A tale of four extremizations

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Introduction

In supersymmetric gauge theories the $U(1)$ R-symmetry current is not necessarily unique and mixes with the global symmetry currents.

If R_0 is a $U(1)$ R-symmetry

$$R = R_0 + \sum_a \Delta_a J_a$$

also is, for all $U(1)$ global symmetries J_a.

On the other hand, a SCFT has an exact R-symmetry sitting in the superconformal algebra.

Introduction

There are many interesting extremization principles to find the exact R-symmetry in the space of all symmetries:

- a-maximization for $\mathcal{N}=1$ in 4d: extremize the trial central charge $a(\Delta_a)$ [Wecht-Intriligator '03]
- F-maximization for $\mathcal{N}=2$ in 3d: extremize the free energy on S^3, $F_{S^3}(\Delta_a)$ [Jafferis '10]
- c-extremization for $(0,2)$ in 2d: extremize the right-moving trial central charge $c_r(\Delta_a)$ [Benini-Bobev '12]
- $\mathcal{I}$-extremization for some superconformal $\mathcal{N}=2$ QM in 1d: extremize the equivariant Witten index? [Benini-Hristov-Zaffaroni '15]

Introduction

There are many interesting holographic duals that are obtained by considering near-horizon geometries of a set of N D3-branes or M2-branes:

- 4d SCFT: dual to $\text{AdS}_5 \times H_5$ in type IIB and $a = \frac{\pi^3 N^2}{4\mathrm{Vol}(H_5)}$
- 3d SCFT: dual to $\text{AdS}_4 \times H_7$ in M-theory and $F_{S^3} = N^{3/2}\sqrt{\frac{2\pi^6}{27\mathrm{Vol}(H_7)}}$

and their twisted compactifications on Riemann surfaces $\Sigma_{\mathfrak{g}}$:

- $(0,2)$ 2d SCFT: dual to fibrations $\text{AdS}_3 \times \Sigma_{\mathfrak{g}} \times H_5$ in type IIB
- 1d SCFT: dual to fibrations $\text{AdS}_2 \times \Sigma_{\mathfrak{g}} \times H_7$ in M theory

and a gravitational dual for the four extremization principles based on the geometry of Sasaki-Einstein manifolds [Martelli-Sparks-Yau '05; Couzens-Gauntlett-Martelli-Sparks '18]

Based on

S. M. Hosseini-AZ; arXiv 1901.05977

S. M. Hosseini-AZ; arXiv 1904.04269

but see also

A. Butti-AZ; arXiv 0506232

F. Benini-K.Hristov-AZ; arXiv 1511.04085 and 1608.07294

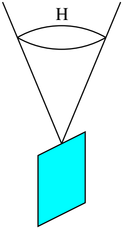
S. M. Hosseini-AZ; arXiv 1604.03122

Part I

D3-branes in type IIB

Superconformal Field Theories Zoo

D3 branes probing a conical Calabi-Yau with base a Sasaki-Einstein H_5:



- Near horizon geometry $\mathrm{AdS}_5 \times H_5$
- Superconformal field theory on the world-volume
- Complete correspondence between CY and CFT in the toric case
  [Franco-Hanany-Kennaway-Vegh-Wecht; Feng-He-Kennaway-Vafa, 2005]

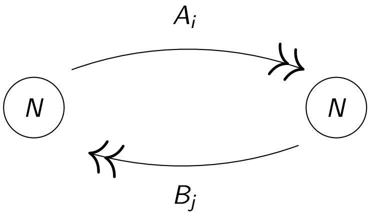
Symmetries come from bulk massless gauge fields:

- isometries of H_5 – mesonic symmetries
- reduction of $A_{(4)}$ on non-trivial three cycles in H_5 – baryonic symmetries

the exact R-symmetry mixes with all

Superconformal Field Theories Zoo

Everyone favorite is the conifold $C(T^{1,1})$: $T^{1,1} = SU(2) \times SU(2)/U(1)$



