: conformal manifold, chiral ring & extremal correlators
   - Relation to Calabi-Yau geometries
2. Computing extremal correlators (ECs) via localization
   - Review of $2d$ localization on $\mathbb{S}^2$
   - Operators mixings from $\mathbb{R}^2$ to $\mathbb{S}^2$
   - Algorithm: Gram-Schmidt othorgonalization
3. Example
   - Toda chain equations & constraints
   - Complete intersections in toric varieties
4. Summary
   - $2d$ v.s. $4d$ extremal correlators
   - Outlook

Introduction and motivation: conformal manifold

- Conformal manifolds of CFTs:

For a given d-dimensional CFT $\mathcal{S}_0$, one may deform it by *exactly* marginal operators $\mathcal{O}_i$'s

$$\mathcal{S}_0 \longrightarrow \mathcal{S} \equiv \mathcal{S}_0 + \lambda^i \int \mathrm{d}^d x\, \mathcal{O}_i\,.$$

Exactness of $\mathcal{O}_i$ guarantees

$$\Delta(\mathcal{O}_i) = d\,, \quad \beta(\lambda_i) = 0\,.$$

$\Longrightarrow$ $\mathcal{S}$ is also conformal. It implies that the theory has *moduli*.

The moduli space $\mathcal{M}$ is parametrized by coordinates $\{\lambda^i\}$, called *conformal manifold* of $\mathcal{S}$.

- Zamolodchikov metric g_{ij} on $\mathcal{M}$:

  We can study the geometry on $\mathcal{M}$. An important geometric data of $\mathcal{M}$ is to measure the variation of the partition function Z of the theory $\mathcal{S}$ along marginal parameters λ^i

$$\delta_\lambda^2 Z \equiv \frac{1}{Z}\frac{\partial^2 Z}{\partial\lambda^i\partial\lambda^j}\,\delta\lambda^i\delta\lambda^j \propto \langle\mathcal{O}_i(0)\mathcal{O}_j(\infty)\rangle\,\delta\lambda^i\delta\lambda^j$$

  Compared to $ds^2 = g_{ij}(\lambda)d\lambda^i d\lambda^j$ on $\mathcal{M}$, the Zamolodchikov metric is defined as

$$g_{ij}(\lambda) \equiv \langle\mathcal{O}_i(0)\mathcal{O}_j(\infty)\rangle$$

  One of tasks: is to exactly compute the function $g_{ij}(\lambda)$ in $2d$ $\mathcal{N}=(2,2)$ SCFTs in this talk.

- What are the exactly marginal operators in $2d$ $\mathcal{N}=(2,2)$ SCFTs?

  In $\mathcal{N}=(2,2)$ SCFTs, there are two R-symmetries, $U(1)_V \times U(1)_A$, corresponding to two types of marginal operators.

chiral primaries ϕ_i :

Def : $\left[\overline{\mathcal{Q}}_{\pm},\, \phi_i\right] = 0$

twisted chiral primaries σ_a :

$\left[\overline{\mathcal{Q}}_{+},\, \sigma_a\right] = [\,\mathcal{Q}_{-},\, \sigma_a] = 0$

R charge : $[q_V,\, \phi_i] = +2\,\phi_i$

$[q_A,\, \phi_i] = 0$

$[q_V,\, \sigma_a] = 0$

$[q_A,\, \sigma_a] = +2\,\sigma_a$

BPS bound : $\Delta(\phi_i) = \frac{1}{2} q_V(\phi_i) = 1$

$\Delta(\sigma_a) = \frac{1}{2} q_A(\sigma_a) = 1$

The conformal manifold $\mathcal{M}$ of a $\mathcal{N}=(2,2)$ SCFT $\mathcal{S}$ is spanned by chiral and twisted chiral primaries, and their hermitian conjugation,

$$\mathcal{S} \equiv \mathcal{S}_0 + \tau^i \int \mathrm{d}^2x\, \mathrm{d}^2\theta\ \Phi_i + \tilde{\tau}^a \int \mathrm{d}^2x\, \mathrm{d}^2\tilde{\theta}\ \Sigma_a + \mathrm{h.c.}\,.$$

where $\Phi_i = (\phi_i,\ \psi_i,\ F_i)$ and $\Sigma_a = (\sigma_a,\ \lambda_a,\ G_a)$ are the supermultiplets of chiral and twisted chiral primaries of ϕ_i and σ_a.

The conformal manifold $\mathcal{S}$ is factorized as (locally) direct product of two Kähler manifolds,

$$\mathcal{M} \simeq \mathcal{M}_{\mathrm{c}}(\tau,\bar{\tau}) \times \mathcal{M}_{\mathrm{tc}}(\tilde{\tau},\bar{\tilde{\tau}})\,.$$

because, by R-charge selection rule, the Zamolodchikov metric computed through correlators are

$$g_{i\bar{j}}(\tau,\bar{\tau}) = \left\langle \phi_i(0)\bar{\phi}_j(\infty)\right\rangle\,, \quad g_{a\bar{b}}(\tilde{\tau},\bar{\tilde{\tau}}) = \langle \sigma_a(0)\bar{\sigma}_b(\infty)\rangle\,, \quad etc. = 0\,,$$

- Chiral ring:

  There are actually more operators saturating the BPS bound,

  $$\mathcal{R} \equiv \left\{ \phi_I \,\middle|\, \Delta(\phi_I) = \frac{1}{2} q_V(\phi_I),\ q_A(\phi_I) = 0 \right\}$$

  BPS bound guarantees the *non-singular* OPE among elements in $\mathcal{R}$

  $$\phi_I(x) \cdot \phi_J(0) = \mathcal{C}_{IJ}^{K}(\tau)\, \phi_K(0) + \textit{superdecendant}$$

  $\Longrightarrow \mathcal{R}$ admits a ring structure under OPE modulo superdecedant, called *chiral ring*. Its hermitian conjugation defines anti-chiral ring $\overline{\mathcal{R}}$.

  One can as well as define (anti-)twisted chiral ring $\widetilde{\mathcal{R}}\,(\overline{\widetilde{\mathcal{R}}})$

  $$\widetilde{\mathcal{R}} \equiv \left\{ \sigma_A \,\middle|\, \Delta(\sigma_A) = \frac{1}{2} q_A(\sigma_A),\ q_V(\sigma_A) = 0 \right\}$$

- Chiral ring bundle:

  $\mathcal{R} \oplus \overline{\mathcal{R}}$, as vector space, can be "planted" on every point $(\tau, \bar{\tau})$ of $\mathcal{M}_c$ to form a vector bundle $\mathcal{V}$ over $\mathcal{M}_c$ graded by $U(1)_V$ charge, or conformal weight Δ,

$$\mathcal{V} = \bigoplus_{\Delta_I=0}^{\hat{c}} \mathcal{V}_I = \mathcal{M}_c \oplus \mathcal{T}\mathcal{M}_c \oplus \cdots$$

- Chiral ring data:

  The metric on bundle $\mathcal{V}$ is similarly determined by the correlator,

$$g_{I\bar{J}}(\tau, \bar{\tau}) \equiv \left\langle \phi_I(0)\bar{\phi}_J(\infty) \right\rangle = \delta_{\Delta_I \Delta_{\bar{J}}} \left\langle \phi_I(0)\bar{\phi}_J(\infty) \right\rangle ,$$

  called *chiral ring data* (CRD), a special type of extremal correlators.

