Heterotic Moduli and Effective Theories

Eirik Eik Svanes (King's College London, ICTP Trieste)

Based on work in collaboration with

A. Ashmore, X. de la Ossa, E. Hardy,
R. Minasian, C. Strickland-Constable.

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Introduction and Motivation

The low energy theory of the heterotic string is ten dimensional supergravity coupled to Yang-Mills gauge theory.

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- Higher curvature corrections induce torsional (non Ricci-flat) geometries [Hull 86, Strominger 86].
- Harder to understand geometries. Often loose toolbox of algebraic geometry and Kähler geometry.
- Harder to understand moduli (the deformation space).

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This talk: Heterotic string on manifolds with reduced $SU(3)$-structure group.

- Review of string compactifications in the context of heterotic string.
- Finite deformations of heterotic $SU(3)$ system, heterotic deformation complex.
- Review cohomology counting infinitesimal moduli.
- Comments on work in progress..

General String Compactifications

Low-energy limit of string theory is ten-dimensional. Common vacuum ansatz:

$$\mathcal{M}_{10} = \mathcal{M}_d \times X_{10-d} \ ,$$

where $\mathcal{M}_d$ is the d-dimensional (usually maximally symmetric) external spacetime, and X is the internal (compact) geometry.

Phenomenology: $d = 4$, require X to admit spinors $\Rightarrow X$ is Calabi-Yau to lowest order.

Formal Geometry: Supersymmetric geometries are often easier to study as they admit extra structure (Complex, Kähler, etc). String theory often leads to new groundbreaking insights: Mirror symmetry, topological string theory, geometric invariants, etc.

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Deformations (moduli):

- Deformations δX preserving supersymmetry $\Leftrightarrow$ moduli fields in external spacetime.
- String Phenomenology: Compact geometries whose moduli contains the Standard Model.
- At this point we have a very good understanding of (type II) Calabi-Yau moduli space.
- Heterotic String: We do not yet understand moduli of generic compactifications. Special cases known (e.g. Standard Embedding [Candelas etal 85]).

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Complications:

- Torsional geometries not well understood.
- Few "non-trivial" examples [Dasgupta etal. 99, Becker etal 06, Halmagyi-Israel-EES 16,..].
- Complicated equations to deal with, e.g. heterotic Bianchi Identity:

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Need a "nicer" description to deal with moduli:
Supergravity: [Anderson etal 10;11;14, delaOssa etal 14;15;18, Garcia-Fernandez etal 13;15;18, Candelas etal 16;18, ..].
Worldsheet $(0,2)$-models: [Melnikov-Sharpe 11, Bertolini etal 13;14;17;18, Fiset etal 17;18, ..]

Given a set of geometric conditions: What the allowed deformations?
Do they obey any structure?

Physical intuition is often helpful in this regard:

Input from Physics	Mathematical Structure
Finite Spectrum	Elliptic system
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Steps in understanding moduli:

- *Step 1:* Start with infinitesimal deformation (linear approximation). Moduli fields $\mathcal{X}$ usually one-forms with values in a bundle $\mathcal{Q}$ (or sheaf), naturally associated to the given moduli problem. $\Rightarrow$ Infinitesimal *massless* spectrum

$$T\mathcal{M} = H^1_{\mathcal{D}}(\mathcal{Q}) \,,$$

cohomology of natural differential $\mathcal{D}$ ($\mathcal{D}^2 = 0$) acting on $\mathcal{Q}$. Infinitesimal deformations are closed

$$\mathcal{D}\mathcal{X} = 0 \,,$$

while exact one-forms, $\mathcal{D}\epsilon$ for $\epsilon \in \Gamma(\mathcal{Q})$, correspond to trivial deformations generated by an infinitesimal symmetry transformation ϵ of the system.

- *Step 2:* Understand geometry of $\mathcal{M}$: Complex structure, Kähler metric, etc? Higher order deformations: Can infinitesimal moduli be integrated: Is the moduli space smooth? Obstructions correspond to Yukawa couplings in effective physics.

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  Physics: Expect a parametrisation of moduli where only a finite number of couplings play a role, as only a finite number of couplings are relevant/marginal in physics.

  E.g. L_2-algebra: Differentially graded Lie Algebra (DGLA). Finite deformations solve Maurer-Cartan equation,

$$\mathcal{D}\mathcal{X} + \tfrac{1}{2}[\mathcal{X}, \mathcal{X}] = 0\,.$$

  Only second order couplings survive.

