Modular Symmetry in Flavors

Morimitsu Tanimoto

Niigata University

November 15, 2018

Kavli IPMU seminar

Collaborated with T. Kobayashi, N. Omoto, Y. Shimizu
K. Takagi and T. Tatsuishi

Outline of my talk

1 Introduction

2 Modular Group

3 Modular A_4 invariance in Flavor

4 Modular S_3 and S_4 Symmetries

5 Summary

1 Introduction

We have a big question since the discovery of Muon

"Who orderd that ?" 1937 Isidor Issac Rabi

What is the principle to control flavors of quarks/leptons ?

The precise measurements of CKM mixing angles and CP violating phase of quarks established the SM (3 families).

Now, the neutrino oscillation experiments are going on observation of lepton mixing angles precisely.
Furthremore, CP violation of lepton sector is within reach
@T2K and Nova experiments T2HK, DUNE.

It may be an important clue for Beyond SM (flavor).

Symmetry Approach

One of the few approaches to flavors, but with several obstacles

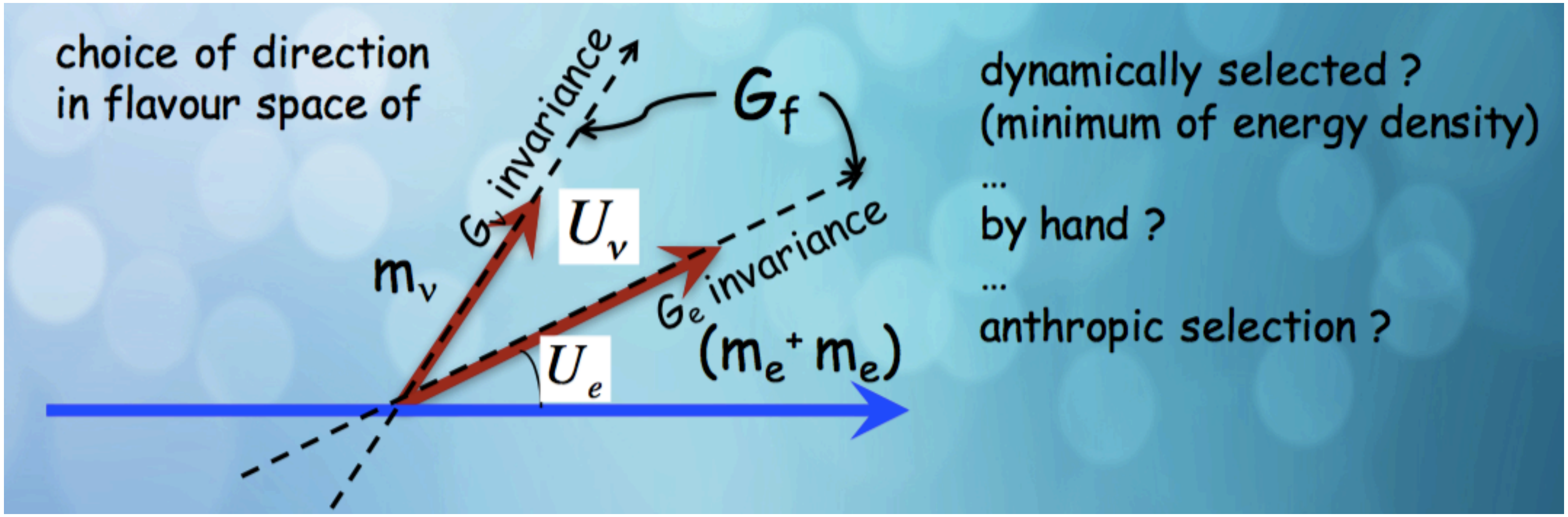
Predictivity ?   Introduce gauge singlet scalars (flavons)

High number of free parameters in effective Lagrangian

☆Lowest order Lagrangian + higher dimensional operators

☆spontaneous breaking in scalar sector

Vacuuum alignment of flavons (Direction of breaking symmetry)?

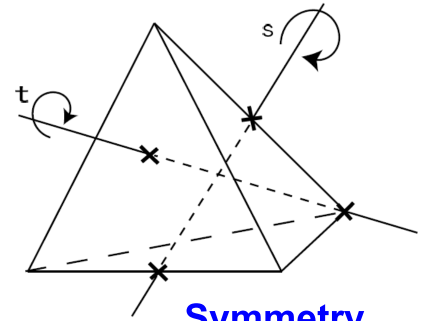


Feruglio FLASY2018

Prototype of A_4 Flavor model

A_4 group

E. Ma, G. Rajasekaran 2001



Even permutation group of four objects (1234)
12 elements (order 12) are generated by
S and T: $S^2=T^3=(ST)^3=1$: S=(14)(23), T=(123)

4 conjugacy classes

C1: 1 h=1
C3: S, T^2ST, TST^2 h=2
C4: T, ST, TS, STS h=3
C4': T^2, ST^2, T^2S, ST^2S h=3

	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	1	3
C_3	2	1	1	1	-1
C_4	3	1	ω	ω^2	0
$C_{4'}$	3	1	ω^2	ω	0

Irreducible representations: 1, 1', 1", 3
The minimum group containing triplet without doublet.

Multiplication rule of A_4 group

Irreducible representations: **1, 1', 1'', 3**

$$S=\frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix},\quad T=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega^2 & 0\\ 0 & 0 & \omega\end{pmatrix};\quad \omega=e^{2\pi i/3}$$

for triplet

$$\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix}_{\mathbf{3}}\otimes\begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}_{\mathbf{3}}=(a_1b_1+a_2b_3+a_3b_2)_{\mathbf{1}}\oplus(a_3b_3+a_1b_2+a_2b_1)_{\mathbf{1'}}$$

$$\oplus(a_2b_2+a_1b_3+a_3b_1)_{\mathbf{1''}}$$

$$\oplus\frac{1}{3}\begin{pmatrix}2a_1b_1-a_2b_3-a_3b_2\\ 2a_3b_3-a_1b_2-a_2b_1\\ 2a_2b_2-a_1b_3-a_3b_1\end{pmatrix}_{\mathbf{3}}\oplus\frac{1}{2}\begin{pmatrix}a_2b_3-a_3b_2\\ a_1b_2-a_2b_1\\ a_3b_1-a_1b_3\end{pmatrix}_{\mathbf{3}}$$

A_4 invariant Majorana neutrino mass term

$$(LL)_1 = L_1L_1 + L_2L_3 + L_3L_2$$

3 x 3

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

A_4 invariant

Effective Lagrangian with A_4 flavor symmetry

Flavor symmetry A_4 is broken by flavon (SU$_2$ singlet scalors) VEV's.
Flavor symmetry A_4 controls Yukaw couplings
among leptons and flavons with special vacuum alignments.

Consider the minimal number of flavons in A_4 model

G. Altarelli, F. Feruglio, Nucl.Phys. B720 (2005) 64

	Leptons	flavons	
A_4 triplets	$\boldsymbol{L}\ (L_e, L_\mu, L_\tau)$	$\phi_\nu(\phi_{\nu1}, \phi_{\nu2}, \phi_{\nu3})$	couples to neutrino sector
		$\phi_E(\phi_{E1}, \phi_{E2}, \phi_{E3})$	couples to charged lepton sector
A_4 singlets	$e_R : \mathbf{1}\quad \mu_R : \mathbf{1}''\quad \tau_R : \mathbf{1}'$		

Mass matrices are given by A_4 invariant Yukawa couplings with flavons

$$\mathbf{L} = y\, \boldsymbol{LL\,\Phi_\nu}\, H_u H_u/\Lambda^2 + y_e\, \boldsymbol{Le^c\,\Phi_E}\, H_d/\Lambda + y_\mu\, \boldsymbol{L\mu^c\,\Phi_E}\, H_d/\Lambda + y_\tau\, \boldsymbol{L\tau^c\,\Phi_E}\, H_d/\Lambda$$

$\mathbf{3}_L \times \mathbf{3}_L \times \mathbf{3}_{\text{flavon}} \to \mathbf{1}$, $\qquad$ $\mathbf{3}_L \times \mathbf{1}_R^{(')('')} \times \mathbf{3}_{\text{flavon}} \to \mathbf{1}$

Majoran neutrino $\qquad$ Charged lepton

Flavor symmetry A_4 is broken by VEV of flavons

$\mathbf{3_L} \times \mathbf{3_L} \times \mathbf{3_{flavon}} \to \mathbf{1}$

$$m_{\nu LL} \sim y \begin{pmatrix} 2\langle\phi_{\nu 1}\rangle & -\langle\phi_{\nu 3}\rangle & -\langle\phi_{\nu 2}\rangle \\ -\langle\phi_{\nu 3}\rangle & 2\langle\phi_{\nu 2}\rangle & -\langle\phi_{\nu 1}\rangle \\ -\langle\phi_{\nu 2}\rangle & -\langle\phi_{\nu 1}\rangle & 2\langle\phi_{\nu 3}\rangle \end{pmatrix}$$

$\mathbf{3_L} \times \mathbf{1_R}\,(\mathbf{1_R}', \mathbf{1_R}'') \times \mathbf{3_{flavon}} \to \mathbf{1}$

$$m_E \sim \begin{pmatrix} y_e\langle\phi_{E1}\rangle & y_e\langle\phi_{E3}\rangle & y_e\langle\phi_{E2}\rangle \\ y_\mu\langle\phi_{E2}\rangle & y_\mu\langle\phi_{E1}\rangle & y_\mu\langle\phi_{E3}\rangle \\ y_\tau\langle\phi_{E3}\rangle & y_\tau\langle\phi_{E2}\rangle & y_\tau\langle\phi_{E1}\rangle \end{pmatrix}$$

Residual symmetries lead to specific Vacuum Alingnments

Z_2 (1,S) in neutrinos $\quad \langle\phi_{\nu 1}\rangle = \langle\phi_{\nu 2}\rangle = \langle\phi_{\nu 3}\rangle$

Z_3 (1,T,T^2) in charged leptons $\quad \langle\phi_{E2}\rangle = \langle\phi_{E3}\rangle = 0$

$$\Rightarrow \quad \langle\phi_\nu\rangle \sim (1,1,1)^T \,, \qquad \langle\phi_E\rangle \sim (1,0,0)^T$$

$$\boxed{S\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}}$$

m_E is a diagonal matrix, on the other hand, $m_{\nu LL}$ is

$$m_{\nu LL} \sim 3y \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - y\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

Rank 2

two generated masses and one massless neutrinos !

(0, 3y, 3y)

Flavor mixing is not fixed !

Z_2 (1,S) is preserved

Adding A_4 singlet flavon $\xi : 1$ ➡ flavor mixing matrix is fixed.