- Extremal correlators (ECs):

  In general, one can consider the following correlation function,

$$\left\langle \phi_1(x_1)\cdots\phi_n(x_n)\,\bar{\phi}_J(y)\right\rangle = \frac{\left\langle \phi_1(x_1')\cdots\phi_n(x_n')\,\bar{\phi}_J(\infty)\right\rangle}{|x_1-y|^{2\Delta_1}\cdots|x_n-y|^{2\Delta_n}}\,.$$

  and $\left\langle \phi_1(x_1')\cdots\phi_n(x_n')\,\bar{\phi}_J(\infty)\right\rangle$ can be shown x_i'-independent by superconformal Ward identities. Thus,

$$\left\langle \phi_1(x_1')\cdots\phi_n(x_n')\,\bar{\phi}_J(\infty)\right\rangle = \left\langle \phi_I(0)\bar{\phi}_J(\infty)\right\rangle = g_{I\bar{J}}(\tau,\bar{\tau})$$

  with $\phi_I(0)\equiv\phi_1(0)\cdots\phi_n(0)=\mathcal{C}_{12}^{i}\mathcal{C}_{i3}^{j}\cdots\mathcal{C}_{ln}^{k}\phi_k(0)$.

  Therefore computing CRD will determine extremal correlators exactly.

- Miscellanies: One can analogously define twisted chiral ring bundle $\widetilde{\mathcal{V}}$, twisted chiral ring data (tCRD) and extremal correlators of twisted chiral primaries with one single twisted anti-chiral primary.

- Moduli spaces of Calabi-Yau manifolds:

  For a given Calabi-Yau manifold $\mathcal{Y}$ with $\dim_{\mathbb{C}} = n$, its moduli space $\mathcal{M}(\mathcal{Y})$ is parametrized by deformations of complex and Kähler structures, J and ω, while keeping $c_1(\mathcal{Y}) = 0$. Hence (at least locally) we have

$$\mathcal{M}(\mathcal{Y}) \simeq \mathcal{M}_C \times \mathcal{M}_K$$

- Metrics on $\mathcal{M}_C \times \mathcal{M}_K$:

  Both $\mathcal{M}_C$ and $\mathcal{M}_K$ are Kähler manifolds, their metric can be determined by their Kähler potentials $K_C(\tau, \bar{\tau})$ and $K_K(\tilde{\tau}, \bar{\tilde{\tau}})$. In local charts, the (Weil-Petersson) metric can be written as,

$$g_{i\bar{j}} = \partial_{\tau^i} \partial_{\bar{\tau}^j} K_C \,, \quad \tilde{g}_{a\bar{b}} = \partial_{\tilde{\tau}^a} \partial_{\bar{\tilde{\tau}}^b} K_K \,.$$

Introduction and motivation: Calabi-Yau manifolds

We can consider more general vector bundles over Calabi-Yau moduli.

- Hodge bundle $\mathcal{H}$ over $\mathcal{M}_C$:

  The Hodge bundle $\mathcal{H}$ is vector bundle over $\mathcal{M}_C$ with fibers

$$H_h = \bigoplus_{\alpha=0}^{n} H^{(n-\alpha,\alpha)}(\mathcal{Y}),$$

  the horizontal cohomologies of $\mathcal{Y}$. The bundle $\mathcal{H}$ is thus graded respect to the holomorphic degree α of differentials

$$\mathcal{H} = \bigoplus_{\alpha=0}^{n} \mathcal{H}_\alpha = \mathcal{M}_C \oplus \mathcal{T}\mathcal{M}_C \oplus \cdots$$

- Metrics on $\mathcal{H}$:

  One can therefore consider Hermitian metrics $g^{(\alpha)}_{I\bar{J}}$ on $\mathcal{H}$, graded by α too.

Introduction and motivation: Calabi-Yau manifolds

- Vector bundle $\widetilde{\mathcal{H}}$ over $\mathcal{M}_K$:

  Similarly, we can construct vector bundle over $\mathcal{M}_K$, corresponding to the K-theory group $K(\mathcal{Y})$, whose non-torsion part is isomorphic to the vertical cohomologies,

$$H_v = \bigoplus_{\tilde{\alpha}=0}^{n} H^{(\tilde{\alpha},\tilde{\alpha})}(\mathcal{Y}),$$

  The bundle $\widetilde{\mathcal{H}}$ is graded respect to α

$$\widetilde{\mathcal{H}} = \bigoplus_{\tilde{\alpha}=0}^{n} \widetilde{\mathcal{H}}_{\tilde{\alpha}} = \mathcal{M}_K \oplus \mathcal{T}\mathcal{M}_{\mathcal{K}} \oplus \cdots$$

- Metrics on $\widetilde{\mathcal{H}}$:

  Parallel to complex moduli, on $\widetilde{\mathcal{H}}$, we have Hermitian metrics $\widetilde{g}^{(\tilde{\alpha})}_{A\bar{B}}$ on $\widetilde{\mathcal{H}}$, graded by α.

- tt^*-equations:

The metrics, e.g. $g^{(\alpha)}_{I\bar{J}}$, are constrained by the Hitchin type integrable system, tt^*-equations,

$$\left[\nabla_i,\, \nabla_{\bar{j}}\right] = -\left[\mathcal{C}_i,\, \bar{\mathcal{C}}_j\right]\,,$$

or on local chart,

$$\partial_{\bar{j}}\left(g^{(\alpha)}_{I\bar{J}}\partial_i g^{(\alpha)\bar{J}K}\right) = \mathcal{C}^{M}_{i\,I}\, g^{(\alpha+1)}_{M\bar{N}}\, \bar{\mathcal{C}}^{\bar{N}}_{\bar{j}\,\bar{J}}\, g^{(\alpha)\bar{J}K} - g^{(\alpha)}_{I\bar{N}}\, \bar{\mathcal{C}}^{\bar{N}}_{\bar{j}\,\bar{J}}\, g^{(\alpha-1)\bar{J}M}\, \mathcal{C}^{K}_{i\,M}$$

where

$$\mathcal{C}:\, \mathcal{H}^{(1)} \times \mathcal{H}^{(\alpha)} \longrightarrow \mathcal{H}^{(\alpha+1)}$$

are holomorphic sections, called chiral OPE coefficients.