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- *Step 3:* Understand **quantum cohomology ring**. Include non perturbative effects such as world-sheet instantons, and quantum corrections (higher genus effects). Topological theory: Witten's topological string, Donaldson-Thomas theory, etc. Compute invariants: Gromov-Witten invariants, Donaldson-Thomas invariants, etc.

The Heterotic $SU(3)$-system

$$\mathcal{M}_{10} = \mathcal{M}_4 \times X_6$$

Physics: Want geometry to preserve supersymmetry $\Rightarrow$ require existence of global spinor $\Rightarrow$ structure group reduces from $SO(6)$ to $SU(3)$, and we have an $SU(3)$-structure:

- $\Omega \in \Omega^3_{\mathbb{C}}(X)$ nowhere vanishing and locally decomposable (defines almost complex structure J).
- $\omega \in \Omega^2(X)$ is of maximal rank (ω^3 nowhere vanishing).
- The forms satisfy the $SU(3)$-structure relations:

$$\omega \wedge \Omega = 0\,, \quad \tfrac{i}{8}\Omega \wedge \overline{\Omega} = \tfrac{1}{6}\omega^3\,.$$

Note that $\Omega \in \Omega^{(3,0)}(X)$ and $\omega \in \Omega^{(1,1)}(X)$ with respect to J.

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Ω and ω are in general non-closed $\Rightarrow$ *intrinsic torsion* $\mathcal{W}_i$:

$$\mathrm{d}\omega = \frac{3i}{4}(\mathcal{W}_1\overline{\Omega} - \overline{\mathcal{W}}_1\Omega) + \mathcal{W}_3 + \omega \wedge \mathcal{W}_4$$

$$\mathrm{d}\Omega = \mathcal{W}_1\omega \wedge \omega + \omega \wedge \mathcal{W}_2 + \overline{\mathcal{W}}_5 \wedge \Omega.$$

Intrinsic torsion measures failure of structure to be covariant with respect to the Levi-Civita connection of the metric defined by the structure ($SU(3)$*-holonomy*). Decomposed into irreducible representations of of $SU(3)$.

Note: $\mathcal{W}_i = 0 \;\; \forall\, i$ implies $\mathrm{d}\Omega = \mathrm{d}\omega = 0$ and X is Calabi-Yau.

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This theory has a superpotential given by [Becker etal 03, Cardoso etal 03, Lukas etal 05, McOrist 16, ..]

$$W = \int_X (H + i\mathrm{d}\omega) \wedge \Omega \, ,$$

where the flux is given by

$$H = \mathrm{d}B + \tfrac{\alpha'}{4}\omega_{CS}(A) \, ,$$

often referred to as anomaly cancellation. We require the flux H to be *gauge invariant* $\Rightarrow$ impose a transformation on B through the Green-Schwarz mechanism [Green-Schwarz 84].

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A four-dimensional $N=1$ Minkowski vacuum requires that:

$$\delta W = W = 0 \,.$$

This implies the "F-term" conditions

$$\mathrm{d}\Omega = 0 \,, \quad F \wedge \Omega = 0 \,, \quad H = i(\partial - \bar{\partial})\omega \,.$$

There are also "D-term" conditions (less relevant for moduli considerations):

$$\mathrm{d}\left(e^{-2\phi}\omega \wedge \omega\right) = 0 \,, \quad \omega \wedge \omega \wedge F = 0 \,.$$

The Heterotic Deformation Algebra

Higher order deformation problems are difficult, and highly dependent on how we parametrise the deformations.

Examples:

- **Ex1**: Linear finite deformation $\delta g \Rightarrow$ deformation of g^{-1} is an infinite expansion in δg.
- **Ex2**: The space of almost complex structures J is a complex manifold, with complex parameters given in terms of $\mu \in \Omega^{(0,1)}(T^{(1,0)}X)$ (Beltrami differential).
- A generic deformation of J is a complicated expression in μ and $\overline{\mu}$. A *holomorphic deformation* Δ of J is however given by $\Delta J = -2i\,\mu$.
- Holomorphic deformations corresponding to integrable complex structures satisfy the Maurer-Cartan equation
$$\overline{\partial}\mu + \tfrac{1}{2}[\mu, \mu] = 0\ ,$$
where $[\ ,\ ]$ is the Lie-bracket on the holomorphic tangent bundle.

Similarly, considering a *generic* finite deformation of the heterotic $SU(3)$-system is in general a very hard problem. Get some complicated L_∞-algebra.

Clues from physics: Superpotential is *holomorphic* in deformations: $\overline{\Delta} W = 0$.