Symmetries:

mesonic: $U(1)_R \times SU(2) \times SU(2)$

baryonic: $+1$ for A_i and -1 for B_i

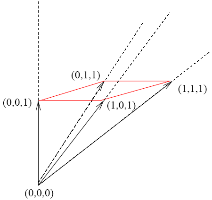
$$W = \mathrm{Tr}\left(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1\right)$$

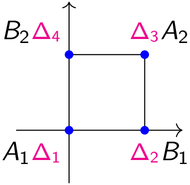
Parameterize general R-symmetry with four numbers Δ_a subject to

$$\sum_{a=1}^{4} \Delta_a = 2 \qquad \mathrm{exact\,R-symmetry}: \Delta_a = \frac{1}{2}$$

R-charge Parameterization for Toric Calabi-Yau

Toric Calabi-Yau are specified by a set of d integer vectors v_a in $\mathbb{R}^3$





- Toric = $U(1)^3$ isometry
- Every v_a associated with $U(1)^3$-invariant three cycles in H: only $d-3$ independent in homology

d symmetries: 3 mesonic + $d-3$ baryonic parameterized by d numbers Δ_a subject to

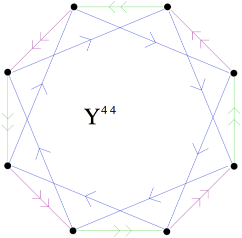
$$\sum_{a=1}^{d} \Delta_a = 2$$

R-charge Parameterization for Toric Calabi-Yau

Chiral fields in the quiver have R-charges of the form $\Delta_a + \Delta_{a+1} + \ldots + \Delta_b$ for some pairs (a, b)

Example: $Y^{p,q}$ with $2p$ $\mathrm{SU}(N)$ gauge groups and $4p + 2q$ chiral fields

$$\boldsymbol{v}_1 = (0,0)\,, \qquad \boldsymbol{v}_2 = (1,0)\,, \qquad \boldsymbol{v}_3 = (0,p)\,, \qquad \boldsymbol{v}_4 = (-1,p+q)$$



(a,b) in Φ_{ab}	multiplicity	$U(1)_R$	fields
$(4,1)$	$p+q$	Δ_1	Y
$(1,2)$	p	Δ_2	U^1
$(2,3)$	$p-q$	Δ_3	Z
$(3,4)$	p	Δ_4	U^2
$(1,3)$	q	$\Delta_2+\Delta_3$	V^1
$(2,4)$	q	$\Delta_3+\Delta_4$	V^2

$$\sum_{a=1}^{4} \Delta_a = 2 \qquad \mathrm{mult}\,\Phi_{ab} = \det(\boldsymbol{v}_a, \boldsymbol{v}_b)$$

Holographic dictionary

Holography teaches us [Gubser,Klebanov 98] :

- The value of the central charge of the CFT is given by

$$a = \frac{\pi^3 N^2}{4\mathrm{Vol}(H_5)}$$

- Wrapping a D3-branes on the three-cycle S_a we obtain a baryon $\det \Phi_{a-1,a}$. This allows to compute the exact R-charge

$$\Delta_a = \frac{\pi \mathrm{Vol}(S_a)}{3\mathrm{Vol}(H_5)}$$

Principle I

a-maximization

a-maximization

The central charge a and the exact R-charges Δ_a in the large N limit can be obtained by extremizing [Wecht-Intriligator '03]

$$a(\Delta_a) = \frac{9}{32}\,\mathrm{Tr}\,R(\Delta_a)^3$$

where the trace runs over all the fermions of the theory. Explicitly,

$$a(\Delta_a) = \frac{9}{32}N^2\left(|G| + \sum_{\Phi_{ab}} \mathrm{mult}(\Phi_{ab})\,(\Delta_{\Phi_{ab}} - 1)^3\right)$$
$$= \frac{9N^2}{64}\sum_{a,b,c=1}^{d} |\det(v_a, v_b, v_c)|\Delta_a\Delta_b\Delta_c$$

[Benvenuti,Pando-Zayas,Tachikawa 05]