One can find similar tt^*-equations for $\widetilde{g}^{(\tilde{\alpha})}_{A\bar{B}}$, in terms of twisted chiral OPE coefficients,

$$\widetilde{\mathcal{C}}:\, \widetilde{\mathcal{H}}^{(1)} \times \widetilde{\mathcal{H}}^{(\tilde{\alpha})} \longrightarrow \widetilde{\mathcal{H}}^{(\tilde{\alpha}+1)}$$

- Calabi-Yau manifolds engineered by $2d$ $\mathcal{N}=(2,2)$ SCFTs :

  Interestingly, all ingredients in $\mathcal{Y}$ have corresponding elements in $\mathcal{S}$,

$\mathcal{Y}$	$\mathcal{S}$
Kähler manifold	admit $\mathcal{N}=(2,2)$ supersymmetries
$c_1(\mathcal{Y})=0$	$\beta(\mathcal{S})=0$
J-deformation	(anti-)chiral operators ϕ_i deformation
ω-deformation	(anti-)twisted chiral operators σ_a deformation
moduli $\mathcal{M}_C\times\mathcal{M}_K$	moduli $\mathcal{M}_c\times\mathcal{M}_{tc}$
$\mathcal{H}$ bundle	(anti-)chiral ring, $\{\phi_I,\ \bar{\phi}_J\}$
$\widetilde{\mathcal{H}}$ bundle	(anti-)twisted chiral ring, $\{\sigma_A,\ \bar{\sigma}_B\}$
$g_{I\bar{J}}$ on $\mathcal{H}$	extremal correlators $\langle\phi_I(0)\bar{\phi}_J(\infty)\rangle$
$\widetilde{g}_{A\bar{B}}$ on $\widetilde{\mathcal{H}}$	extremal correlators $\langle\sigma_A(0)\bar{\sigma}_B(\infty)\rangle$

Computing ECs via localization: review on localization

- Pestun's supersymmetric localization (general):

  The partition function, $Z(\lambda, \bar{\lambda})$, can be regarded as the generating function of all ECs or metrics. Using supersymmetries $\mathcal{Q}$, one can evaluate it exactly,

$$Z(\lambda, \bar{\lambda}, t) = \int \mathcal{D}\varphi \, \mathrm{e}^{-\mathcal{S}[\varphi;\lambda,\bar{\lambda}] + t\mathcal{Q}\mathcal{V}[\varphi]} .$$

  If $\mathcal{Q}^2 = 0$ up to total derivative,

$$\frac{d}{dt} Z(\lambda, \bar{\lambda}, t) = 0 \Longrightarrow Z(\lambda, \bar{\lambda}) = Z(\lambda, \bar{\lambda}, 0) = Z(\lambda, \bar{\lambda}, \infty)$$

  So saddle-point approximation turns to be exact, and we localize the infinitely dimensional integral onto finite loci, $\mathcal{N}_0 = \{\varphi_0 | \mathcal{Q}\mathcal{V}[\varphi_0] = 0\}$

$$Z(\lambda, \bar{\lambda}) = \int_{\mathcal{N}_0} d\varphi_0 \, \mathrm{e}^{-\mathcal{S}[\varphi_0;\lambda,\bar{\lambda}]} Z_{1-\mathrm{loop}}[\varphi_0] \, Z_{\mathrm{inst.}}[\varphi_0]$$

- Review of $2d$ supersymmetric localization:
  [Benini & Cremonesi], [Doroud, Gomis, Le Floch & Lee]

  For a given $\mathcal{N}=(2,2)$ SCFT $\mathcal{S}$ with Lagrangian description (GLSM),

$$\mathcal{S}=S_{\mathrm{g}}+S_{\mathrm{m}}+S_{\mathrm{p}}+S_{\mathrm{tp}}$$

  $S_{\mathrm{g}},\ S_{\mathrm{m}}$ — gauge and matter sectors

  $S_{\mathrm{p}}=\tau^I\!\int\!\mathrm{d}^2x\,\mathrm{d}^2\theta\ \Phi_I+\mathrm{h.c.}$ — superpotential encoding ECs

  $S_{\mathrm{tp}}=\tilde{\tau}^A\!\int\!\mathrm{d}^2x\,\mathrm{d}^2\tilde{\theta}\ \Sigma_A+\mathrm{h.c.}$ — twisted superpotential encoding ECs

  One can put the theory from $\mathbb{R}^2$ to $\mathbb{S}^2$ of radius R,

$$\mathcal{S}\longrightarrow\mathcal{S}[\mathbb{S}^2]=\mathcal{S}+\mathcal{O}(1/R)\,,$$

  while preserving a subsuperalgebra $\mathfrak{su}(2|1)$ of the full $\mathcal{N}=(2,2)$ superconformal algebra. However it contains only a single $U(1)$ R-symmetry. One thus has to choose breaking either $U(1)_V$ or $U(1)_A$.

Computing ECs via localization: review on localization

- Localization respect to $\mathfrak{su}(2|1)_A$:

  $\mathfrak{su}(2|1)_A$ contains: isometries of $\mathbb{S}^2$, supercharges $\widetilde{\mathcal{Q}}$, and $U(1)_V$.
  Thanks to $\widetilde{\mathcal{Q}}$, $\mathcal{S}[\mathbb{S}^2]$ is $\widetilde{\mathcal{Q}}$-exact, except for S_{tp},

$$\begin{aligned}\mathcal{S}_A[\mathbb{S}^2] &= \cdots + \tilde{\tau}^A \int_{\mathbb{S}^2} \mathrm{d}^2x\, \mathrm{d}^2\tilde{\theta}\; \widetilde{\mathcal{E}}(x,\tilde{\theta})\, \Sigma_A + \mathrm{h.c.} + \mathcal{O}(1/R) \\ &= \widetilde{\mathcal{Q}}(\cdots) + \tilde{\tau}^A \int_{\mathbb{S}^2} \mathrm{d}^2x \sqrt{g}\left(G_A(x) - \frac{\Delta_A - 1}{R}\sigma_A(x)\right) + \mathrm{h.c.}\end{aligned}$$

  Using the technique of localization, one can exactly compute the deformed partition function on $\mathbb{S}^2$,

$$Z_A[\mathbb{S}^2](\tilde{\tau}^A, \bar{\tilde{\tau}}^A) = \int \mathcal{D}\varphi\, \mathrm{e}^{-\mathcal{S}[\mathbb{S}^2][\varphi]}\,,$$

  and the undeformed one [Jockers, Kumar, Lapan, Morrison & Romo],

$$Z_A[\mathbb{S}^2](\tilde{\tau}^A, \bar{\tilde{\tau}}^A)\Big|_{\tilde{\tau}^A = \bar{\tilde{\tau}}^A = 0,\, \Delta_A \geq 2} \;=\; \mathrm{e}^{-K_{\mathrm{tc}}(\tilde{\tau}^a, \bar{\tilde{\tau}}^a)}\,.$$

Computing ECs via localization: review on localization

- Localization respect to $\mathfrak{su}(2|1)_B$:

  $\mathfrak{su}(2|1)_B$ contains: isometries of $\mathbb{S}^2$, supercharges $\mathcal{Q}$, and $U(1)_A$. Thanks to $\mathcal{Q}$, $\mathcal{S}[\mathbb{S}^2]$ is $\mathcal{Q}$-exact, except for S_{p},

$$\begin{aligned}\mathcal{S}_B[\mathbb{S}^2] &= \cdots + \tau^I \int_{\mathbb{S}^2} \mathrm{d}^2x\, \mathrm{d}^2\theta\; \mathcal{E}(x,\theta)\, \Phi_I + \mathrm{h.c.} + \mathcal{O}(1/R) \\ &= \mathcal{Q}(\cdots) + \tau^I \int_{\mathbb{S}^2} \mathrm{d}^2x \sqrt{g}\, \left(F_I(x) - \frac{\Delta_I - 1}{R} \phi_I(x) \right) + \mathrm{h.c.}\end{aligned}$$