A *finite* and *holomorphic* deformation of the heterotic $SU(3)$-system can be represented as a $(0,1)$-form

$$y = (x, \alpha, \mu) \in \Omega^{(0,1)}(Q)\ , \quad Q = T^{*(1,0)}X \oplus \text{End}(V) \oplus T^{(1,0)}X$$

where $\mu \in \Omega^{(0,1)}(T^{(1,0)}X)$, $\alpha \in \Omega^{(0,1)}(\text{End}(V))$ and $x \in \Omega^{(0,1)}(T^{*(1,0)}X)$ now correspond to finite deformations of the structure.

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Deforming the superpotential away from supersymmetric locus, one finds

$$\Delta W = \int_X \left(\langle y, \overline{D} y \rangle - \tfrac{1}{3} \langle y, [y, y] \rangle - \mu^a \partial_a b \right) \wedge \Omega\ ,$$

where $\overline{D}$ is the heterotic differential, the pairing for $y_1, y_2 \in \Omega^{(0,*)}(Q)$ is given by

$$\langle y_1, y_2 \rangle = \mu_1^a x_{2\,a} + \mu_2^a x_{1\,a} + \text{tr}\,(\alpha_1 \alpha_2)\ ,$$

$b \in \Omega^{(0,2)}(X)$ is some auxiliary field, and the bracket

$$[\,,\,] \ :\ \ \Omega^{(0,p)}(Q) \times \Omega^{(0,q)}(Q) \to \Omega^{(0,p+q)}(Q)$$

satisfies Leibniz rule w.r.t. $\overline{D}$ and Jacobi identity modulo ∂_a-exact terms. Holomorphic generalisation of Dorfman bracket including bundles.

Note the close similarity to holomorphic Chern-Simons theory.

Supersymmetric solutions $\Rightarrow \delta\Delta W = \Delta W = 0$. We derive the following equations

$$\begin{aligned} \partial\Omega(\mu) &= 0 \\ \overline{D}y - \tfrac{1}{2}[y,y] - \tfrac{1}{2}\partial b &= 0 \\ \overline{\partial}b - \tfrac{1}{2}y^a\partial_a b + \tfrac{1}{3!}\langle y,[y,y]\rangle &= 0\,. \end{aligned}$$

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The last two equations can be rephrased as the Maurer-Cartan equation of a heterotic L_3 algebra $(\mathcal{Y}_*, \ell_1, \ell_2, \ell_3)$, where

$$\mathcal{Y}_n = \Omega^{(0,n)}(Q) \oplus \Omega^{(0,n+1)}(X)\,,$$

and where the L_3 multilinear products are given by

$$\ell_1(Y) = (\overline{D}y - \tfrac{1}{2}\partial b, \overline{\partial}b)\,,\ \ell_2(Y, Y) = ([y, y], \langle y, \partial b\rangle)\,,\ \ell_3(Y, Y, Y) = (0, -\langle y, [y, y]\rangle)\,,$$

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for $Y = (y,b)$. Higher products vanish. The Maurer-Cartan equation is then given by

$$\mathcal{F}(Y) = \ell_1(Y) - \tfrac{1}{2}\ell_2(Y) - \tfrac{1}{3!}\ell_3(Y) = 0\ ,$$

which is invariant under symmetry transformations, for $\Lambda \in \mathcal{Y}_0$

$$\delta_\Lambda Y = \ell_1(\Lambda) + \ell_2(\Lambda, Y) - \tfrac{1}{2}\ell_3(\Lambda, Y, Y)\ .$$

Integrating out the auxiliary field b from the action ΔW gives the condition $\partial\Omega(\mu) = 0$, similar to Kodaira-Spencer gravity [Bershadsky etal 93].

The action becomes

$$\Delta W \to \Delta W = \int_X \left(\langle y, \overline{D} y \rangle - \tfrac{1}{3} \langle y, [y, y] \rangle \right) \wedge \Omega \, .$$

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This action is invariant under gauge transformations

$$\delta y_a = \partial_a \kappa$$
$$\delta y = \overline{D}\epsilon - [y, \epsilon] \, ,$$

for $\kappa \in \Omega^0(X)$, and $\epsilon \in \Omega^0(Q)$ satisfying $\partial\Omega(\epsilon) = 0$, where

$$\Omega(\epsilon) = \tfrac{1}{2} \Omega_{abc} \, \epsilon^a \, \mathrm{d}z^{bc} \, .$$

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Question: Can we use this theory to define generalisations of Donaldson-Thomas invariants for heterotic geometries and holomorphic Courant algebroids?