G. Altarelli, F. Feruglio, Nucl.Phys. B720 (2005) 64

$3_L \times 3_L \times 1_{flavon} \rightarrow 1$

$$m_{\nu LL} \sim y_1 \begin{pmatrix} 2\langle\phi_{\nu 1}\rangle & -\langle\phi_{\nu 3}\rangle & -\langle\phi_{\nu 2}\rangle \\ -\langle\phi_{\nu 3}\rangle & 2\langle\phi_{\nu 2}\rangle & -\langle\phi_{\nu 1}\rangle \\ -\langle\phi_{\nu 2}\rangle & -\langle\phi_{\nu 1}\rangle & 2\langle\phi_{\nu 3}\rangle \end{pmatrix} + y_2\langle\xi\rangle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\langle\phi_{\nu 1}\rangle = \langle\phi_{\nu 2}\rangle = \langle\phi_{\nu 3}\rangle$, which preserves S symmetry.

$$m_{\nu LL} = 3a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - a \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Flavor mixing is determined:

Tri-bimaximal mixing

$$V_{\rm TBM} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\theta_{13}=0$

Exp : $\theta_{13}=0.14\text{-}0.16$

$$m_\nu = 3a+b,\ b,\ 3a-b \ \Rightarrow m_{\nu_1} - m_{\nu_3} = 2m_{\nu_2}$$

This is a minimal framework of A_4 symmetry predicting mixing angles and masses.

Prototype A_4 flavor model should be modified !

Triplet flavon   1 flavon   Additional 1' flavon

$$M_\nu = a\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + b\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} + c\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} + d\boxed{\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}}$$

$$a = \frac{y^\nu_{\phi_\nu}\alpha_\nu v_u^2}{\Lambda},\qquad b = -\frac{y^\nu_{\phi_\nu}\alpha_\nu v_u^2}{3\Lambda},\qquad c = \frac{y^\nu_{\xi}\alpha_\xi v_u^2}{\Lambda},\qquad d = \frac{y^\nu_{\xi'}\alpha_{\xi'} v_u^2}{\Lambda}\qquad a = -3b$$

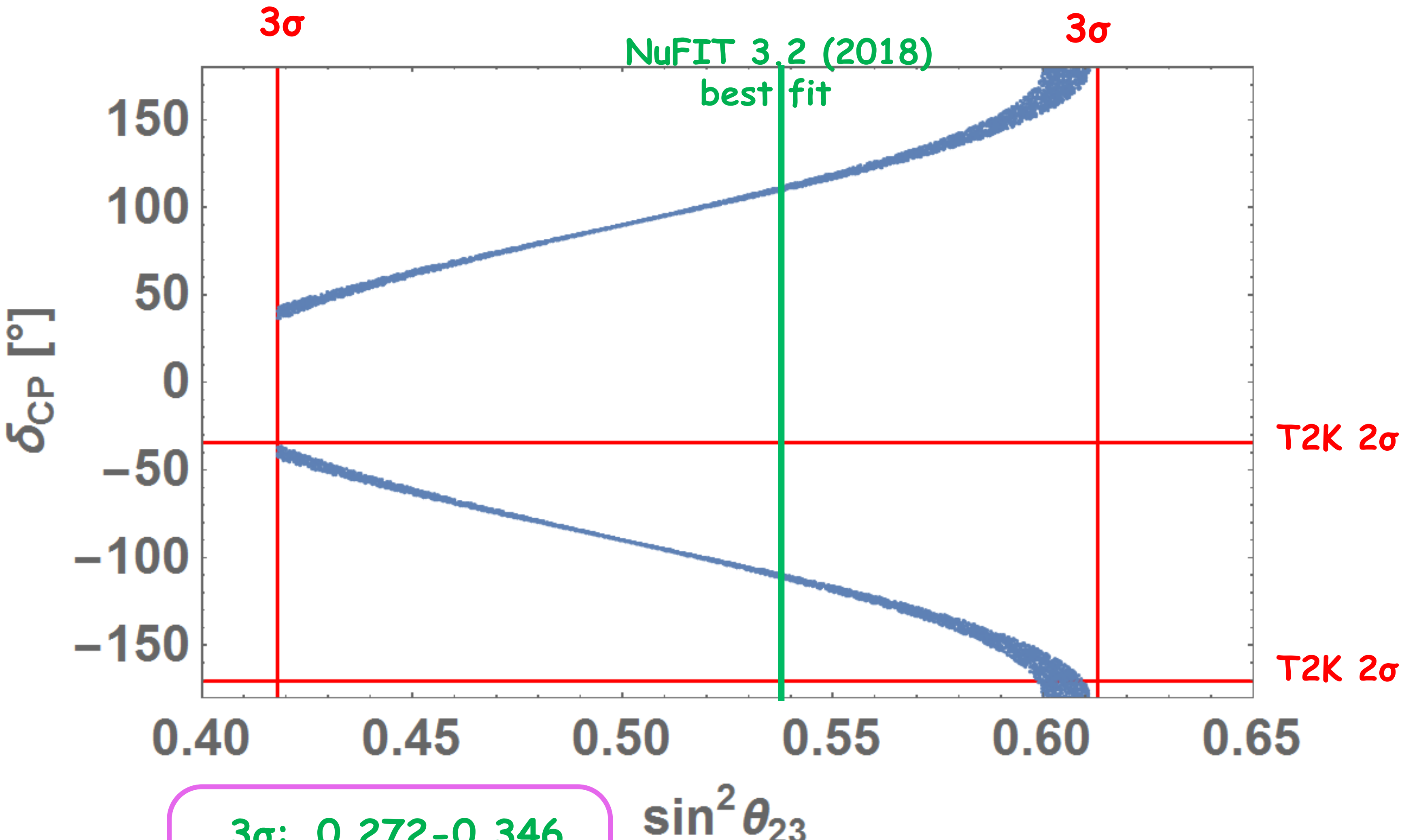
$$M_\nu = V_{\text{tri-bi}}\begin{pmatrix} a+c-\frac{d}{2} & 0 & \frac{\sqrt{3}}{2}d \\ 0 & a+3b+c+d & 0 \\ \frac{\sqrt{3}}{2}d & 0 & a-c+\frac{d}{2}\end{pmatrix} V^T_{\text{tri-bi}}$$

$$\boxed{V_{\text{tri-bi}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}}$$

Predictions are consistent with the data of mixing angles for both normal and inverted mass hierarchies.

Predictability is reduced because of additional parameters.

Prediction CP violating phase by using sum rules.



3σ: 0.272-0.346

$$\sin^2\theta_{12} = \frac{1}{3}\frac{1}{\cos^2\theta_{13}} \geq \frac{1}{3}\ ,$$

$$\cos\delta_{CP}\tan 2\theta_{23} \simeq \frac{1}{\sqrt{2}\sin\theta_{13}}\left(1-\frac{5}{4}\sin^2\theta_{13}\right)$$

Another aspect of A_4 model building

New Approach

Flavor symmetry comes from a finite subgroup of the modular group.

Flavor symmetry acts non-linealy (Modular function).

Lepton masses and mixing depend on a modulus τ,
which is selected by some unknown mechanism.

2 Modular Group

What is the origin of finite groups ?

It is well known that the superstring theory
on certain compactifications lead to
non-Abelian finite groups.

Indeed, torus compactification leads to Modular symmetery,
which includes S_3, A_4, S_4, A_5 as its subgroup.

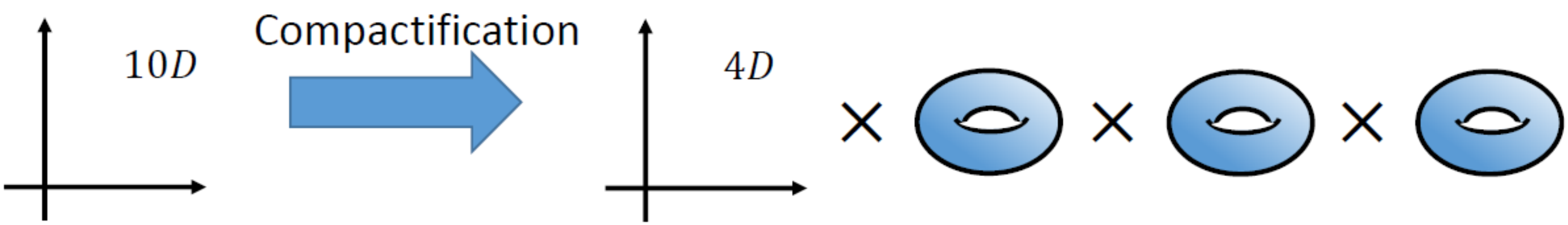
R.Toorop, F.Feruglio, C.Hagedorn, arXiv:1112.1340;
F.Feruglio, arXiv:1706.08749; A_4 J.C.Criado, F.Feruglio, arXiv:1807.01125; A_4
J.T.Penedo, S.T.Petcov, arXiv:1806.11040; S_4
T.Kobayashi, K.Tanaka, T.H.Tatsuishi, arXiv:1803.10391; S_3
T.Kobayashi,N.Omoto,Y.Shimizu,K.Takagi,M.T,T.H.Tatsuishi, arXiv:1808.03012; A_4

P.P. Novichkov, J. T. Penedo, S. T. Petcov, A. V. Titov, arXiv : 1811.04933: S_4

Superstring theory 10D
Our universe is 4D

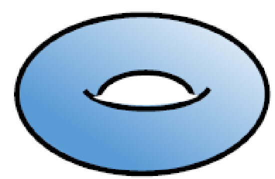
The extra 6D should be compactified.

Torus compactification



We get 4D effective Lagrangian by integrating out over 6D.

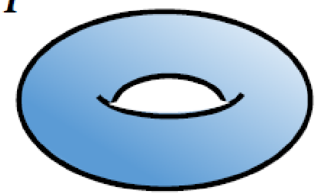
$$S = \int d^4x d^6y\, \mathcal{L}_{10D} \to \int d^4x\, \mathcal{L}_{\text{eff}}$$

$\mathcal{L}_{\text{eff}}$ depends on the structure of 

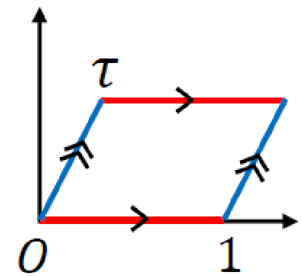
- 4D effective theory depends on internal space

$2D$ torus (T^2) is equivalent to parallelogram with identification of confronted sides.

T^2

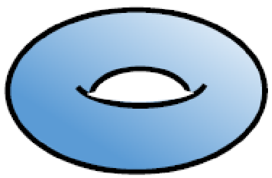


$\simeq$

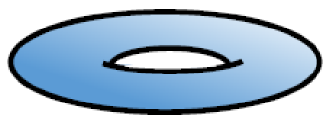


Two-dimensional torus T^2 is obtained as $T^2 = \mathbb{R}^2 / \Lambda$
Λ is two-dimensional lattice

The shape of torus is represented by a modulus $\tau \in \mathbb{C}$.



$\tau = \tau_1$



$\tau = \tau_2$

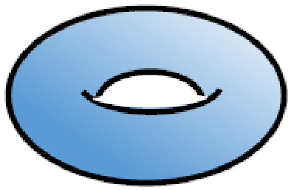
The different value of τ realize the different shape of T^2

$\mathcal{L}_{\text{eff}}$ depends on τ.