  Using the technique of localization, one can exactly compute the deformed partition function on $\mathbb{S}^2$,

$$Z_B[\mathbb{S}^2](\tau^I, \bar{\tau}^I) = \int \mathcal{D}\varphi\, \mathrm{e}^{-\mathcal{S}[\mathbb{S}^2][\varphi]}\,,$$

  and the undeformed one [Gomis & Lee], [Doroud & Gomis],

$$Z_B[\mathbb{S}^2](\tau^I, \bar{\tau}^I)\Big|_{\tau^I=\bar{\tau}^I=0,\, \Delta_I \geq 2} = \mathrm{e}^{-K_{\mathrm{c}}(\tau^i, \bar{\tau}^i)}\,.$$

Computing ECs via localization: operators mixing on $\mathbb{S}^2$

- Extremal correlators on $\mathbb{S}^2$: [Gerchkovitz, Gomis & Komargodsk], [Gomis, Hsin, Komargodski, Schwimmer, Seiberg & Theise]
  By supersymmetric Ward identity on $\mathbb{S}^2$, one can show two important equations:

$$\int_{\mathbb{S}^2} \mathrm{d}^2x\sqrt{g}\,\left(F_I(x) - \frac{\Delta_I - 1}{R}\phi_I(x)\right) = 2\pi R\,\phi_I(N) + \mathcal{Q}(\dots)$$

$$\int_{\mathbb{S}^2} \mathrm{d}^2x\sqrt{g}\,\left(\bar{F}_J(x) - \frac{\Delta_{\bar{J}} - 1}{R}\bar{\phi}_J(x)\right) = -2\pi R\,\bar{\phi}_J(S) + \mathcal{Q}(\dots)\,,$$

where "N", "S" are the north and south poles of $\mathbb{S}^2$. So, for $R=1$,

$$\langle\phi_I(N)\rangle_{\mathbb{S}^2} = \frac{1}{2\pi}\frac{1}{Z[\mathbb{S}^2]}\partial_{\tau^I}Z[\mathbb{S}^2]\Big|_{\tau^I=\bar{\tau}^I=0,\Delta_I\geq 2}$$

$$\langle\bar{\phi}_J(S)\rangle_{\mathbb{S}^2} = -\frac{1}{2\pi}\frac{1}{Z[S^2]}\partial_{\bar{\tau}^J}Z[\mathbb{S}^2]\Big|_{\tau^J=\bar{\tau}^J=0,\Delta_J\geq 2}$$

Computing ECs via localization: operators mixing on $\mathbb{S}^2$

One can localize arbitrarily many ϕ_I ($\bar{\phi}_J$) on north (south) pole to compute their correlator on $\mathbb{S}^2$, for example,

$$\left\langle \phi_I(N)\,\bar{\phi}_J(S)\right\rangle_{\mathbb{S}^2} = -\frac{1}{4\pi^2}\frac{1}{Z[\mathbb{S}^2]}\partial_{\tau^I}\partial_{\bar{\tau}^J} Z[\mathbb{S}^2]\Big|_{\tau^I=\bar{\tau}^J=0,\Delta_{I,J}\geq 2}$$

However, it is non-trivial to map correlators on $\mathbb{S}^2$ to those on $\mathbb{R}^2$,

$$\left\langle \phi_I(N)\,\bar{\phi}_J(S)\right\rangle_{\mathbb{S}^2} \longrightarrow \left\langle \phi_I(0)\,\bar{\phi}_J(\infty)\right\rangle_{\mathbb{R}^2},$$

even though $\mathbb{S}^2$ can be conformally mapped to $\mathbb{R}^2$, and $\{N, S\} \mapsto \{0, \infty\}$.

To understand how the correlator on $\mathbb{S}^2$ is related to that on $\mathbb{R}^2$, one needs to study the non-trivial operators mixing on $\mathbb{S}^2$.

- Operators mixing on $\mathbb{S}^2$:
  [Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski & Pufu on $\mathbb{S}^4$]

  On flat space,

  $$\langle \phi_I(0) \rangle_{\mathbb{R}^2} = 0$$

  is required by conformal symmetrie, or say unwanted nonzero can be offset by counter terms.

  On the other hand, putting the theory $\mathcal{S}$ on $\mathbb{S}^2$ respect to $\mathfrak{su}(2|1)$ superalgebra will lead conformal anomalies. Put in other words,

  $$\langle \phi_I(N) \rangle_{\mathbb{S}^2} \propto R^{-\Delta_I} \neq 0$$

  cannot be removed by turning on counter terms at will meanwhile simultaneously preserving $\mathfrak{su}(2|1)$ superalgebra on $\mathbb{S}^2$

Computing ECs via localization: operators mixing on $\mathbb{S}^2$

The counter terms,

$$\Gamma_{\mathrm{c.t.}} = \frac{1}{2}\int \mathrm{d}^2x\mathrm{d}^2\theta\, \mathcal{E}\, \mathfrak{R}\, \mathcal{F}(\tau) + \mathrm{h.c.}\,,$$

added to regularize sphere partition function $Z[\mathbb{S}^2]$, must respect to $\mathfrak{su}(2|1)$ superalgebra, where $\mathfrak{R}$ is the $\mathfrak{su}(2|1)$ supergravity multiplet of dimension $\Delta_{\mathfrak{R}} = 1$, with $\mathfrak{R}|_{\mathrm{bot.}} \sim 1/R$.

$\mathfrak{R}$ will lead to mixing between a given chiral operator Φ_Δ of dimension Δ and all other operators of lower dimensions,

$$\Phi_\Delta \longrightarrow \Phi_\Delta + \gamma_{\Delta-1}(\tau,\bar{\tau};\Delta)\,\mathfrak{R}\,\Phi_{\Delta-1} +, \cdots, +\gamma_0(\tau,\bar{\tau};\Delta)\,\mathfrak{R}^\Delta\,\mathbb{1}\,,$$

where γ_i's, as conformal anomalies, are computable explicitly via localization.