Infinitesimal Moduli: The Cohomology $H^{(0,1)}_{\overline{D}}(Q)$

This talk: "Technical assumptions": either $\partial\overline{\partial}$-lemma or $h^{(0,1)} = 0$, and *stable bundles* (so $h^{(0,1)}(\text{End}(V)) = 0$).

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Infinitesimal moduli preserving SUSY conditions	$\Leftrightarrow$	Massless fields in 4d theory

Preserving a holomorphic top-form $\text{d}\Omega = 0$ gives

$$\text{d}\delta\Omega = 0 \;\Rightarrow \delta\Omega \in H^{(2,1)}_{\bar{\partial}}(X) \Leftrightarrow \mu \in H^{(0,1)}_{\bar{\partial}}(T^{(1,0)}X)\ ,$$

where μ is defined as

$$\delta\Omega = \Omega(\mu) = \tfrac{1}{2}\,\mu^a\,\Omega_{abc}\text{d}z^{bc}\ .$$

Here μ can be thought of as the deformation of the complex structure, often called the *Beltrami differential*.

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The deformations of the holomorphic bundle gives

$$\delta(F\wedge\Omega) = 0 \quad\Leftrightarrow\quad F_{a\overline{b}}\,\mathrm{d}z^{\overline{b}}\wedge\mu^a = \mathcal{F}(\mu) = -\overline{\partial}_A\alpha\,,$$

where $\alpha = \delta A^{(0,1)} \in \Omega^{(0,1)}(\text{End}(V))$ are deformations of the bundle.

Kernels and the Atiyah Algebroid

It follows that μ is in the kernel of Atiyah map [Atiyah 57, Anderson etal 10]

$$\mathcal{F} \ : \quad H^{(0,1)}_{\bar\partial}(T^{(1,0)}X) \to H^{(0,2)}_{\bar\partial_A}(\text{End}(V)) \ .$$

We thus see that the infinitesimal moduli of a complex manifold with holomorphic bundle is

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From general arguments, this should correspond to the first cohomology of some differential.

Indeed, consider the $(0,1)$-differential

$$\bar\partial_1=\begin{pmatrix}\bar\partial_A & \mathcal{F}\\ 0 & \bar\partial\end{pmatrix}\ :\ \Omega^{(q,p)}\begin{pmatrix}\text{End}(V)\\ T^{(1,0)}X\end{pmatrix}\to\Omega^{(q,p+1)}\begin{pmatrix}\text{End}(V)\\ T^{(1,0)}X\end{pmatrix}\ ,$$

on the bundle $Q_1=\text{End}(V)\oplus T^{(1,0)}X$. Note that $\bar\partial_1^2=0$ due to the Bianchi identity $\bar\partial_A F=0$.

One then finds

$$T\mathcal{M}_1=H^{(0,1)}_{\bar\partial_1}(Q_1)=..=H^{(0,1)}(\text{End}(V))\oplus\ker(\mathcal{F})\ .$$

Hunag 93: The finite deformations are described by a DGLA.

From a variation of the condition $H = i(\partial - \overline{\partial})\omega$ we find

$$\mathcal{H}(\mu, \alpha)_b \mathrm{d}z^b = 2\mu^a \wedge i\partial_{[a}\omega_{b]\overline{c}}\mathrm{d}z^{b\overline{c}} - \frac{\alpha'}{2}\mathrm{tr}\,\alpha \wedge F = \overline{\partial}x^{(1,1)}\ .$$

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$\Rightarrow (\mu,\alpha) \in H^{(0,1)}(Q_1)$ is in the kernel of

$$\mathcal{H}\ :\quad H^{(0,1)}_{\overline{\partial}_1}(Q_1) \to H^{(0,2)}_{\overline{\partial}}(T^{*(1,0)}X)\ .$$

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The massless moduli are then given by

$$T\mathcal{M} = H^{(0,1)}(T^{*(1,0)}X) \oplus \mathrm{ker}(\mathcal{H})\ ,\quad \mathrm{ker}(\mathcal{H}) \subseteq H^{(0,1)}(Q_1)\ ,$$

where $H^{(0,1)}(T^{*(1,0)}X) \cong H^{(1,1)}(X)$ are hermitian moduli.

What differential computes the massless moduli?