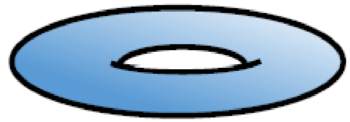
e.g.) $\mathcal{L}_{\text{eff}} \supset Y(\tau)_{ij}\phi\overline{\psi_i}\psi_j + \cdots$

- $4D$ effective theory depends on a modulus τ

The different value of τ realize the different shape of T^2



$\tau = \tau_1$



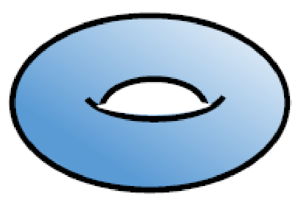
$\tau = \tau_2$

However,

there are specific transformations of τ which don't change T^2

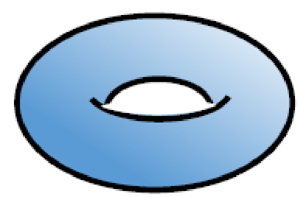
Modular transformation

$\tau \to \tau'$



τ

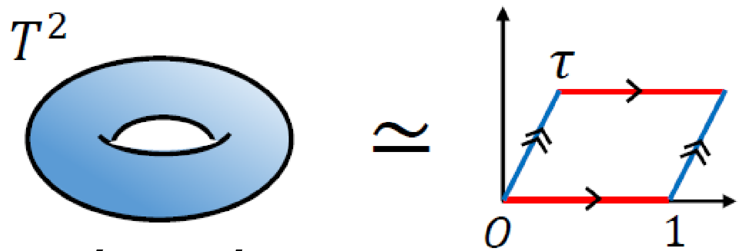
$=$

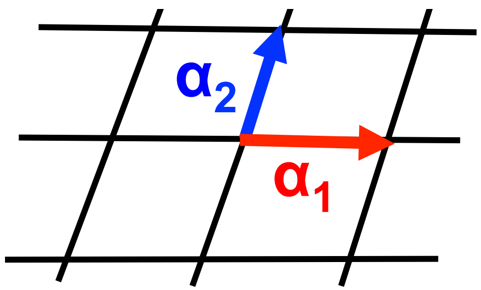


τ'

Modular transformation

The shape of a torus $T^2 \simeq$ The shape of a lattice on $\mathbb{C}$-plane





$(x,y) \sim (x,y) + n_1\alpha_1 + n_2\alpha_2$

Two-dimensional torus T^2 is obtained as

$$T^2 = \mathbb{R}^2 / \Lambda$$

Λ is two-dimensional lattice,
which is spanned by two lattice vectors

$$\alpha_1 = 2\pi R \quad \text{and} \quad \alpha_2 = 2\pi R\,\tau$$

$\tau = \alpha_2 / \alpha_1$ **is a modulus parameter (complex).**

The same lattice is spanned by other bases under the transformation

$$\boxed{\begin{pmatrix} \alpha'_2 \\ \alpha'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}}$$

ad-bc=1
a,b,c,d are integer SL(2,Z)

$$\begin{pmatrix} \alpha_2' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}$$

$\tau = \alpha_2 / \alpha_1$

$$\boxed{\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}}$$

Modular transformation

Modular transf. does not change the lattice (torus)

$4D$ effective theory (depends on τ)
must be invariant under modular transf.

The modular transformation is generated by S and T .

$$\boxed{\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}}$$

$$S : \tau \longrightarrow -\frac{1}{\tau}$$

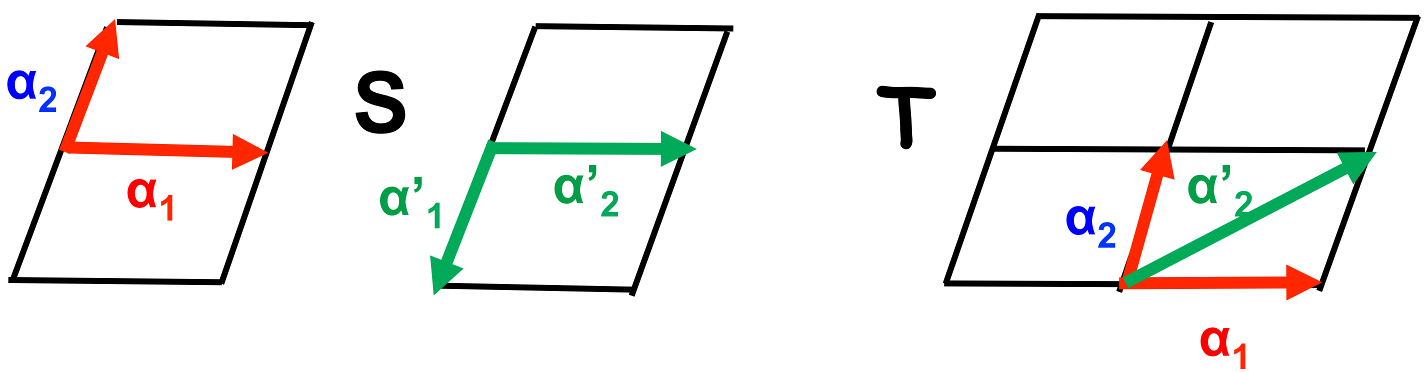
duality

$$T : \tau \longrightarrow \tau + 1$$

Dicrete shift symmetry

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}$$



$$\tau = \alpha_2 / \alpha_1$$

$$S : \tau \longrightarrow -\frac{1}{\tau}, \qquad S^2 = 1, \qquad (ST)^3 = 1.$$
$$T : \tau \longrightarrow \tau + 1.$$

generate infinite discrete group

Modular group
$\Gamma \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}\}$

4D effective theory

- **depends on a modulus** τ
- **is independent under modular transformation**

An example

$\mathcal{L}_1 = f(\tau)\phi_1\phi_2\cdots\phi_n$

$f(\tau)$: coupling constant
ϕ_i: scalar fields

$f(\tau) \to (c\tau + d)^k f(\tau)$ ⟸ Modular form with weight k

$\phi_i \to (c\tau + d)^{-k_i}\phi_i$

When $k = \sum_i k_i$, $\mathcal{L}_1$ is modular invariant.

Dedekind eta-function

$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$

Another example

$$\mathcal{L}_1 = f(\tau)\phi_1\phi_2\cdots\phi_n$$

- $f(\tau)$ and ϕ_i can be $\boxed{\text{non-trivial representations of modular group } \Gamma}$

Modular transformation:

SL(2, Z)

$$\gamma \in \Gamma \qquad \tau \to \frac{a\tau+b}{c\tau+d}, \qquad ad-bc=1$$

$$f(\tau) \to (c\tau+d)^k \rho(\gamma) f(\tau)$$

$$\phi_i' \to (c\tau+d)^{-k} \rho^{(i)}(\gamma)\phi_i$$

vanishing total modular weight

$\rho \times \rho^{I_1} \times \ldots \times \rho^{I_n}$ contains an invariant singlet

Representation matrix of Γ

L_1 is modular invariant.

Kinetic term is given by $\boxed{\dfrac{|\partial_\mu \phi^{(I)}|^2}{\langle -i\tau + i\bar{\tau}\rangle^{k_I}}}$

which is also invariant under modular transformation

- **Superpotential should be invariant under modular transformation in global SUSY model.**

Modular group has interesting subgroups

Modular group

$\Gamma \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}\}$ Infinite discrete group

Impose congruence condition

$$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N\}$$

called principal congruence subgroups

$\Gamma_N \equiv \Gamma / \Gamma(N)$ quotient group finite group

$\Gamma(N) \rightarrow \Gamma(N) / \{\pm \mathbb{1}\}$ for N=1, 2

$$\Gamma_N \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

$\Gamma_2 \simeq S_3$ $\Gamma_3 \simeq A_4$ $\Gamma_4 \simeq S_4$ $\Gamma_5 \simeq A_5$

We can consider effective theories with Γ_N symmetry.

$$\mathcal{L}_{\mathrm{eff}} \in f(\tau)\phi^{(1)}\cdots\phi^{(n)}$$

$f(\tau), \phi^{(I)}$: non-trivial rep. of Γ_N

In some cases, explicit form of function f(τ) have been obtained.

Famous modular function : Dedekind eta-function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q = e^{2\pi i\tau}$$

$$\eta(-1/\tau) = \boxed{\sqrt{-i\tau}\,\eta(\tau)}, \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

So called Modular weight 1/2

Modular transformation of chiral superfields in MSSM

$$\phi^{(I)} \to (c\tau+d)^{-k_I}\rho^{(I)}(\gamma)\phi^{(I)}$$

Modular weight

Representation matrix

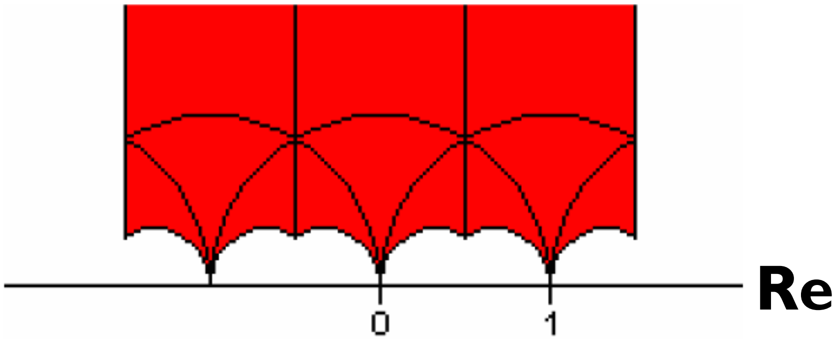
4 Modular A_4 invariance in flavor

$$\Gamma_N \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

Taking $T^3=1$, we have $\Gamma_3 \simeq A_4$ group.

N	g	$d_{2k}(\Gamma(N))$	μ_N	Γ_N
2	0	$k+1$	6	S_3
3	0	$2k+1$	12	A_4
4	0	$4k+1$	24	S_4
5	0	$10k+1$	60	A_5
6	1	$12k$	72	
7	3	$28k-2$	168	

2k is weight



Fundamental domain of τ

There are 3 linealy independent modular forms for 2k=2 (weight 2)

Dimension $d_{2k}(\Gamma(3))=2k+1$

Triplet !

How to find A_4 triplet modular functions.