A closer look at marginal operators ϕ_i and $\bar{\phi}_j$ of dimension 1:

$$\textit{operators mixing}: \qquad \phi_i \to \phi_i + \langle \phi_i(N) \rangle_{\mathbb{S}^2} \, \mathbb{1}$$

$$\Longrightarrow \textit{ if we define}: \qquad \hat{\phi}_i \equiv \phi_i - \langle \phi_i(N) \rangle_{\mathbb{S}^2} \cdot \mathbb{1}$$

$$\bar{\phi}_j \to \hat{\bar{\phi}}_j \equiv \bar{\phi}_j - \langle \bar{\phi}_j(S) \rangle_{\mathbb{S}^2} \cdot \bar{\mathbb{1}}$$

$$\Longrightarrow \left\langle \hat{\phi}_i(N) \right\rangle_{\mathbb{S}^2} = \left\langle \hat{\bar{\phi}}_j(S) \right\rangle_{\mathbb{S}^2} = 0 \quad \textit{disentangled by definition}$$

Thus we use disentangled operator $\hat{\phi}_i$ and $\hat{\bar{\phi}}_j$ to replace ϕ_i and $\bar{\phi}_j$ in the evaluation of correlators on $\mathbb{S}^2$,

$$g^{(1)}_{i\bar{j}}(\tau, \bar{\tau}) = \left\langle \hat{\phi}_i(N) \hat{\bar{\phi}}_j(S) \right\rangle_{\mathbb{S}^2} = -\partial_i \partial_{\bar{j}} \log Z[\mathbb{S}^2] \Big|_{\tau^I = \bar{\tau}^I = 0,\, \Delta_I \geq 2}$$

$$\Longrightarrow \qquad Z[\mathbb{S}^2] \Big|_{\tau^I = \bar{\tau}^I = 0,\, \Delta_I \geq 2} = \mathrm{e}^{-K(\tau, \bar{\tau})}, \quad \textit{the result from localization!}$$

Computing ECs via localization: variation of Hodge structure

- Operators mixing v.s. Griffiths transversality:

  For simplicity, consider chiral ring $\mathcal{R} = \langle \phi \rangle$ generated by a single chiral primary ϕ, corresponding to a CY-$n$fold, with all middle cohomologies,

$$\dim H^{(n-\alpha,\,\alpha)}(\mathcal{Y}_\tau) = 1\,, \quad \alpha = 1, 2, \dots, n$$

  We can establish a $1-1$ correspondence between states and cohomologies,

$$\underbrace{\phi \cdots \phi}_{\alpha}(0)|\mathbb{1};\,\tau\rangle_{\mathbb{R}^2} \equiv |\phi^\alpha;\,\tau,\bar{\tau}\rangle_{\mathbb{R}^2} \in H^{(n-\alpha,\,\alpha)}(\mathcal{Y}_\tau)$$

  Then how states prepared on $\mathbb{S}^2$ related to these cohomologies, say for example $|\phi;\,\tau,\bar{\tau}\rangle_{\mathbb{S}^2} \equiv \phi(N)|\mathbb{1};\,\tau\rangle_{\mathbb{S}^2}$

Computing ECs via localization: operators mixing on $\mathbb{S}^2$

- PHYSICS SIDE: On $\mathbb{S}^2$, a state respect to chiral primary ϕ,

$$|\phi;\,\tau,\bar{\tau}\rangle_{\mathbb{S}^2} = \phi(N)|\mathbb{1};\,\tau\rangle_{\mathbb{S}^2} = \int_{\mathbb{S}^2} \mathrm{d}^2x\sqrt{g}\left(F(x) - \frac{\Delta-1}{R}\phi(x)\right)|\mathbb{1};\,\tau\rangle_{\mathbb{S}^2}$$
$$= \partial_\tau|\mathbb{1};\,\tau\rangle_{\mathbb{S}^2}.$$

- MATH SIDE: $|\mathbb{1};\,\tau\rangle_{\mathbb{R}^2} = |\mathbb{1},\,\tau\rangle_{\mathbb{S}^2} \in H^{(n,0)}(\mathcal{Y}_\tau)$. However

$$\partial_\tau|\mathbb{1},\,\tau\rangle_{\mathbb{S}^2} \in H^{(n,0)}(\mathcal{Y}_\tau) \oplus H^{(n-1,1)}(\mathcal{Y}_\tau)$$

is called the Griffiths transversality,

$$\Longrightarrow \quad |\phi;\,\tau,\bar{\tau}\rangle_{\mathbb{S}^2} = \partial_\tau|\mathbb{1};\,\tau\rangle_{\mathbb{S}^2} = |\phi;\,\tau,\bar{\tau}\rangle_{\mathbb{R}^2} + \gamma_0|\mathbb{1};\,\tau\rangle_{\mathbb{R}^2}$$

Using $\left\langle\bar{\mathbb{1}},\bar{\tau}|\mathbb{1},\tau\right\rangle = Z[\mathbb{S}^2]$, one projects out γ_0 as,

$$\gamma_0 = \partial_\tau \log Z[\mathbb{S}^2] = \langle\phi(N)\rangle_{\mathbb{S}^2}\,.$$

We thus recover the disentangled formula in terms of states,

$$|\phi;\,\tau,\bar{\tau}\rangle_{\mathbb{R}^2} = |\phi;\,\tau,\bar{\tau}\rangle_{\mathbb{S}^2} - \langle\phi(N)\rangle_{\mathbb{S}^2}\,|\mathbb{1};\,\tau\rangle_{\mathbb{S}^2}$$

Computing ECs via localization: variation of Hodge structure

One can proceed further, by Griffiths transversality,

$$|\phi^\alpha;\,\tau,\bar\tau\rangle_{\mathbb{S}^2}\equiv\partial_\tau^\alpha|\mathbb{1};\,\tau\rangle_{\mathbb{S}^2}\in\bigoplus_{\beta=0}^{\alpha}H^{(n-\beta,\beta)}(\mathcal{Y}_\tau)\,,$$

and find,

$$|\phi^\alpha;\,\tau,\bar\tau\rangle_{\mathbb{R}^2}=|\phi^\alpha;\,\tau,\bar\tau\rangle_{\mathbb{S}^2}-\sum_{\beta=0}^{\alpha-1}\gamma_\beta|\phi^\beta;\,\tau,\bar\tau\rangle_{\mathbb{R}^2}\,,$$

with,

$$\begin{aligned}\gamma_\beta &= \frac{{}_{\mathbb{R}^2}\big\langle\bar\phi^\beta;\,\tau,\bar\tau|\phi^\alpha;\,\tau,\bar\tau\big\rangle_{\mathbb{S}^2}}{\big\langle\bar\phi^\beta;\,\tau,\bar\tau|\phi^\beta;\,\tau,\bar\tau\big\rangle_{\mathbb{R}^2}}=\frac{{}_{\mathbb{R}^2}\big\langle\bar\phi^\beta;\,\tau,\bar\tau|\phi^\alpha;\,\tau,\bar\tau\big\rangle_{\mathbb{S}^2}}{\big\langle\bar{\mathbb{1}};\,\tau|\mathbb{1};\,\tau\big\rangle_{\mathbb{R}^2}}\cdot\left(g^{(\beta)}\right)^{-1}\\ &= \Big\langle\phi^\alpha(N)\,\hat{\bar\phi}^\beta(S)\Big\rangle_{\mathbb{S}^2}\cdot\left(g^{(\beta)}\right)^{-1}\end{aligned}$$

Computing ECs via localization: general algorithm

Algorithm: Gram-Schmidt orthogonalization (induction by dimension Δ_ϕ)

- Step 1: For $\Delta_\phi = 0$, only identity operator $\mathbb{1}$, need no disentangle,

$$g^{(0)} = \langle \mathbb{1}(N)\bar{\mathbb{1}}(S)\rangle_{\mathbb{S}^2} \equiv 1$$