Volume Minimization

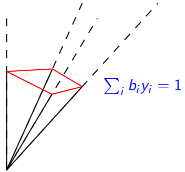
The gravitational dual is volume minimization: impose supersymmetry but relax equations of motions [Martelli-Sparks-Yau '05]

$$\mathrm{Sasaki} - \mathrm{Einstein}\; H \Longrightarrow \mathrm{Sasaki}\; H(b_i)$$

Volumes functions of the Reeb vector $\zeta = \sum_{i=1}^{3} b_i \partial_{\phi_i}$ with $b_1 = 3$ that specify the direction of R-symmetry inside the isometries of H.

$$\mathrm{Vol}(H) = \frac{\pi^3}{b_1} \sum_{a=1}^{d} \frac{(v_{a-1}, v_a, v_{a+1})}{(v_{a-1}, v_a, b)(v_a, v_{a+1}, b)}$$

$$\mathrm{Vol}(S_a) = 2\pi^2 \frac{(v_{a-1}, v_a, v_{a+1})}{(v_{a-1}, v_a, b)(v_a, v_{a+1}, b)}$$



Extremization of

$$a(b_i) = \frac{\pi^3 N^2}{4\mathrm{Vol}(H)}$$

gives Reeb vector $\bar{b} = (3, \bar{b}_2, \bar{b}_3)$ and volumes of the Sasaki-Einstein manifold.

a-maximization $=$ Volume Minimization

Parameterizing $\Delta_a(b_i) = \frac{\pi \mathrm{Vol}(S_a)}{b_1 \mathrm{Vol}(H)}$

$$a(b_i) \equiv a(\Delta_a)\Big|_{\Delta_a(b_i)} \equiv \frac{9N^2}{64} \sum_{a,b,c} |\det(v_a, v_b, v_c)| \Delta_a \Delta_b \Delta_c \Big|_{\Delta_a(b_i)}$$

[Butti, A.Z; 05]

However

- a-extremization is over $d-1$ independent parameters (mesonic+baryonic)
- volume minimization is over b_2, b_3 (mesonic only)

$a(\Delta_a)$ is automatically extremized with respect to the baryonic directions

$$\sum_a B_a \frac{\partial a(\Delta_a)}{\partial \Delta_a}\Big|_{\Delta_a(b)} \equiv 0$$

Proof simplified in [Lee-Rey; 05] and generalized to other quivers [Eager 10]

Principle II

c-extremization

Two-dimensional $(0,2)$ CFTs

We can obtain a two-dimensional $(0,2)$ CFT by compactifying the D3-brane theory on a Riemann surface $\Sigma_{\mathfrak{g}}$ with a topological A-twist.

- We obtain a family of twisted compactifications by turning turn on magnetic fluxes for all symmetries:

$$\int_{\Sigma_{\mathfrak{g}}} F_I = \mathfrak{n}_I$$

- We can parameterize the inequivalent twists using integers

$$\text{vertex } v_a \qquad \Longrightarrow \qquad \mathfrak{n}_a$$

$$(\nabla_\mu - iA_\mu^R)\epsilon \equiv \partial_\mu \epsilon = 0 \qquad \Longrightarrow \qquad \sum_{a=1}^{d} \mathfrak{n}_a = 2 - 2\mathfrak{g}.$$

The dual is a fibration $AdS_3 \times \Sigma_{\mathfrak{g}} \times H$ [Benini-Bobev 12]

c-extremization

The exact R-symmetry and the right-moving central charge can be found by extremizing [Benini-Bobev 12]

$$c_r(\Delta_a, \mathfrak{n}_a) = 3 \,\mathrm{Tr}\, \gamma_3 R(\Delta_a)^2$$

where the trace runs over all the two-dimensional fermions.