Prepare 4 Dedekind eta-functions as Modular functions

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

$$\eta(3\tau) \to \sqrt{\frac{-i\tau}{3}}\,\eta(\tau/3),$$

$$\eta(\tau/3) \to \sqrt{-i3\tau}\,\eta(3\tau),$$

$$\eta((\tau+1)/3) \to e^{-i\pi/12}\sqrt{-i\tau}\,\eta((\tau+2)/3),$$

$$\eta((\tau+2)/3) \to e^{i\pi/12}\sqrt{-i\tau}\,\eta((\tau+1)/3).$$

S : $\tau \longrightarrow -1/\tau$

$$\eta(3\tau) \to e^{i\pi/4}\eta(3\tau),$$

$$\eta(\tau/3) \to \eta((\tau+1)/3),$$

$$\eta((\tau+1)/3) \to \eta((\tau+2)/3),$$

$$\eta((\tau+2)/3) \to e^{i\pi/12}\eta(\tau/3),$$

T : $\tau \longrightarrow \tau+1$

Modular function with weight 2 by using Dedekind eta-function

$$Y(\alpha,\beta,\gamma,\delta|\tau)=\frac{d}{d\tau}\left(\alpha\log\eta(\tau/3)+\beta\log\eta((\tau+1)/3)+\gamma\log\eta((\tau+2)/3)+\delta\log\eta(3\tau)\right)$$

$$\alpha+\beta+\gamma+\delta=0$$

$$\boxed{\begin{aligned} S &: \tau \longrightarrow -\frac{1}{\tau}, \\ T &: \tau \longrightarrow \tau+1. \end{aligned}} \qquad \begin{aligned} S &: \quad Y(\alpha,\beta,\gamma,\delta|\tau)\to\tau^2 Y(\delta,\gamma,\beta,\alpha|\tau), \\ T &: \quad Y(\alpha,\beta,\gamma,\delta|\tau)\to Y(\gamma,\alpha,\beta,\delta|\tau). \end{aligned}$$

In A_4 group, $T^3=1$

$$\rho(S)=\frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix}, \qquad \rho(T)=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix},$$

F. Feruglio, arXiv:1706.08749

A_4 triplet of modular function with weight 2

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \\ Y_3(-1/\tau) \end{pmatrix} = \tau^2 \rho(S) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \\ Y_3(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}.$$

$$\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right), \\
Y_2(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \\
Y_3(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right),
\end{aligned}$$

$$\begin{aligned}
Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\
Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\
Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots).
\end{aligned}$$

$q = e^{2\pi i \tau}$

$|q| \ll 1$

$Y_2^2 + 2Y_1Y_3 = 0$

Comment : Two special sets of τ

T($\tau \rightarrow \tau$ +1) preserved : $\langle \tau \rangle = i\,\infty$ ($q=0$) $(Y_1, Y_2, Y_3) = (1, 0, 0)$

S($\tau \rightarrow -1/\tau$) preserved : $\langle \tau \rangle = i$ ($q=e^{-2\pi}$) $(Y_1, Y_2, Y_3) = Y_1(i)\,(1, 1-\sqrt{3}, -2+\sqrt{3})$

Another eigenvector of S

Eigenvector of S: (1,1,1) cannot be realized

$$
\begin{aligned}
Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\
Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\
Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots).
\end{aligned}
$$

$$q = e^{2\pi i \tau}$$

$$Y_2^2 + 2Y_1Y_3 = 0$$

Simplest Model

left-handed leptons **L(3) (L_e, L_μ, L_τ)**
right-handed leptons **e_R (1);μ_R (1”);τ_R (1’)**

$-k_I$ is weight

	$SU(2)_L \times U(1)_Y$	A_4	k_I
$e^c_{R_1}$	$(1,+1)$	1	k_{e1}
$e^c_{R_2}$	$(1,+1)$	$1''$	k_{e2}
$e^c_{R_3}$	$(1,+1)$	$1'$	k_{e3}
L	$(2,-1/2)$	3	k_L
H_u	$(2,+1/2)$	1	k_{H_u}
H_d	$(2,-1/2)$	1	k_{H_d}
ϕ	$(1,0)$	3	k_ϕ

Sum of weights should vanish
$-2k_L-2k_{Hu}+\underset{Y}{2} = 0$, $-k_L-k_{ei}-k_{Hd}+\underset{Y}{2} = 0$

$$f(\tau) \to (c\tau+d)^k \rho(\gamma) f(\tau)$$

$$\phi^{(I)} \to (c\tau+d)^{-k_I} \rho^{(I)}(\gamma)\phi^{(I)}$$

$k_L=1$, $k_{ei}=1$
$k_{Hu}=k_{Hd}=0$

Modular invariant superpotential

$$w_e = \alpha e_R H_d(LY) + \beta \mu_R H_d(LY) + \gamma \tau_R H_d(LY)$$

$1_R^{(')('')} \times 3_L \times 3_Y \rightarrow 1$

$$w_\nu = -\frac{1}{\Lambda}(H_u H_u L L Y)_1$$ Weinberg Operator

$3_L \times 3_L \times 3_Y \longrightarrow 1$

$$M_E = \mathrm{diag}[\alpha,\beta,\gamma]\begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}$$

α, β, γ are fixed by the charged lepton masses

$$M_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}$$

Only source of breaking of the modular symmetry is the VEV of τ .

Unfortunately, the prediction is too large θ_{13}!

Seesaw model

Introduce right-handed neutrinos: A_4 Triplet

$$w_e = \alpha\ E_1^c H_d (L\ Y)_1 + \beta\ E_2^c H_d (L\ Y)_{1'} + \gamma\ E_3^c H_d (L\ Y)_{1''}$$

$$w_\nu = g(N^c H_u L\ Y)_1 + \Lambda (N^c N^c Y)_1$$

Sum of weights vanish.

$$Y = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix} \qquad q = e^{2\pi i \tau}$$

$M_E =$ $\alpha e_R H_d (LY) + \beta \mu_R H_d (LY) + \gamma \tau_R H_d (LY)$

A_4 1 1 3 3   1" 1 3 3   1' 1 3 3

$M_D =$ $g(\nu_R H_u L Y)_1$ $M_N =$ $\Lambda(\nu_R \nu_R Y)_1$

A_4 3 1 3 3 A_4 3 3 3

seesaw $\quad M_\nu = -M_D^T M_N^{-1} M_D$

ν_L ν_R

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{\mathbf{1'}}$$

$$\oplus (a_2b_2 + a_1b_3 + a_3b_1)_{\mathbf{1''}}$$

$$\oplus \boxed{\frac{1}{3}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}}} \oplus \boxed{\frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}}}.$$

symmetric × 3_Y anti-symmetric × 3_Y

Consider the case of Normal neutrino mass hierarchy

$m_1 < m_2 < m_3$

A_4 triplet 3 (L_e, L_μ, L_τ) 3 (ν_{eR}, $\nu_{\mu R}$, $\nu_{\tau R}$)

A_4 singlets e_R 1 ; μ_R 1″ ; τ_R 1′

$$\mathcal{Y}_e = \begin{pmatrix} \alpha Y_1 & \alpha Y_3 & \alpha Y_2 \\ \beta Y_2 & \beta Y_1 & \beta Y_3 \\ \gamma Y_3 & \gamma Y_2 & \gamma Y_1 \end{pmatrix}$$

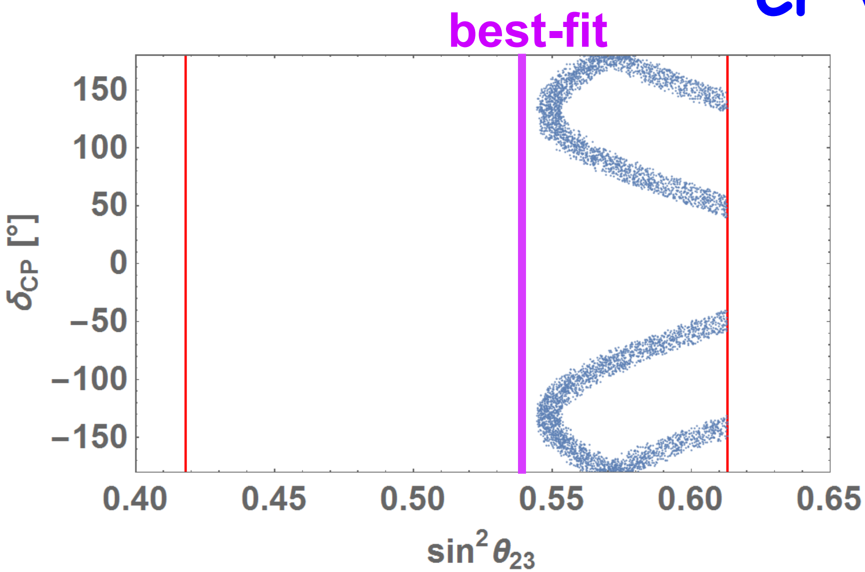
$$\mathcal{Y}_\nu = \begin{pmatrix} 2g_1 Y_1 & (-g_1+g_2)Y_3 & (-g_1-g_2)Y_2 \\ (-g_1-g_2)Y_3 & 2g_1 Y_2 & (-g_1+g_2)Y_1 \\ (-g_1+g_2)Y_2 & (-g_1-g_2)Y_1 & 2g_1 Y_3 \end{pmatrix}$$

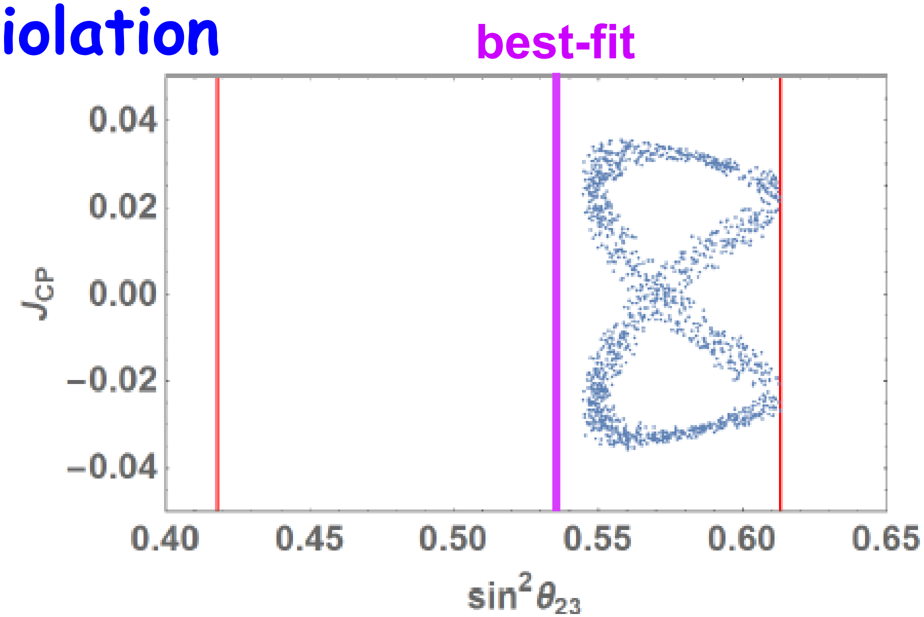
$$\mathcal{M}_R = \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \Lambda$$