- Step 2: For $\Delta_\phi \leq k-1$, assume we have disentangle all operators,

$$\hat{\phi}_{K_\alpha} \quad and \quad \hat{\bar{\phi}}_{L_\alpha} \quad with \quad \alpha \leq k-1\,.$$

  and compute their ECs,

$$g^{(0)}\,,\ g^{(1)}\,,\cdots\ g^{(k-1)}$$

- Step 3: For $\Delta_{\phi_I} = k$, disentangle chiral primaries ϕ_{I_k} by defining,

$$\hat{\phi}_{I_k} \equiv \phi_{I_k} - \sum_{\alpha=0}^{k-1}\sum_{K_\alpha,\bar{L}_\alpha}\left(g^{(\alpha)}\right)^{-1\,\bar{L}_\alpha K_\alpha}\left\langle \phi_{I_k}(N)\,\hat{\bar{\phi}}_{L_\alpha}(S)\right\rangle_{\mathbb{S}^2}\bar{\phi}_{K_\alpha}$$

- Step 4: Compute ECs of $\Delta_{\phi_I} = k$,

$$\begin{aligned} g^{(k)}_{I_k\bar{J}_k} &= \langle \phi_{I_k}(N)\,\bar{\phi}_{J_k}(S)\rangle_{\mathbb{S}^2} \\ &-\sum_{\alpha=0}^{k-1}\sum_{K_\alpha,\bar{L}_\alpha}\left(g^{(\alpha)}\right)^{-1\,\bar{L}_\alpha K_\alpha}\left\langle \phi_{I_k}(N)\,\hat{\bar{\phi}}_{L_\alpha}(S)\right\rangle_{\mathbb{S}^2}\left\langle \hat{\phi}_{K_\alpha}(N)\,\bar{\phi}_{J_k}(S)\right\rangle_{\mathbb{S}^2} \end{aligned}$$

Example: Toda chain equations

- Toda chain equations:

We focus on the case of chiral ring $\mathcal{R}$ generated by single chiral primary ϕ. The (normalized) extremal correlators are given as,

$$g^{(\alpha)}(\tau,\bar{\tau}) = \frac{\left\langle \bar{\phi}^{\alpha} \middle| \phi^{\alpha} \right\rangle_{\mathbb{R}^2}}{\left\langle \bar{\mathbb{1}} \middle| \mathbb{1} \right\rangle_{\mathbb{R}^2}} .$$

We will establish general differential eqs. that $g^{(\alpha)}$'s must satisfy.

First, we interpret operators mixing of higher dimensions as connections on various vector bundles. It can be shown that,

$$|\phi^{\alpha+1}\rangle_{\mathbb{R}^2} = \partial_\tau^{\alpha+1}|\mathbb{1}\rangle_{\mathbb{R}^2} - \sum_{\beta=0}^{\alpha}\gamma_\beta|\phi^\beta\rangle_{\mathbb{R}^2} = \partial_\tau|\phi^\alpha\rangle_{\mathbb{R}^2} - \Gamma_\alpha\,|\phi^\alpha\rangle_{\mathbb{R}^2}$$

$$\Longrightarrow \Gamma_\alpha = \partial_\tau \log \left\langle \bar{\phi}_\alpha \middle| \phi_\alpha \right\rangle_{\mathbb{R}^2} = -\partial_\tau K_{\mathrm{C}} + \left(g^{(\alpha)}\right)^{-1}\partial_\tau g^{(\alpha)}$$

Γ_α is the connection defined on $\mathcal{H}_0 \otimes \mathcal{H}_\alpha$

Example: Toda chain equations

Now computing $g^{(\alpha+1)}$,

$$g^{(\alpha+1)} = \frac{\left\langle \bar{\phi}^{\alpha+1} \middle| \phi^{\alpha+1} \right\rangle_{\mathbb{R}^2}}{\left\langle \bar{\mathbb{1}} \middle| \mathbb{1} \right\rangle_{\mathbb{R}^2}} = -\left(\sum_{\beta=0}^{\alpha} \partial_{\bar{\tau}} \Gamma_\beta \right) \frac{\left\langle \bar{\phi}^{\alpha} \middle| \phi^{\alpha} \right\rangle_{\mathbb{R}^2}}{\left\langle \bar{\mathbb{1}} \middle| \mathbb{1} \right\rangle_{\mathbb{R}^2}}$$

Resolving $\partial_{\bar{\tau}} \Gamma_\alpha$, achieving closed Toda chian eqs.

$$\partial_{\bar{\tau}} \partial_\tau \log g^{(\alpha)} = \frac{g^{(\alpha)}}{g^{(\alpha-1)}} - \frac{g^{(\alpha+1)}}{g^{(\alpha)}} + g^{(1)}\,, \quad \text{for} \quad 1 \leq \alpha \leq n-1\,,$$

$$\partial_{\bar{\tau}} \partial_\tau \log g^{(n)} = \frac{g^{(n)}}{g^{(n-1)}} + g^{(1)}\,, \ \text{with} \ \ g^{(0)} = 1\,, \ g^{(1)} = -\partial_{\bar{\tau}} \partial_\tau \log Z[\mathbb{S}^2]\,.$$

- Constraints on $g^{(\alpha)}$:

  There are additional constraints due to symmetry of horizontal cohomologies, or charge conjugation from physics perspective, e.g.

$$|\phi^n;\, \tau, \bar{\tau}\rangle_{\mathbb{R}^2} = \mathcal{C}^{(n)}\, \mathrm{e}^{K_{\mathrm{C}}}\, |\bar{\mathbb{1}};\, \bar{\tau}\rangle_{\mathbb{R}^2} \in H^{(0,n)}(\mathcal{Y}_\tau)\,,$$

$$\Longrightarrow\ \mathcal{C}^{(n)}(\tau) = \langle \mathbb{1};\, \tau |\, \phi^n;\, \tau, \bar{\tau}\rangle_{\mathbb{R}^2} = \langle \phi \cdot \phi \cdot \cdots \cdot \phi \rangle_{\mathrm{TQFT}}\,,$$

  $\mathcal{C}^{(n)}(\tau)$ turns out to be only holomorphically τ-dependent, and in fact the chiral correlator determined by TQFT.

  We further have

$$|\phi^\alpha\rangle_{\mathbb{R}^2} = \mathcal{C}^{(n)}\, \mathrm{e}^{K_{\mathrm{C}}}\, \left(g^{(n-\alpha)}\right)^{-1}\, \left|\bar{\phi}^{n-\alpha}\right\rangle_{\mathbb{R}^2}\,,$$

$$\Longrightarrow\ \textit{constraints}:\ g^{(\alpha)}\, g^{(n-\alpha)} = \mathrm{e}^{2K_{\mathrm{C}}}\, \left|\mathcal{C}^{(n)}(\tau)\right|^2 \quad \text{for} \quad \alpha = 1, 2, \ldots, n\,.$$

Example: complete intersections in toric varieties

- The sextic fourfold: $X_6 \subset \mathbb{P}^5$
  [Jockers, Kumar, Lapan, Morrison & Romo], [Honma & Manabe]

  The sextic fourfold, defined by a degree six hypersurface in $\mathbb{P}^5$, can be realized as a $U(1)$ abelian $\mathcal{N} = (2,2)$ GLSM at UV regime, with following ingredients,

Field	$U(1)$	$U(1)_V$	$U(1)_A$
Φ_i	$+1$	$2\mathfrak{q}$	0
P	-6	$2-12\mathfrak{q}$	0
Σ	0	0	2

Table: The $U(1)$ gauge charge, $U(1)_V$ and $U(1)_A$ R-charge of matter fields P, Φ_i for $i = 1, 2..., 6$, and gauge field strength Σ

We aim to compute its ECs, corresponding to the vertical cohomologies.