There is a simple formula valid at large N

$$c_r(\Delta, \mathfrak{n}) = -\frac{32}{9} \sum_{a=1}^{d} \mathfrak{n}_a \frac{\partial a(\Delta)}{\partial \Delta_a}$$

[Hosseini, A.Z; 05]

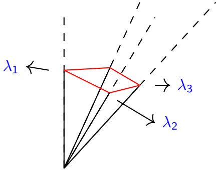
Explicitly,

$$c_r(\Delta, \mathfrak{n}) = \frac{N^2}{2} \sum_{a,b,c} |\det(v_a, v_b, v_c)| (\mathfrak{n}_a \Delta_b \Delta_c + \mathfrak{n}_b \Delta_a \Delta_c + \mathfrak{n}_c \Delta_a \Delta_b)$$

CGMS-extremization

The gravitational dual of c-extremization can be obtained by imposing susy but relaxing the equation of motion [Couzens,Gauntlett,Martelli,Sparks, 18]

$$ds^2_{10} = L^2 e^{-B/2}\left(ds^2_{\mathrm{AdS}_3} + ds^2_{H\to\Sigma_{\mathfrak{g}}}\right)$$
$$F_5 = -L^4(\mathrm{Vol}_{\mathrm{AdS}_3}\wedge F + \star F)$$



depending on:

- the Reeb vector (b_1, b_2, b_3) inside the $U(1)^3$ isometry with $b_1 = 2$
- parameters A, λ_a specifying the Kähler class of $\Sigma_{\mathfrak{g}}$ and H and entering in F
- 2 mesonic fluxes specifying the fibration of H over $\Sigma_{\mathfrak{g}}$
- $d-3$ baryonic fluxes coming from F_5

Fluxes can be all combined into d integers

$$\int_{\Sigma_{\mathfrak{g}}\times S_a} F_5 = \mathfrak{n}_a \qquad \sum_{a=1}^{d}\mathfrak{n}_a = 2-2\mathfrak{g}$$

GMS-extremization

Supersymmetry and flux quantization conditions for the off-shell background

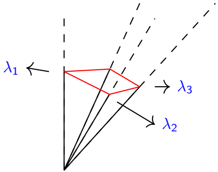
$$N = -\sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} \,,$$

$$\mathfrak{n}_a N = -\frac{A}{2\pi} \sum_{b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b} - b_1 \sum_{i=1}^{3} n^i \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial b_i} \,, \qquad n^i = \sum_{a=1}^{d} v_a^i \mathfrak{n}_a$$

$$A \sum_{a,b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b} = 2\pi n^1 \sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} - 2\pi b_1 \sum_{i=1}^{3} n^i \sum_{a=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial b_i}$$

with the master volume [Gauntlett,Martelli,Sparks, 18]

$$\mathcal{V} = 4\pi^3 \sum_{a=1}^{d} \lambda_a \frac{\lambda_{a-1}(v_a, v_{a+1}, b) - \lambda_a(v_{a-1}, v_{a+1}, b) + \lambda_{a+1}(v_{a-1}, v_a, b)}{(v_{a-1}, v_a, b)(v_a, v_{a+1}, b)}$$



c is obtained by extremizing

$$c(b_i, \mathfrak{n}_a) = -48\pi^2 \left(A \sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} + 2\pi b_1 \sum_{i=1}^{3} n^i \frac{\partial \mathcal{V}}{\partial b_i} \right) \Bigg|_{\lambda_a(b,\mathfrak{n}),\, A(b,\mathfrak{n})}$$

c-extremization = GMS extremization

Parameterizing $\Delta_a(b_i, \mathfrak{n}_a) = -\frac{2}{N}\frac{\partial \mathcal{V}}{\partial \lambda_a}$

$$c(b_i, \mathfrak{n}_a) \equiv c_r(\Delta_a, \mathfrak{n}_a)\Big|_{\Delta_a(b,\mathfrak{n})} \equiv -\frac{32}{9}\sum_{a=1}^{d} \mathfrak{n}_a \frac{\partial a(\Delta_a)}{\partial \Delta_a}\Big|_{\Delta_a(b,\mathfrak{n})}$$

[Hosseini, A.Z; 19]

However, as before,

- c-extremization is over $d-1$ independent parameters (mesonic+baryonic)
- volume minimization is over b_2, b_3 (mesonic only)