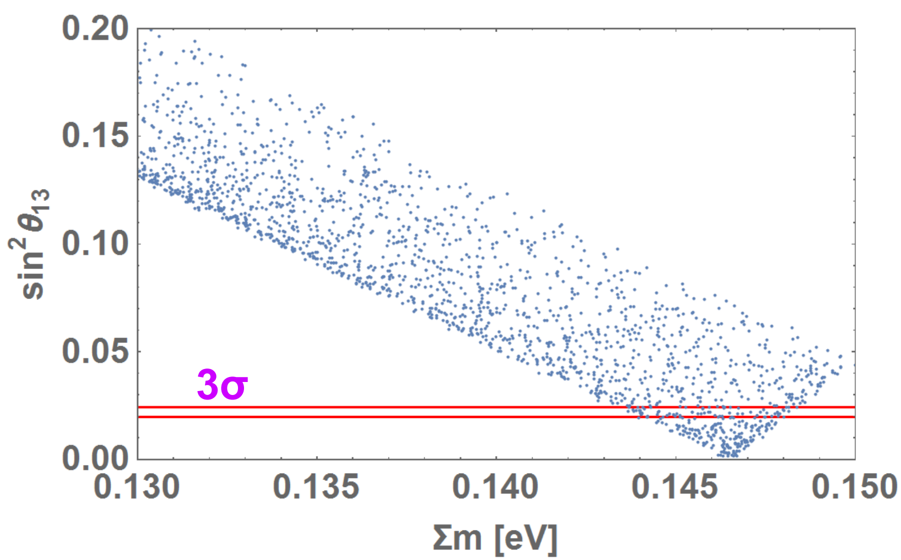
Parameters:
α, β, γ, $g_2/g_1=g$, τ

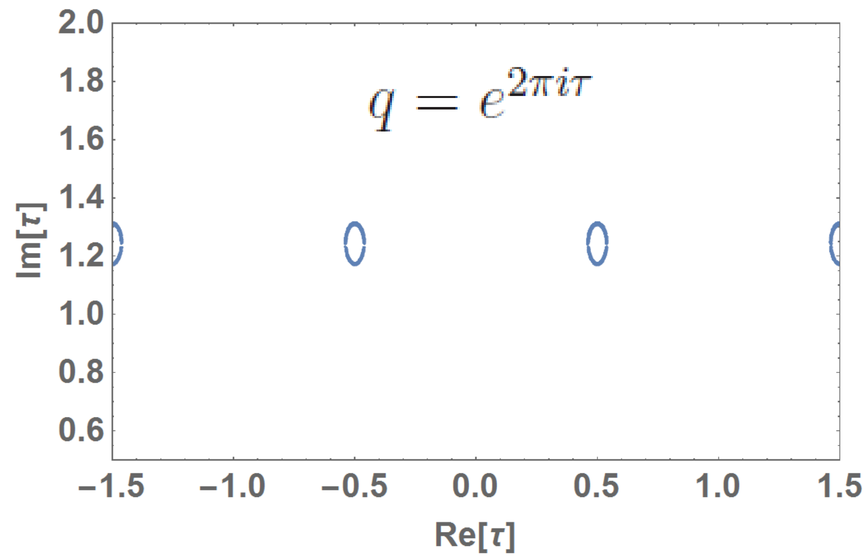
m_e, m_μ, m_τ fix α, β, γ .
$\Delta m^2_{sol} / \Delta m^2_{atm}$ and θ_{23}, θ_{12}, θ_{13} fix $g_2/g_1=g$ and τ .

CP violation



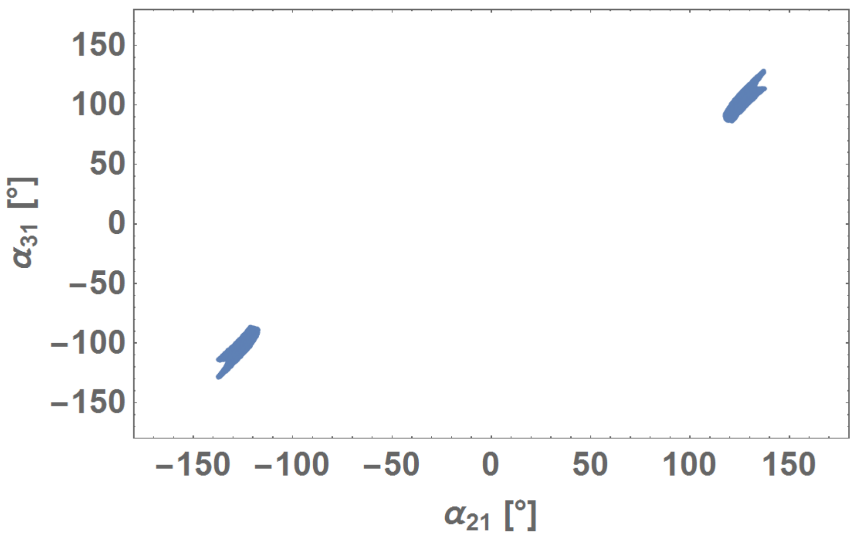


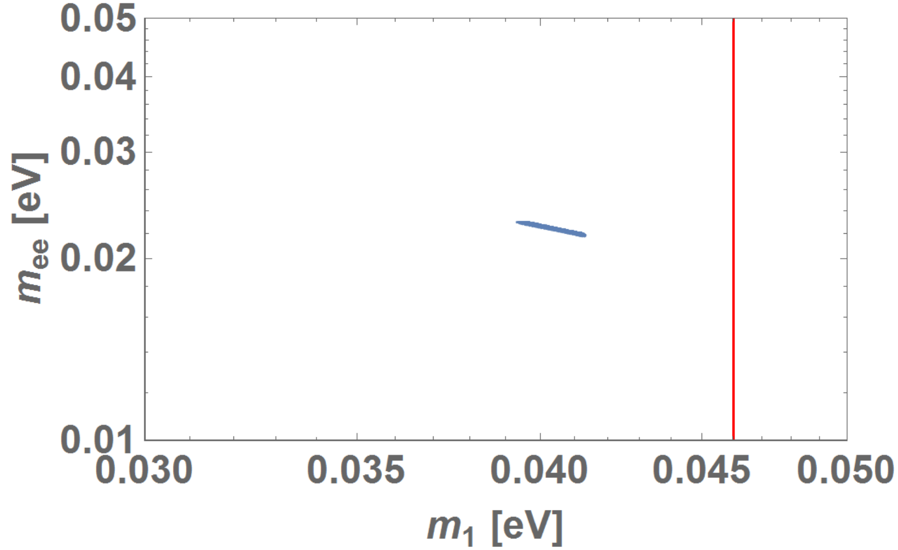




Planck 2018 results <0.12 eV@ΛCDM model ?

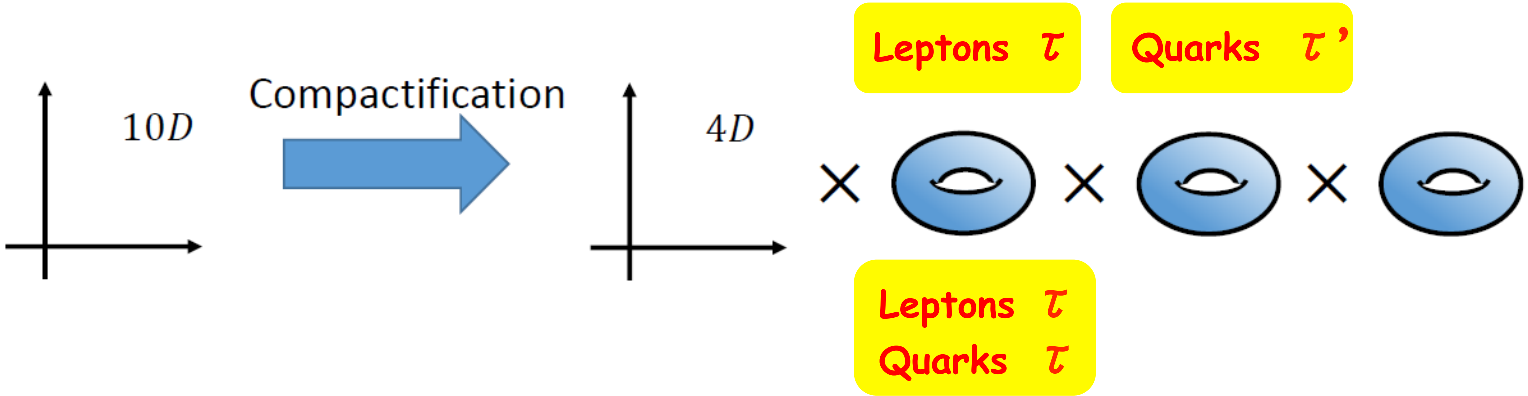
Predicted Majorana Phases and $\langle m_{ee} \rangle$





$m_1 \simeq m_2 \simeq 40\text{meV}$ and $m_3 \simeq 60\text{meV}$

How is the quark mass matrix in modular A_4 symmetry ?



Simple model: left-handed doublet 3, right-handed singlet 1, 1", 1'

$$\mathrm{diag}[\alpha, \beta, \gamma] \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}$$

for both up- and down-quarks

Coefficients α, β, γ are different for up- and down-quarks.

After fixing α, β, γ by inputting quark masses, one can examine CKM matrix elements by scanning modulus parameter τ .

A_4 model (triplet)? but S_3 model (doublet+singlet) may be OK

4 Modular S_3 and S_4 Symmetries

$\Gamma_2 \simeq$ S_3 group

Irreducible representations: 1, 1', 2

T. Kobayashi, K. Tanaka, T.H. Tatsuishi, arXiv:1803.10391

There are 2 linealy independent modular forms for weight 2

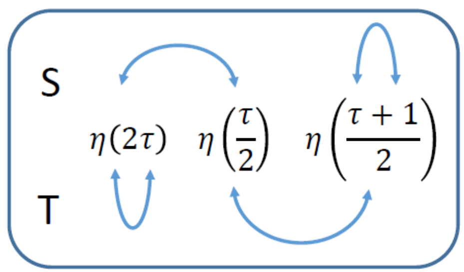
because of Dimension 2.

Doublet !

Prepare 3 Dedekind eta-functions as Modular functions

S

$$\eta(2\tau) \to \sqrt{\frac{-i\tau}{2}}\eta(\tau/2),$$
$$\eta(\tau/2) \to \sqrt{-i3\tau}\eta(2\tau),$$
$$\eta((\tau+1)/2) \to e^{-i\pi/12}\sqrt{-i\tau}\eta((\tau+1)/2).$$



T

$$\eta(2\tau) \to e^{i\pi/6}\eta(2\tau),$$
$$\eta(\tau/2) \to \eta((\tau+1)/2),$$
$$\eta((\tau+1)/2) \to e^{i\pi/12}\eta(\tau/2).$$

$$Y(\alpha,\beta,\gamma|\tau) = \frac{d}{d\tau}\left(\alpha\log\eta(\tau/2) + \beta\log\eta((\tau+1)/2) + \gamma\log\eta(2\tau)\right).$$

$$S:\quad Y(\alpha,\beta,\gamma|\tau) \to \tau^2 Y(\gamma,\beta,\alpha|\tau),$$
$$T:\quad Y(\alpha,\beta,\gamma|\tau) \to Y(\gamma,\alpha,\beta|\tau).$$

$\alpha+\beta+\gamma=0$

$$\rho(S) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \end{pmatrix} = \tau^2\rho(S)\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \end{pmatrix} = \rho(T)\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}.$$

$$\begin{aligned} Y_1(\tau) &= \frac{i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} - \frac{8\eta'(2\tau)}{\eta(2\tau)}\right), \\ Y_2(\tau) &= \frac{\sqrt{3}i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)}\right), \end{aligned}$$

$$\begin{aligned} Y_1(\tau) &= \frac{1}{8} + 3q + 3q^2 + 12q^3 + 3q^4\cdots, \\ Y_2(\tau) &= \sqrt{3}q^{1/2}(1 + 4q + 6q^2 + 8q^3\cdots). \end{aligned}$$

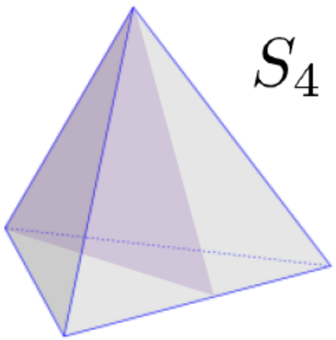
Phenomenological analyses are not enough .