Example: complete intersections in toric varieties

The action, more explicitly, is given by

$$\mathcal{S} = S_{\rm g} + S_{\rm m} + S_{\rm p} + S_{\rm tp}$$

with

$$S_{\rm g} = \frac{1}{g_{\rm YM}^2}\int \mathrm{d}^2x\,\mathrm{d}^4\theta\;\overline{\Sigma}\,\Sigma\,,\quad S_{\rm m} = \int \mathrm{d}^2x\,\mathrm{d}^4\theta\;\overline{\Phi_i}\,\mathrm{e}^{V}\Phi_i + \overline{P}\,\mathrm{e}^{-6V}P$$

$$S_{\rm p} = \int \mathrm{d}^2x\,\mathrm{d}^2\theta\;P\,G_6(\Phi) + \mathrm{h.c.}$$

$$S_{\rm tp} = \tilde{\tau}\int \mathrm{d}^2x\,\mathrm{d}^2\tilde{\theta}\;\Sigma + \mathrm{h.c.}$$

where W_6 is a homogeneous polynomial of degree 6 to determine the sextic fourfold, and the twisted potential Σ is the FI-term, with marginal coupling $\tilde{\tau} = \frac{\theta}{2\pi} + i\,r$.

Example: complete intersections in toric varieties

- Twisted chiral ring and its ECs:

  To compute tCRD, consider the twisted chiral ring

$$\widetilde{\mathcal{R}} = \langle\sigma\rangle/\{\sigma^5 = 0\} = \{\mathbb{1},\, \sigma,\, \sigma^2,\, \sigma^3,\, \sigma^4\}\,.$$

  There are thus five extremal correlators,

$$g^{(\alpha)}\left(\tilde{\tau}, \bar{\tilde{\tau}}\right) \equiv \left\langle \sigma^\alpha(0)\, \bar{\sigma}^\alpha(\infty)\right\rangle, \quad \text{for} \quad \alpha = 0, 1, 2, 3, 4\,,$$

  with $g^{(0)} = 1$ by normalization.

Example: complete intersections in toric varieties

- tt^*-equations and constraints of extremal correlators:

$$\partial_{\bar\tau}\partial_\tau \log Z_A[\mathbb{S}^2] = -g^{(1)}\,,$$
$$\partial_{\bar\tau}\partial_\tau \log g^{(\alpha)} = \frac{g^{(\alpha)}}{g^{(\alpha-1)}} - \frac{g^{(\alpha+1)}}{g^{(\alpha)}} + g^{(1)}\,, \quad \text{for} \quad 1 \le \alpha \le 3\,,$$
$$\partial_{\bar\tau}\partial_\tau \log g^{(4)} = \frac{g^{(4)}}{g^{(3)}} + g^{(1)}\,, \quad \text{with} \quad g^{(0)} = 1$$

tt^*-equations reduce to celebrated Toda chain equations.

- Additional constraints:

$$g^{(4)} = g^{(1)}\,g^{(3)} = \left(g^{(2)}\right)^2\,, \quad \text{and} \quad g^{(\alpha)} = 0\,, \quad \text{for} \quad \alpha \ge 5\,,$$

due to symmetries of Hodge structure.

To compute ECs, localize the theory respect to $\mathfrak{su}(2|1)_A$,

$$Z_A[\mathbb{S}^2] = \sum_{m\in\mathbb{Z}} \mathrm{e}^{-i\theta m} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma}{2\pi} \mathrm{e}^{-4\pi i\, r\sigma} \frac{\Gamma(\mathfrak{q} - i\sigma - \frac{1}{2}m)^6\, \Gamma(1 - 6\mathfrak{q} + 6i\sigma + 3m)}{\Gamma(1 - \mathfrak{q} + i\sigma - \frac{1}{2}m)^6\, \Gamma(6\mathfrak{q} - 6i\sigma + 3m)}$$

$g^{(\alpha)}$ can be computed either by the algorithm in Sec.2 or equivalently using $Z_A[\mathbb{S}^2]$ to solve Toda chain eqs..

We express $g^{(\alpha)}$ in two phases:

Calabi-Yau phase: $r \gg 0$

Landau-Ginzburg phase: $r \ll 0$

Example: complete intersections in toric varieties

- Calabi-Yau phase:

In CY phase, $Z_A[\mathbb{S}^2]$ can be simplified in large volume expansion, up to 1-instanton correction,

$$\begin{aligned} Z_A[\mathbb{S}^2](t,\bar{t}\,) &= \frac{1}{4}\,\xi^{-4} + 840\,\zeta(3)\,\xi^{-1} \\ &+ 30248\,(\bar{q}+q\,)\left(\xi^{-2}+2\xi^{-1}\right) + 609638400\,\bar{q}q + \mathcal{O}(q^2) + \text{c.c.} \end{aligned}$$

with $\xi \equiv \frac{1}{4\pi\,\mathrm{Im}\,t}$, $q \equiv \mathrm{e}^{2\pi i\,t}$, and new coordinates

$$t = \tilde{\tau} + 6264\,\mathrm{e}^{2\pi i\tilde{\tau}} + 67484340\,\mathrm{e}^{4\pi i\tilde{\tau}} + 1272752107200\,\mathrm{e}^{6\pi i\tilde{\tau}} + \cdots .$$

in large volume limit.

There are both perturbative contribution and non-perturbative instanton correction to $Z_A[\mathbb{S}^2]$, as well as ECs.

Example: complete intersections in toric varieties

ECs, up to 1-instanton, are computed as

$$
\begin{aligned}
g^{(1)} &= \frac{4\xi^2\left(1-1680\,\zeta(3)\,\xi^3\right)^2}{\left(1+3360\,\zeta(3)\,\xi^3\right)^2} - 241920\,(\bar{q}+q)\,\left(\xi^3+\mathcal{O}(\xi^4)\right) \\
&\quad +12192768000\,\bar{q}q\,\left(\xi^4+\mathcal{O}(\xi^5)\right)+\mathcal{O}(q^2)+\text{c.c.}\,, \\
g^{(2)} &= \frac{24\xi^4}{1+3360\,\zeta(3)\,\xi^3} + 241920\,(\bar{q}+q)\,\left(\xi^4+\mathcal{O}(\xi^5)\right) \\
&\quad +2438553600\,\bar{q}q\,\left(\xi^4+\mathcal{O}(\xi^6)\right)+\mathcal{O}(q^2)+\text{c.c.}\,, \\
g^{(3)} &= \frac{144\xi^6}{\left(1-1680\,\zeta(3)\,\xi^3\right)^2} + 2903040\,(\bar{q}+q)\,\left(\xi^6+\mathcal{O}(\xi^7)\right) \\
&\quad +58525286400\,\bar{q}q\,\left(\xi^6+\mathcal{O}(\xi^7)\right)+\mathcal{O}(q^2)+\text{c.c.}\,, \\
g^{(4)} &= \frac{576\xi^8}{\left(1+3360\,\zeta(3)\,\xi^3\right)^2} + 11612160\,(\bar{q}+q)\,\left(\xi^8+\mathcal{O}(\xi^9)\right) \\
&\quad +234101145600\,\bar{q}q\,\left(\xi^8+\mathcal{O}(\xi^9)\right)+\mathcal{O}(q^2)+\text{c.c.}\,,
\end{aligned}
$$