$c(\Delta_a, \mathfrak{n}_a)$ is automatically extremized with respect to the baryonic directions

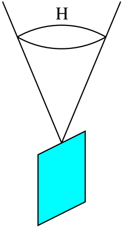
$$\sum_a B_a \frac{\partial c_r(\Delta, \mathfrak{n})}{\partial \Delta_a}\Big|_{\Delta_a(b,\mathfrak{n})} = \sum_{a,b} B_a \mathfrak{n}_b \frac{\partial^2 a(\Delta_a)}{\partial \Delta_a \partial \Delta_b}\Big|_{\Delta_a(b,\mathfrak{n})} \equiv 0$$

Part II

M2-branes in M theory

Superconformal Field Theories Zoo

M2 branes probing a conical Calabi-Yau with base a Sasaki-Einstein H_7:

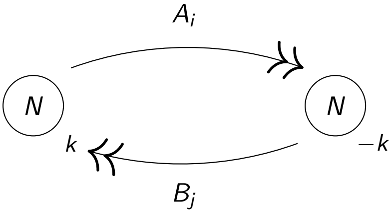


- Near horizon geometry $\mathrm{AdS}_4 \times H_7$
- Superconformal field theory on the world-volume
- Correspondence between CY and CFT still missing even in the toric case

Most examples are obtained by reducing dimensionally D3-branes and adding Chern-Simons couplings and/or flavoring with fundamentals.

Superconformal Field Theories Zoo

Everyone favorite is the ABJM theory: $C(H_7) = \mathbb{C}^4/\mathbb{Z}_k$



Symmetries:

mesonic: $U(1)_R \times SU(2) \times SU(2) \times U(1)$

$$W = \text{Tr}\left(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1\right)$$

Parameterize general R-symmetry with four numbers Δ_a subject to

$$\sum_{a=1}^{4} \Delta_a = 2 \qquad \mathrm{exact\,R-symmetry}: \Delta_a = \frac{1}{2}$$

Principle III

F-maximization

F-maximization = Volume Minimization

The exact R-charges Δ_a of a three-dimensional $\mathcal{N} = 2$ CFT can be obtained by extremizing the free energy on S^3, $F_{S^3}(\Delta_a)$, that depends on a trial R-charge

[Jafferis '10]

- The large N limit $F_{S^3}(\Delta_a)$ has been computed only for a subclass of theories and curiously depends only on mesonic symmetries
- In all examples

$$\boxed{F_{S^3}(\Delta_I)\Big|_{\Delta_a(b_i)} = N^{3/2}\sqrt{\frac{2\pi^6}{27\mathrm{Vol}_S(H_7(b_i))}}} \qquad \Delta_a(b_i) = \frac{2\pi}{3b_1}\frac{\mathrm{Vol}_S(S_a(b_i))}{\mathrm{Vol}_S(H_7(b_i))}$$

where Δ_I are linear combinations of the toric Δ_a found case by case.

[Martelli-Sparks;Herzog-Jafferis-Klebanov-Pufu-Safdi;Amariti-Klare-Siani, ...]

Principle IV

$\mathcal{I}$-extremization

Quantum mechanics BH horizon

We can obtain a quantum mechanics by compactifying the M2-brane theory on a Riemann surface $\Sigma_\mathfrak{g}$ with a topological A-twist

- Again we obtain a family of twisted compactifications by turning turn on magnetic fluxes for all symmetries:

$$\int_{\Sigma_\mathfrak{g}} F_a = \mathfrak{n}_a \qquad \sum_{a=1}^{d} \mathfrak{n}_a = 2 - 2\mathfrak{g}.$$

The dual is a fibration $\mathrm{AdS}_2 \times \Sigma_\mathfrak{g} \times H_7$ that can be interpreted as the horizon of four-dimensional magnetically charged black holes in $\mathrm{AdS}_4 \times H_7$ and the quantum mechanics describes their microstates.