$\Gamma_4 \simeq$ S$_4$ group

Irreducible representations: 1, 1', 2, 3, 3'

Modular S4

$$\boxed{S^2 = (ST)^3 = T^4 = 1}$$



$$\mathbf{1}: \quad \rho(S) = 1, \quad \rho(T) = 1$$

$$\mathbf{1}': \quad \rho(S) = -1, \quad \rho(T) = -1$$

$$\mathbf{2}: \quad \rho(S) = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{3}: \quad \rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad \rho(T) = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix}$$

$$\mathbf{3}': \quad \rho(S) = -\frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad \rho(T) = -\frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix}$$

we have adapted group theoretical results from
Bazzocchi, Merlo, Morisi, 0901.2086

$$\omega = e^{2\pi i/3}$$

J. Penedo

J. Penedo, S. Petcov arXiv:1806.11040

There are 5 linealy independent modular forms for weight 2 because of Dimension 5.

Doublet + Triplet !

Prepare 6 Dedekind eta-functions as Modular functions

$$
S:\quad \begin{cases}
\eta\left(\tau+\frac{1}{2}\right) & \rightarrow \quad \frac{1}{\sqrt{2}}\sqrt{-i\tau}\,\eta\left(\frac{\tau+2}{4}\right) \\
\eta\left(4\tau\right) & \rightarrow \quad \frac{1}{2}\sqrt{-i\tau}\,\eta\left(\frac{\tau}{4}\right) \\
\eta\left(\frac{\tau}{4}\right) & \rightarrow \quad 2\sqrt{-i\tau}\,\eta\left(4\tau\right) \\
\eta\left(\frac{\tau+1}{4}\right) & \rightarrow \quad e^{-i\pi/6}\sqrt{-i\tau}\,\eta\left(\frac{\tau+3}{4}\right) \\
\eta\left(\frac{\tau+2}{4}\right) & \rightarrow \quad \sqrt{2}\sqrt{-i\tau}\,\eta\left(\tau+\frac{1}{2}\right) \\
\eta\left(\frac{\tau+3}{4}\right) & \rightarrow \quad e^{i\pi/6}\sqrt{-i\tau}\,\eta\left(\frac{\tau+1}{4}\right)
\end{cases}
\qquad
T:\quad \begin{cases}
\eta\left(\tau+\frac{1}{2}\right) & \rightarrow \quad e^{i\pi/12}\,\eta\left(\tau+\frac{1}{2}\right) \\
\eta\left(4\tau\right) & \rightarrow \quad e^{i\pi/3}\,\eta\left(4\tau\right) \\
\eta\left(\frac{\tau}{4}\right) & \rightarrow \quad \eta\left(\frac{\tau+1}{4}\right) \\
\eta\left(\frac{\tau+1}{4}\right) & \rightarrow \quad \eta\left(\frac{\tau+2}{4}\right) \\
\eta\left(\frac{\tau+2}{4}\right) & \rightarrow \quad \eta\left(\frac{\tau+3}{4}\right) \\
\eta\left(\frac{\tau+3}{4}\right) & \rightarrow \quad e^{i\pi/12}\,\eta\left(\frac{\tau}{4}\right)
\end{cases}
.
$$

$$Y(a_1,\dots,a_6|\tau) \equiv \frac{d}{d\tau}\left(\sum_{i=1}^{6} a_i \,\log \eta_i(\tau)\right) \qquad \Sigma\, a_i = 0$$

$$= a_1\frac{\eta'(\tau+1/2)}{\eta(\tau+1/2)} + 4\,a_2\frac{\eta'(4\tau)}{\eta(4\tau)} + \frac{1}{4}\left[a_3\frac{\eta'(\tau/4)}{\eta(\tau/4)} + a_4\frac{\eta'((\tau+1)/4)}{\eta((\tau+1)/4)} + a_5\frac{\eta'((\tau+2)/4)}{\eta((\tau+2)/4)} + a_6\frac{\eta'((\tau+3)/4)}{\eta((\tau+3)/4)}\right]$$

$$S:\; Y(a_1,\dots,a_6|\tau) \;\to\; Y(a_1,a_2,a_3,a_4,a_5,a_6|-1/\tau) \;=\; \tau^2\,Y(a_5,a_3,a_2,a_6,a_1,a_4|\tau)\,,$$

$$T:\; Y(a_1,\dots,a_6|\tau) \;\to\; Y(a_1,a_2,a_3,a_4,a_5,a_6|\tau+1) \;=\; Y(a_1,a_2,a_6,a_3,a_4,a_5|\tau)\,.$$

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \end{pmatrix} = \tau^2\rho(S)\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \end{pmatrix} = \rho(T)\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}.$$

$$\begin{pmatrix} Y_3(-1/\tau) \\ Y_4(-1/\tau) \\ Y_5(-1/\tau) \end{pmatrix} = \tau^2\rho(S)\begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_3(\tau+1) \\ Y_4(\tau+1) \\ Y_5(\tau+1) \end{pmatrix} = \rho(T)\begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}.$$

$$S^2 = (ST)^3 = T^4 = \mathbb{1}$$

$$\mathbf{2}: \quad \rho(S) = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\mathbf{3}: \quad \rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad \rho(T) = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix},$$

$$\mathbf{3'}: \quad \rho(S) = -\frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad \rho(T) = -\frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix},$$

$$-\frac{8i}{3\pi}Y_1(\tau) = 1 - 24y - 72y^2 + 288y^3 + 216y^4 + \ldots,$$

$$-\frac{8i}{3\pi}Y_2(\tau) = 1 + 24y - 72y^2 - 288y^3 + 216y^4 + \ldots,$$

$$\frac{4i}{\pi}Y_3(\tau) = 1 - 8z + 64z^3 + 32z^4 + 192z^5 - 512z^7 + 384z^8 + \ldots,$$

$$\frac{2i}{\pi}\left[Y_4(\tau) + Y_5(\tau)\right] = 1 + 4z - 32z^3 + 32z^4 - 96z^5 + 256z^7 + 384z^8 + \ldots,$$

$$\frac{i}{\pi}\left[Y_4(\tau) - Y_5(\tau)\right] = 2\sqrt{3}\,z\left(1 + 8z^2 - 24z^4 - 64z^6 + \ldots\right),$$

$y \equiv i\sqrt{q/3}$, $z \equiv e^{i\pi/4}(q/4)^{1/4}$, and as usual $q = e^{2\pi i\tau}$.

$$Y_{\mathbf{2}}(\tau) \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}$$

$$Y_{\mathbf{3'}}(\tau) \equiv \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}$$

3'

No solution of weight 2 for S_4 triplet

S_4 Triplet appears in Weight 4

3' x 3' → 3

J. Penedo

Modular forms of higher weight

From tensor products of $Y_{\mathbf{2}}(\tau)$ and $Y_{\mathbf{3'}}(\tau)$:

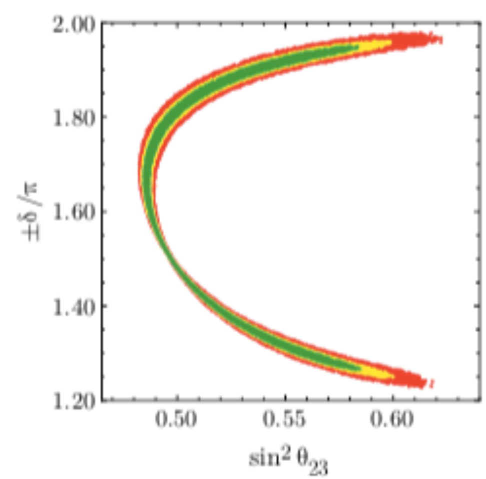
Non-zero $\mathbf{1'}$ arises only at weight 6

$$Y^{(4)}_{\mathbf{1}} = Y_1 Y_2 \sim \mathbf{1}$$

$$Y^{(4)}_{\mathbf{2}} = (Y_2^2,\, Y_1^2)^T \sim \mathbf{2}$$

$$Y^{(4)}_{\mathbf{3}} = (Y_1Y_4 - Y_2Y_5,\, Y_1Y_5 - Y_2Y_3,\, Y_1Y_3 - Y_2Y_4)^T \sim \mathbf{3}$$

$$Y^{(4)}_{\mathbf{3'}} = (Y_1Y_4 + Y_2Y_5,\, Y_1Y_5 + Y_2Y_3,\, Y_1Y_3 + Y_2Y_4)^T \sim \mathbf{3'}$$



Phenomenological implications are seen in 1811.04933 (Novichkov, Penedo, Petcov, Titov)

5 Summary

- Footprint of the non-Abelian discrete symmetry is expected to be seen in the neutrino mixing matrix.
  It is an imprint of generators of finite groups. A_4 S_4.....
- A_4 is a subgroup of the modular group, which may come from superstring theory on certain compactifications.
- Mass matrices of A_4 model are determined essentially by the modular parameter τ .
- There is a consistent neutrino mass matrix with NH (no IH). Predictions are sharp and testable in the future.
- Is Modulus τ common in both quarks and leptons ?
- S_3 and S_4 are also subgroups of the modular group.

We need more phenomenological discussions.

Need additional flavons in A_4 model

A_4 model realizes non-vanishing θ_{13} .

Y. Simizu, M. Tanimoto, A. Watanabe, PTP 126, 81(2011)

Add 1' or 1" flavon which couples to neutrinos.