Let $Q = T^{*(1,0)}X \oplus \text{End}(V) \oplus T^{(1,0)}X$. Can define holomorphic structure on the differential complex $\Omega^{(q,p)}(Q)$

$$\overline{D} = \begin{pmatrix} \overline{\partial} & \mathcal{H} \\ 0 & \overline{\partial}_1 \end{pmatrix} \; : \; \Omega^{(q,p)} \begin{pmatrix} T^{*(1,0)}X \\ Q_1 \end{pmatrix} \; \to \; \Omega^{(q,p+1)} \begin{pmatrix} T^{*(1,0)}X \\ Q_1 \end{pmatrix} ,$$

Note that $\overline{D}^2 = 0$ iff the heterotic Bianchi Indentity is satisfied. This is the heterotic differential appearing in ΔW.

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Compute first cohomology [Anderson etal 14, delaOssa-EES 14]

$$T\mathcal{M} = H^{(0,1)}_{\overline{D}}(Q) = H^{(1,1)}(X) \oplus \text{ker}(\mathcal{H}) ,$$

Computed from long exact sequence

$$0 \to H^{(0,1)}(T^{*(1,0)}X) \to H^{(0,1)}(Q) \to H^{(0,1)}(Q_1)$$
$$\xrightarrow{\mathcal{H}} H^{(0,2)}(T^{*(1,0)}X) \to ..$$

Let $\alpha_h \in H^{(0,1)}(\text{End}(V))$ correspond to the closed part of α, i.e. a bundle modulus. Then there is an effective super-potential coupling generated

$$\int_X \text{tr}(F \wedge \alpha_h) \wedge \Omega(\mu) \in \Delta W \ .$$

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Similarly, there is then an F-term generated for μ in the effective theory provided there exists an α_h such that

$$\text{tr}(F \wedge \alpha_h) \neq 0$$

in cohomology.

This relation is symmetric in μ and α_h, and we can conclude that for every μ lifted by the Atiyah constraint, there is a corresponding lifted bundle modulus. We conclude

$$h^{(0,1)}(Q) \leq h^{(1,1)} + h^{2,1} + h^{(0,1)}(\text{End}(V)) - 2\,\text{Im}(\mathcal{F}) \ .$$

.

We note that the structure $\overline{D}$ corresponds to a complex structure on the total space of Q.

Complex structures tend to be size independent, while α'-corrections correspond to $1/Volume$ corrections.

Makes it plausable that most of the first order system survives to higher orders in α'.

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For $\partial\overline{\partial}$-manifolds one can also show that [delaOssa-Hardy-EES]

$$\dim(\mathcal{M}) \geq \{\text{Number of massless Kähler amd complex structure moduli}\}\ .$$

That is, the number of unobstructed directions is bounded from below by the infinitesimal geometric moduli.

Remarks on α'-corrections, $\dim(\mathcal{M})$ and Kähler potential

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A Kähler metric on the moduli space can be obtained from dimensional reduction. It is given by the Kähler potential [Candelas etal 16;18, McOrist 16]

$$K = -\log\left(i\int_X \Omega\wedge\overline{\Omega}\right) - \log\left(\frac{3}{4}\int_X \omega^3\right)\ .$$

Note: Hidden dependence on bundle moduli through ω and Bianchi Identity.

Conclusions

Conclusions:

- Heterotic geometries give nice generalisations of torsion-free geometries when bundles are included, but the moduli problem gets harder.

- We discussed higher order deformations of the heterotic $SU(3)$-system and the heterotic deformation algebra (an L_3-algebra).

- We have reviewed the cohomology describing the infinitesimal moduli of the heterotic $SU(3)$-system.

Outlook

- So far mostly a mathematical investigation into the structures. Interesting to look for applications in particle physics, AdS/CFT, black hole entropy, and other areas of string theory and differential geometry?
- Further investigation into higher order deformations and obstructions. What are the *integrable* deformations? Is there an analog of Tian-Todorov?
- What about non-perturbative effects, world sheet instantons, NS5-branes? Correct the Bianchi Identity

$$\mathrm{d}H + W_5 = \frac{\alpha'}{4}(\mathrm{tr}\, F^2 - \mathrm{tr}\, R^2)\ , \quad [W_5] \in H^{(2,2)}(X)\ .$$

$\Rightarrow$ Spoils integrability of differential $\overline{D}^2 \neq 0$.

- Connect with developments of $(0,2)$ moduli from the world-sheet point of view [Melnikov-Sharpe 11, Bertolini etal 13;14;17;18, Fiset etal 17;18, ..].
- Quantum corrections: Quantise quasi-topological action ΔW? Is there a corresponding topological world-sheet theory (e.g. ala Witten's topological string for Chern-Simons, or $\beta\gamma$-systems)? Compute invariants for heterotic geometries such as generalisations of Gromov-Witten and Donaldson-Thomas invariants?

Thank you for your attention!