Representations of quivers over Frobenius algebras

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Representations of quivers over fields

- $\Bbbk$ field, $\Bbbk = \overline{\Bbbk}$ or $|\Bbbk| < \infty$
- $Q = (V, E)$ quiver $E \subset V \times V$ $\alpha : V \to \mathbb{N}$ dimension vector
- $\rho \in Rep^{\alpha}(Q, \Bbbk) := \bigoplus_{(i,j)\in E} \mathrm{Hom}_{\Bbbk}(\Bbbk^{\alpha_i}, \Bbbk^{\alpha_j})$
- Problem: classify $\coprod_{\alpha\in\mathbb{N}^V} Rep^{\alpha}(Q, \Bbbk)/G_{\alpha}$!
- building blocks: ρ (abs.) indecomposable representations , i.e. $\rho = \rho_1 \oplus \rho_2 \rightsquigarrow \rho_1 = 0$ or $\rho_2 = 0$ (over $\overline{\Bbbk}$)
- Problem: classify (abs.) indecomposable representations, and their dimension vectors!

Theorem (Kac 1982)

There exists $\rho \in Rep^{\alpha}_{a.i}(Q, \Bbbk)/G_{\alpha} \Leftrightarrow \alpha \in \mathbb{N}^V$ *is a root of* $\mathfrak{g}_Q$

- $|\coprod_{\alpha} Rep^{\alpha}_{a.i}(Q, \Bbbk)/G_{\alpha}| < \infty$ i.e. Q is finite $\Leftrightarrow$ Q is *ADE* type (Gabriel 1971), when $\mathfrak{g}_Q$ is finite dimensional simple Lie algebra

Representations of quivers over finite fields

- $a_\alpha(Q, \mathbb{F}_q) := |Rep^{\alpha}_{a.i}(Q, \mathbb{F}_q)/G_\alpha|$ Kac polynomial
- e.g. for Jordan quiver $a_n(J, \mathbb{F}_q) = q$

Theorem (Kac 1982)

- $a_\alpha(Q, \mathbb{F}_q) \in \mathbb{Z}[q]$
- $a_\alpha(Q, \mathbb{F}_q)$ *independent of orientation of Q*
- $a_\alpha(Q, \mathbb{F}_q) \neq 0 \Leftrightarrow \alpha \in \mathbb{N}^V$ *is a root of* $\mathfrak{g}_Q$

Conjecture (Kac 1982)

1. $a_\alpha(Q, \mathbb{F}_q)|_{q=0} = m_\alpha$ *multiplicity of* α *in* $\mathfrak{g}_Q = \oplus_\alpha \mathfrak{g}_\alpha \oplus \mathfrak{h}$
2. $a_\alpha(Q, \mathbb{F}_q) \in \mathbb{N}[q]$

- Conjecture 1 completed by (Hausel 2010)
- Conjecture 2 completed by (Hausel, Letellier, Villegas 2013)

Fourier transform over $\Bbbk$

- $\Bbbk = \mathbb{F}_q$ and $\Psi : \mathbb{F}_q \to \mathbb{C}^\times$ non-trivial additive character
- $\mathbb{V}$ finite dimensional vector space over $\mathbb{F}_q$; $\mathbb{C}[\mathbb{V}] := \{f : \mathbb{V} \to \mathbb{C}\}$
- $$\begin{array}{cccc} \mathcal{F} : & \mathbb{C}[\mathbb{V}] & \to & \mathbb{C}[\mathbb{V}^*] \\ & \mathcal{F}(f)(w) & \mapsto & \sum_{v \in V} f(v)\Psi(\langle v, w\rangle) \end{array}$$
- Fourier inversion formula: $\mathcal{F}(\mathcal{F}(f))(x) = |V| f(-x) \Rightarrow \mathcal{F}$ is iso
- finite group $G \to \mathrm{GL}(\mathbb{V}) \rightsquigarrow \mathcal{F}$ is G equivariant $\rightsquigarrow$ $|\mathbb{V}/\mathrm{G}| = \dim(\mathbb{C}[\mathbb{V}]^G) = \dim(\mathbb{C}[\mathbb{V}^*]^G) = |\mathbb{V}^*/\mathrm{G}|$
- $\rightsquigarrow |\mathrm{Rep}^\alpha(Q, \mathbb{F}_q)/\mathrm{G}_\alpha| = |\mathrm{Rep}^\alpha(Q', \mathbb{F}_q)/G_\alpha|$ where Q' is Q with one arrow reversed $\rightsquigarrow$ independence of orientation
- G algebraic $/\mathbb{F}_q$ and $\rho : \mathrm{G} \to \mathrm{GL}(\mathbb{V}) \rightsquigarrow \varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$
- $$\begin{array}{cccc} \mu : & \mathbb{V} \times \mathbb{V}^* & \to & \mathfrak{g}^* \\ & (v, w) & \mapsto & x \mapsto \langle \varrho(x) v, w\rangle \end{array}$$ moment map
- $\#_\mu(\xi) := |\mu^{-1}(\xi)|$ count function $a_\varrho(x) := |\ker(\varrho(x))|$