LL $\boxed{\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1}} = a_1 * b_1 + a_2 * b_3 + a_3 * b_2$

LL $\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1}' = a_1 * b_2 + a_2 * b_1 + a_3 * b_3$

LL $\boxed{\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1}''} = a_1 * b_3 + a_2 * b_2 + a_3 * b_1$

$$\overset{\xi}{\mathbf{1} \times \mathbf{1} \Rightarrow \mathbf{1}} \quad , \quad \overset{\xi'}{\mathbf{1}'' \times \mathbf{1}' \Rightarrow \mathbf{1}}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \boxed{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}$$

In 2012, θ_{13} was measured by Daya Bay, RENO, Double Chooz, T2K, MINOS,

Tri-bimaximal mixing was ruled out !

$$\theta_{13} \simeq 9^\circ \simeq \theta_c/\sqrt{2}$$

Rather large θ_{13} promoted to search for *CP* violation !

$$J_{CP} = s_{23}c_{23}s_{12}c_{12}s_{13}c_{13}^2 \sin\delta_{CP} \simeq 0.0327 \sin\delta$$

J_{CP}(quark)$\sim 3\times10^{-5}$

CP violating phase δ_{CP} is a key parameter to understand flavors as well as two large mixing angles θ_{12} and θ_{23}.

Neutrino mixing matrix

$$\nu_\alpha = (U_{PMNS})_{\alpha i}\, \nu_i$$

α =e, μ , τ i=1, 2, 3

flavor eigenstates **mass eigenstates**

$$U_{\rm PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\rm CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix}$$

c_{ij} and s_{ij} denote $\cos\theta_{ij}$ and $\sin\theta_{ij}$, respectively.

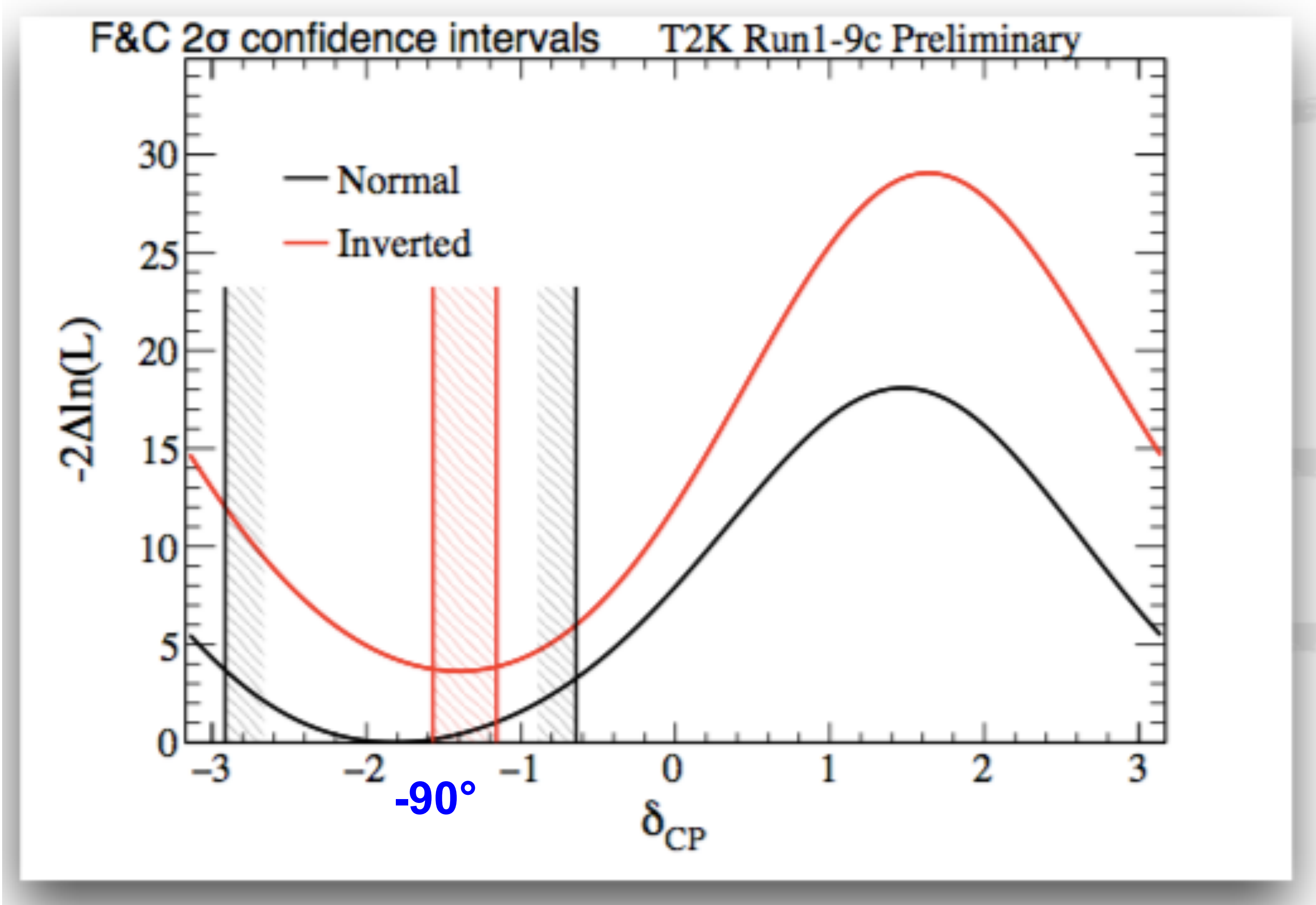
$m_1 < m_2 < m_3$ $m_3 < m_1 < m_2$

observable	3σ range for NH	3σ range for IH
$\Delta m^2_{\rm atm}$	$(2.399 \sim 2.593) \times 10^{-3}\mathrm{eV}^2$	$(-2.562 \sim -2.369) \times 10^{-3}\mathrm{eV}^2$
$\Delta m^2_{\rm sol}$	$(6.80 \sim 8.02) \times 10^{-5}\mathrm{eV}^2$	$(6.80 \sim 8.02) \times 10^{-5}\mathrm{eV}^2$
$\sin^2\theta_{23}$	$0.418 \sim 0.613$	$0.435 \sim 0.616$
$\sin^2\theta_{12}$	$0.272 \sim 0.346$	$0.272 \sim 0.346$
$\sin^2\theta_{13}$	$0.01981 \sim 0.02436$	$0.02006 \sim 0.02452$

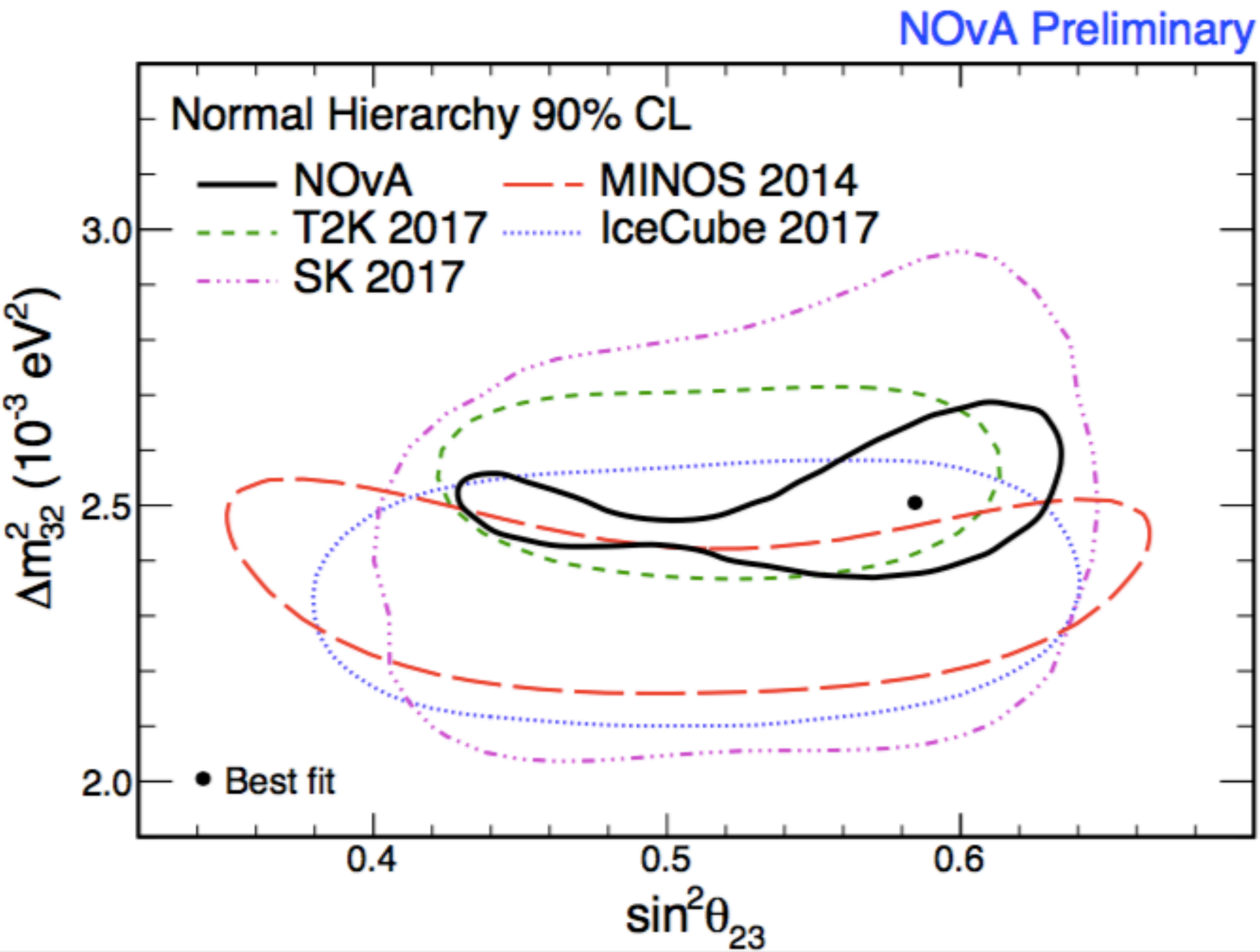
NuFIT3.2 (2018) $\Delta m^2_{\rm atm} = m_3^2 - m_1^2$, $\Delta m^2_{\rm sol} = m_2^2 - m_1^2$

Neutrino 2018

DATA FIT with reactor constraint



- **CP conserving values of δ_{CP} lie outside 2σ region.**



NOvA Preliminary
Normal Hierarchy 90% CL
NOvA
MINOS 2014
T2K 2017
IceCube 2017
SK 2017
Best fit
3.0
2.5
2.0
0.4
0.5
0.6
Δm^2_{32} (10^{-3} eV^2)
$\sin^2\theta_{23}$

If θ_{23} is rather less than 45°
it could be related neutrino masses.

For example,

$$\boxed{\sin^2\theta_{23} \simeq \sqrt[4]{\frac{\Delta m^2_{\rm sol}}{\Delta m^2_{\rm atm}}} = 0.40 \sim 0.43}$$

FTY(2003), FSTY(2012)

Just like GST relation

GST 1968 Weinberg 1977

$$M_{\rm d} = \begin{pmatrix} 0 & A \\ A & B \end{pmatrix} \quad \Rightarrow \quad \theta_{12} \simeq \sqrt{\frac{m_d}{m_s}}$$

However, the closer θ_{23} =45° or > 45°
the more likely that some symmetry/structure behind it.