$\mathcal{I}$-extremization

The entropy of such black holes can be found by extremizing

$$\boxed{\mathcal{I}(\Delta_I, \mathfrak{n}_I) = \log Z_{\Sigma_\mathfrak{g} \times S^1}(\Delta_a, \mathfrak{n}_a)}$$

where the topologically twisted index of the M2 theory [Benini-Hristov-AZ 15]

$$Z_{\Sigma_\mathfrak{g} \times S^1}(\Delta_I, \mathfrak{n}_I) = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{iJ_I \Delta_I} e^{-\beta H} = \mathrm{Tr}_{\mathcal{H}}(-1)^{R(\Delta_I)}$$

computes the equivariant Witten index of the IR quantum mechanics. It is also argued that this selects the exact R-symmetry.

Even here there is is a simple formula valid at large N [Hossein-AZ 16]

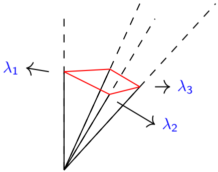
$$\boxed{\mathcal{I}(\Delta_I, \mathfrak{n}_I) = -\frac{1}{2} \sum_I \mathfrak{n}_I \frac{\partial F_{S^3}(\Delta_I)}{\partial \Delta_I}}$$

puzzling enough, all baryonic symmetries are invisible at large N.

CGMS-extremization

The gravitational dual of $\mathcal{I}$-extremization can be obtained by imposing susy but relaxing the equation of motion [Couzens,Gauntlett,Martelli,Sparks, 18]

$$ds^2_{11} = L^2 e^{-2B/3}\left(ds^2_{\text{AdS}_2} + ds^2_{H_7\to\Sigma_{\mathfrak{g}}}\right)$$
$$G = L^3 \text{Vol}_{\text{AdS}_2} \wedge F$$



depending on:

- the Reeb vector (b_1, b_2, b_3, b_4) inside the $U(1)^4$ isometry with $b_1 = 1$
- parameters A, λ_a specifying the Kähler class of $\Sigma_{\mathfrak{g}}$ and H_7 and entering in G
- 3 mesonic fluxes specifying the fibration of H_7 over $\Sigma_{\mathfrak{g}}$
- $d-4$ baryonic fluxes coming from G

This times the functional $S(b_i, \mathfrak{n}_a)$ proposed by CGMS on-shell computes the entropy of the black hole.

$\mathcal{I}$-extremization = GMS extremization ?

Partial answer. Large N computations exists only for few theories and are blind to baryonic symmetries ...

However,

- equivalence holds for all theories without baryonic symmetries, S^7, $V^{5,2}$, ...
- for all toric Calabi-Yau one can identify a twist along the mesonic directions only where the two principles are again equivalent

$$S(b_i, \mathfrak{n}_a) \equiv \mathcal{I}(\Delta_a, \mathfrak{n}_a)\Big|_{\Delta_a(b)} \equiv -\frac{1}{2}\sum_{a=1}^{d} \mathfrak{n}_a \frac{\partial F_{S^3}(\Delta_a)}{\partial \Delta_a}\Big|_{\Delta_a(b)}$$

[Hosseini, A.Z; see also Gauntlett, Martelli, Sparks; Kim, Kim 19]

where $\Delta_a(b_i) = -\frac{2}{N}\frac{\partial \mathcal{V}}{\partial \lambda_a} \equiv \frac{2\pi}{3b_1}\frac{\mathrm{Vol}_S(S_a)}{\mathrm{Vol}_S(H_7)}$

Conclusions

Rich geometrical structure of Sasaki-Einstein manifolds and interplay with QFT extremization principles.

- Still many geometrical relations to uncover
- Constructions are based on complex geometry: compute stuff even when the metric on H is not known or the black hole solution has not be found

In three dimensions we only have partial answers and puzzles

- why the known large N limit works only for few quivers?
- why in the large N limit baryonic symmetries disappear?
- black holes with only baryonic symmetries exist ...

Other saddle points to be discovered? Or not all solutions in CGMS really correspond to black hole horizons?