Theorem (Hausel 2010)

$$\#_\mu = \mathcal{F}(a_\varrho)\frac{|\mathbb{V}|}{|\mathfrak{g}|}$$

Example of Eguchi-Hanson

- $\varrho : \mathfrak{gl}_1 \to \mathfrak{gl}(\Bbbk^2)$ by $(\alpha) \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$
  $a_\varrho : \mathbb{F}_q \to \mathbb{C}$ is $a_\varrho(\alpha) = 1$ unless $\alpha = 0$ when $a_\varrho(0) = q^2$
  $a_\varrho = 1 + (q^2 - 1)\delta_0$ and so
  $\mathcal{F}(a_\rho) = q\delta_0 + (q^2 - 1)$.
  Now $\mu : \Bbbk^2 \times \Bbbk^2 \to \mathfrak{gl}_1^*$ is given by $x_1 y_1 + x_2 y_2$.
  let $\mathcal{U} := \mu^{-1}(1)$. Indeed
  $\#\mathcal{U}(\mathbb{F}_q) = \#_\mu(1) = \frac{q^2}{q}\hat{a}_\rho(1) = q(q^2 - 1) = (q - 1)(q^2 + q)$
- $\rightsquigarrow |\mu^{-1}(1)/\mathrm{GL}_1(\mathbb{F}_q)| = q^2 + q \overset{\text{Katz}}{\rightsquigarrow} b_i(\mu^{-1}(1)/\mathrm{GL}_1(\mathbb{C})) = 1$ for $i = 0, 2$ and 0 ow
- $\mu^{-1}(1)/\mathrm{GL}_1(\mathbb{C})$ is the Eguchi-Hanson gravitational instanton; the A_1 ALE space

Betti numbers of Nakajima quiver varieties

Theorem (Hausel 2006)

For any quiver Q, and $\mathbf{w} \in \mathbb{N}^I$ the Betti numbers of Nakajima quiver varieties $\mathcal{M}(\mathbf{v},\mathbf{w})$ are:

$$\sum_{\mathbf{v}\in\mathbb{N}^I}\sum_i \dim(b_{2i}(\mathcal{M}(\mathbf{v},\mathbf{w})))q^{d(\mathbf{v},\mathbf{w})-i}X^{\mathbf{v}} =$$

$$= \frac{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\left(\prod_{(i,j)\in E} q^{\langle\lambda^i,\lambda^j\rangle}\right)\left(\prod_{i\in I} q^{\langle\lambda^i,(1^{\mathbf{w}_i})\rangle}\right)}{\prod_{i\in I}\left(q^{\langle\lambda^i,\lambda^i\rangle}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-q^{-j})\right)}}{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^I} X^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\prod_{(i,j)\in E} q^{\langle\lambda^i,\lambda^j\rangle}}{\prod_{i\in I}\left(q^{\langle\lambda^i,\lambda^i\rangle}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-q^{-j})\right)}},$$

- combinining (Nakajima 1998) with Weyl-Kac character formula and Kac-Stanley-Hua formula for $a_v(Q,\mathbb{F}_q) \Rightarrow$ Kac's Conjecture 1