Example: complete intersections in toric varieties

All these ECs satisfies the constraints,

$$g^{(4)} = g^{(1)}\, g^{(3)} = \left(g^{(2)}\right)^2 \,, \quad \text{and} \quad g^{(\alpha)} = 0\,, \quad \text{for} \quad \alpha \geq 5\,.$$

As a spinoff, from

$$\overline{\mathcal{C}^{(4)}}\,\mathcal{C}^{(4)} = \left|\langle \sigma \cdot \sigma \cdot \sigma \cdot \sigma \rangle_{\mathrm{TQFT}}\right|^2 = g^{(4)}\ \left(Z_A[\mathbb{S}^2]\right)^2$$

$$\Longrightarrow \mathcal{C}^{(4)}(q) \ = \ 6\Big(1 + 20160\, q + 689472000\, q^2 + 24691154100480\, q^3 + \ 903369974818590720\, q^4 + \mathcal{O}(q^5)\Big)\,,$$

chiral correlator computed from TQFT [Greene, Morrison & Plesser], up to 4-instanton, is recovered!

Example: complete intersections in toric varieties

- Landau-Ginzburg phase:

In the limit $r \ll 0$, $Z_A[\mathbb{S}^2]$ is recast as

$$Z_A[\mathbb{S}^2] = \sum_{\alpha=0}^{4} Z_{\rm cl}^{(\alpha)}\, Z_{1-{\rm loop}}^{(\alpha)}\, \overline{Z_{\rm vortex}^{(\alpha)}(z)}\, Z_{\rm vortex}^{(\alpha)}(z)$$

with

$$Z_{\rm cl}^{(\alpha)} = {\rm e}^{4\pi\, r\cdot\frac{\alpha}{6}} = (\bar{z}z)^{-\frac{\alpha}{6}} \ ,$$

$$Z_{1-{\rm loop}}^{(\alpha)} = \frac{(-1)^\alpha}{6}\frac{\Gamma\left(\frac{1+\alpha}{6}\right)^6}{\Gamma\left(1+\alpha\right)^2\,\Gamma\left(\frac{5-\alpha}{6}\right)^6} ,$$

$$Z_{\rm vortex}^{(\alpha)}(z) = {}_5F_4\left(\left\{\frac{1+\alpha}{6},...,\frac{1+\alpha}{6}\right\};\left\{\frac{2+\alpha}{6},...,\widehat{1},...,\frac{6+\alpha}{6}\right\};\frac{1}{6^6 z}\right)$$

where $z = {\rm e}^{2\pi i\tilde{\tau}}$.

Example: complete intersections in toric varieties

The partition function $Z_A[\mathbb{S}^2]$ of X_6 is exactly its mirror

$$\widetilde{X}_6 \equiv \left\{ \{Z_i\} \in \mathbb{C}^6 \middle| \sum_{i=1}^{6} Z_i^6 + \tau \prod_{i=1}^{6} Z_i = 0 \right\} / \mathbb{Z}_6 \,,$$

after blowing up singularities. Engineer a SCFT (LG-model) on $\widetilde{X}_6$ and localize it respect to $\mathfrak{su}(2|1)_B$, we have,

$$Z_A[\mathbb{S}^2](X_6) = Z_B[\mathbb{S}^2](\widetilde{X}_6)$$

It is equivalent to compute the ECs of complex moduli in $\widetilde{X}_6$. Here a closed formula is proposed for ECs in terms of the periods,

$$\mathcal{F}^{(\alpha)}(z) \equiv z^{-\frac{\alpha}{6}} Z^{(\alpha)}_{\mathrm{vortex}}(z) \,,$$

of the complex moduli in $\widetilde{X}_6$.

Example: complete intersections in toric varieties

Rewrite

$$Z_B[\mathbb{S}^2](\widetilde{X}_6) = \sum_{\alpha=0}^{4} c_\alpha \, \overline{\mathcal{F}^{(\alpha)}} \, \mathcal{F}^{(\alpha)} = G^{(0)} \equiv \mathfrak{D}_0 \, ,$$

with $c_\alpha \equiv Z^{(\alpha)}_{1-\mathrm{loop}}$. Further define,

$$\mathfrak{D}_n \equiv \sum_{0\le\alpha_0<\ldots<\alpha_n\le 4} c_{\alpha_0} \cdots c_{\alpha_n} \left| \mathcal{W}(\mathcal{F}^{(\alpha_0)}, \ldots, \mathcal{F}^{(\alpha_n)}) \right|^2 \, ,$$

with

$$\mathcal{W}\left(\mathcal{F}^{(\alpha_0)}, \ldots, \mathcal{F}^{(\alpha_n)}\right) = \begin{vmatrix} \mathcal{F}^{(\alpha_0)} & \mathcal{F}^{(\alpha_1)} & \ldots & \mathcal{F}^{(\alpha_n)} \\ \partial_\tau \mathcal{F}^{(\alpha_0)} & \partial_\tau \mathcal{F}^{(\alpha_1)} & \ldots & \partial_\tau \mathcal{F}^{(\alpha_n)} \\ \vdots & \vdots & \ddots & \vdots \\ \partial^n_\tau \mathcal{F}^{(\alpha_0)} & \partial^n_\tau \mathcal{F}^{(\alpha_1)} & \ldots & \partial^n_\tau \mathcal{F}^{(\alpha_n)} \end{vmatrix}$$

the n-th Wronskian. One can show, by solving Toda chain eqs.,

$$g^{(n)} = \frac{G^{(n)}}{G^{(0)}} \, , \quad G^{(n)} = (-1)^n \frac{\mathfrak{D}_n}{\mathfrak{D}_{n-1}} \, , \quad \text{for} \quad n = 1 \cdots 4$$

Summary: $2d$ v.s. $4d$ ECs

- $2d$ chiral ring is nilpotent, and thus not freely generated.

  $\Longrightarrow$ ECs satisfy many non-trivial constraints, need modified algorithm from 4d case, and so forth...

- There is a geometric interpretation of $2d$ operators mixings in complex moduli. One may expect to find ones in $2d$ Kähler moduli and $4d$ case.

- $2d$ ECs of high dimensional operators can be fully computed exactly, while the non-perturbative part of $4d$ ones are still missing...

Summary: outlook

- More detailed physical understanding on conformal anomalies of operators mixing [in progress]
- General closed form for ECs/twisted ECs in general non-abelian GLSMs, (in)complete intersections in (non)toric varieties
- Computing ECs/twisted ECs of off-critical theories perturbed from SCFTs, or say general Kähler manifolds with $c_1 > 0$. [in progress]
- Applications to bootstraps, integrability, test of resurgence, and etc..

THANK YOU!