2 Towards Non-Abelian Flavor Symmetry

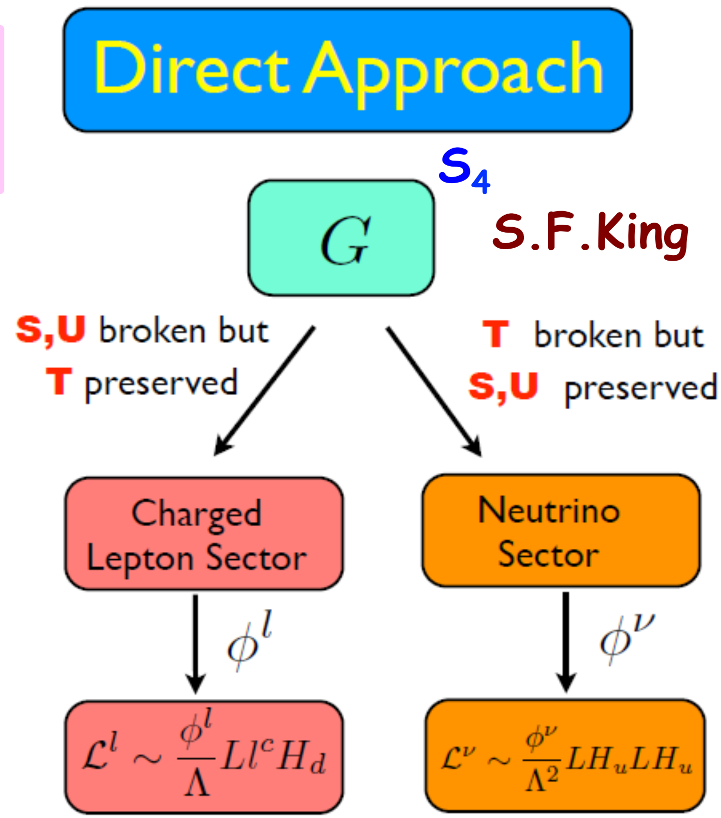
Footprint of the non-Abelian discrete symmetry is expected to be seen in the neutrino mixing matrix.

How to find an imprint of generators of finite groups

Generators of G (S,T,U) determine the flavor mixing directly.

Suppose group G for flavors at high energy.

At low energy, different subgroups of G are preserved in Yukawa sectors of Neutrinos and Charged leptons, respectively.



Consider the case of A_4 flavor symmetry:

A_4 has subgroups:

three Z_2, four Z_3, one $Z_2 \times Z_2$ (klein four-group)

$S^2 = T^3 = (ST)^3 = 1$

Z_2: {1, S}, {1, T^2ST}, {1, TST^2}
Z_3: {1, T, T^2}, {1, ST, T^2S}, {1, TS, ST^2}, {1, STS, ST^2S}
K_4: {1, S, T^2ST, TST^2}

Suppose A_4 is spontaneously broken to one of subgroups:
Neutrino sector preserves Z_2: {1, S}
Charged lepton sector preserves Z_3: {1, T, T^2}

$$S^T m^\nu_{LL} S = m^\nu_{LL}, \quad T^\dagger Y_e Y_e^\dagger T = Y_e Y_e^\dagger$$

$$\Downarrow$$

$$[S, m^\nu_{LL}] = 0, \quad [T, Y_e Y_e^\dagger] = 0$$

Mixing matrices diagonalise m^ν_{LL}, $Y_e Y_e^\dagger$ also diagonalize S and T, respectively !

For the triplet, the representations are given as

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}; \quad \omega = e^{2\pi i/3}$$

$$V_\nu^T S V_\nu = \mathrm{diag}(-1, 1, -1)$$

$$V_\nu = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

Tri-bimaximal Mixing

Independent of mass eigenvalues !

Freedom of the rotation between 1st and 3rd column because a column corresponds to a mass eigenvalue.

Finally, we obtain PMNS matrix.

$$V_\nu = \begin{pmatrix} 2c/\sqrt{6} & 1/\sqrt{3} & 2s/\sqrt{6} \\ -c/\sqrt{6}+s/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6}-c/\sqrt{2} \\ -c/\sqrt{6}-s/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6}+c/\sqrt{2} \end{pmatrix}$$

$c = \cos\theta \quad s = \sin\theta e^{-i\sigma}$

CP violating phase appears accidentally.

Tri-maximal mixing : so called TM_2

θ and σ are not fixed.

Since two parameters appear, there are two relations among mixing angles and CP violating phase.

Mixing sum rules

$$\sin^2\theta_{12} = \frac{1}{3}\frac{1}{\cos^2\theta_{13}} \geq \frac{1}{3}\,, \qquad \cos\delta_{CP}\tan 2\theta_{23} \simeq \frac{1}{\sqrt{2}\sin\theta_{13}}\left(1-\frac{5}{4}\sin^2\theta_{13}\right)$$

Direct Approach

☆ Flavor Structure of Yukawa Interactions is directly related with the Generators of Finite groups. Predictions are testable.

★ One cannot discuss the related phenomena without Lagrangian.
Leptogenesis, Quark CP violation, Lepton flavor violation

Model building is required.

☆ Conventional model building :
Introduce flavons (gauge singlet scalars) to discuss dynamics of flavors. Write down an effective Lagrangian including flavons. Flavor symmetry is broken spontaneously by VEV of flavons.

★ The number of parameters of Yukawa interactions increases. Predictability of model is considerably reduced.

$\mathcal{N}$=1 SUSY modular invariant theories

focus on Yukawa interactions and $\mathcal{N}$=1 global SUSY
[extension to $\mathcal{N}$=1 SUGRA straightforward]

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar\theta \; K(\Phi, \bar\Phi) + \int d^4x d^2\theta \; w(\Phi) + h.c.$$

$\Phi = (\tau, \varphi)$

S invariant if
$$\begin{cases} w(\Phi) \to w(\Phi) \\ K(\Phi,\bar\Phi) \to K(\Phi,\bar\Phi) + f(\Phi) + f(\bar\Phi) \end{cases}$$

- $$K(\Phi,\bar\Phi) = -h \log(-i\tau + i\bar\tau) + \sum_I (-i\tau + i\bar\tau)^{+k_I} |\varphi^{(I)}|^2$$

  "minimal" Kahler potential

- $$w(\Phi) = \sum_n Y_{I_1 \dots I_n}(\tau)\, \varphi^{(I_1)} \dots \varphi^{(I_n)}$$

  field-dependent Yukawa couplings

invariance of w(Φ) guaranteed by an holomorphic Y such that

$$Y_{I_1 \dots I_n}(\gamma\tau) = (c\tau + d)^{k_Y(n)} \rho(\gamma)\, Y_{I_1 \dots I_n}(\tau)$$

modular forms of level N and weight k_Y

1. the weights sum to zero: $k_Y(n) + k_{I_1} + \dots + k_{I_n} = 0$
2. the product $\rho \times \rho^{I_1} \times \dots \times \rho^{I_n}$ contains an invariant singlet

Feruglio FLASY 2018

J. Penedo, S. Petcov arXiv:1806.11040

Normal hierarchy

$$M_e^{\mathrm{I}} = v_d \begin{pmatrix} \alpha\, Y_3 & \alpha\, Y_5 & \alpha\, Y_4 \\ \beta\,(Y_1 Y_4 - Y_2 Y_5) & \beta\,(Y_1 Y_3 - Y_2 Y_4) & \beta\,(Y_1 Y_5 - Y_2 Y_3) \\ \gamma\,(Y_1 Y_4 + Y_2 Y_5) & \gamma\,(Y_1 Y_3 + Y_2 Y_4) & \gamma\,(Y_1 Y_5 + Y_2 Y_3) \end{pmatrix}^\dagger$$

$$M_\nu^{\mathrm{II}} = \frac{2g' v_u^2}{\Lambda} \left[\begin{pmatrix} (g/g')Y_1 Y_2 & Y_2^2 & Y_1^2 \\ Y_2^2 & Y_1^2 & (g/g')Y_1 Y_2 \\ Y_1^2 & (g/g')Y_1 Y_2 & Y_2^2 \end{pmatrix} \right.$$
$$\left. + \frac{1}{2}\frac{g''}{g'} \begin{pmatrix} 2(Y_1 Y_4 - Y_2 Y_5) & -(Y_1 Y_3 - Y_2 Y_4) & -(Y_1 Y_5 - Y_2 Y_3) \\ -(Y_1 Y_3 - Y_2 Y_4) & 2(Y_1 Y_5 - Y_2 Y_3) & -(Y_1 Y_4 - Y_2 Y_5) \\ -(Y_1 Y_5 - Y_2 Y_3) & -(Y_1 Y_4 - Y_2 Y_5) & 2(Y_1 Y_3 - Y_2 Y_4) \end{pmatrix} \right]$$

Kinetic Term

Kinetic term of the modulus τ $$\frac{|\partial_\mu \tau|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$

Modular transformation $$\tau' = \frac{a\tau + b}{c\tau + d}, ad - bc = 1$$

- numerator

$$\partial_\mu \tau' = \frac{(a\partial_\mu \tau)(c\tau + d) - (a\tau + b)(c\partial_\mu \tau)}{(c\tau + d)^2} = \frac{(ad - bc)\partial_\mu \tau}{(c\tau + d)^2} = \frac{\partial_\mu \tau}{(c\tau + d)^2}$$

- denominator

$$\tau' - \bar{\tau}' = \frac{(a\tau + b)(c\bar{\tau} + d) - (a\bar{\tau} + b)(c\tau + d)}{|c\tau + d|^2} = \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} = \frac{\tau - \bar{\tau}}{|c\tau + d|^2}$$

$$\Rightarrow \quad \frac{|\partial_\mu \tau'|^2}{\langle -i\tau' + i\bar{\tau}'\rangle^2} = \frac{|\partial_\mu \tau|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$ Modular invariant

Modular Form

How to find the concrete form of modular form with weight 2 and non-trivial rep. of $\Gamma(N)$

- Suppose functions $f_i(\tau)$ to be modular forms with weight k_i
- Also suppose $\sum_i k_i = 0$

$\Rightarrow$ $\frac{d}{d\tau}\sum_i \log f_i(\tau)$ is a modular form with weight 2

$\boxed{\text{Proof}}$ Modular transformation: $\tau' = \dfrac{a\tau + b}{c\tau + d}, ad - bc = 1$

$$\frac{d}{d\tau'} = \frac{d\tau}{d\tau'}\frac{d}{d\tau} = (c\tau + d)^2 \frac{d}{d\tau}, \qquad f_i(\tau') = (c\tau + d)^{k_i} f_i(\tau)$$

$$\frac{d}{d\tau'}\sum_i \log f_i(\tau') = (c\tau + d)^2 \frac{d}{d\tau}\sum_i [\log f_i(\tau) + \underbrace{k_i(c\tau + d)}_{=0}]$$

$$= (c\tau + d)^2 \frac{d}{d\tau}\sum_i \log f_i(\tau)$$

➢ When we find a set of $f_i(\tau)$,
we can construct modular form with weight 2