Representations of quivers over R

- R Frobenius $\mathbb{k}$-algebra i.e. commutative unital finite dimensional $\mathbb{k}$-algebra with a *Frobenius* 1*-form* $\lambda : R \to \mathbb{k}$ which is not zero on any non-trivial ideal in R
- examples: $\mathbb{k}_d = \mathbb{k}[t]/(t^d)$
  more generally local algebras with unique minimal ideal
- when R is Frobenius so is $R[\epsilon] := R[\epsilon]/(\epsilon^2)$
- a *locally free* representation of quiver Q over R of rank $\alpha \in \mathbb{N}^V$ is $\in \mathrm{Rep}^\alpha(Q, R) := \bigoplus_{(i,j)\in E} \mathrm{Hom}_R(R^{\alpha_i}, R^{\alpha_j})$
- Is there an interesting theory for $a_\alpha(Q, R)$ for $|\mathbb{k}| = \mathbb{F}_q$?
- $|M_{n\times n}(R)/\mathrm{GL}_n(R)|$ have been studied for $R = \mathbb{k}_d$ computed for $n \leq 3$ and all d and for $n = 4$ and $d = 2$ (Avni, Prasad, Vaserstein 2009)
- Conjecture: $|M_{n\times n}(\mathbb{k}_d)/\mathrm{GL}_n(\mathbb{k}_d)| \in \mathbb{Z}[q] \rightsquigarrow a_\alpha(J, \mathbb{k}_d) \in \mathbb{Z}[q]$
- (Geiss, Leclerc, Schröer 2017) studied $Rep(Q, \mathbb{k}_d)$ including non-locally free ones, but found that representation type can vary when one is changing orientation of quiver
- there is a nice theory for R Frobenius algebra and locally free representations

Finite representation type quivers $/R$

Theorem (Hausel, Letellier, Villegas 2018)

Let $R/\mathbb{k}$ Frobenius, Q connected quiver. There are finitely many (abs.) indecomposable locally-free representations of Q/R $\Leftrightarrow$

1. $Q = ADE$ $R = \mathbb{k}$
2. $Q = A_1$ R *arbitrary*
3. $Q = A_2$ $R = \mathbb{k}_d$
4. $Q = A_3$ $R = \mathbb{k}_2$ *or* $\mathbb{k}_3$
5. $Q = A_4$ $R = \mathbb{k}_2$

- (3) follows from Schmitt normal form for $\mathbb{k}_d$
- (4) and (5) follow from (GLS 2017) $+$ independence of orientation for $|\mathrm{Rep}^{\alpha}_{a.i}(Q, R)/\mathrm{G}_{\alpha}(R)|$
- (GLS 2017) $\rightsquigarrow$ for $A_3/\mathbb{k}_2$ the number of *all* indecomposable representations could differ when changing orientation
- for $R = \mathbb{k}_2$ finite quivers are exactly the same as for preprojective algebra $\Pi_{\mathbb{k}}(Q)$ by (GLS 2005)

Fourier transform over finite Frobenius algebras

- $\Bbbk = \mathbb{F}_q$, $\lambda : R \to \Bbbk$ Frobenius, $1 \neq \Psi : \Bbbk \to \mathbb{C}^\times$ additive
- $\mathbb{M} \cong R^n$ finite rank free R-module, X finite set
- $$\begin{array}{rccc} \mathcal{F}: & \mathbb{C}[X \times \mathbb{M}] & \to & \mathbb{C}[X \times \mathbb{M}^\vee] \\ & \mathcal{F}(f)(w) & \mapsto & \sum_{v \in M} f(v)\Psi\lambda(\langle v, w\rangle) \end{array}$$
- Fourier inversion holds, thus $\mathcal{F}$ is iso (in fact $\Leftrightarrow$ λ Frobenius)
- finite group $G \circlearrowright X \times \mathbb{M} \rightsquigarrow \mathcal{F}$ is G equivariant $\rightsquigarrow$ $|(X \times \mathbb{M})/G| = |(X \times \mathbb{M}^\vee)/G|$
- $\rightsquigarrow |\mathrm{Rep}^\alpha(Q, R)/\mathrm{G}_\alpha(R)| = |\mathrm{Rep}^\alpha(Q', R)/\mathrm{G}_\alpha(R)|$ where Q' is Q with one arrow reversed $\rightsquigarrow$

Theorem (HLV 2018)

The number of (abs.) indecomposable locally-free representations of Q over Frobenius algebra R is independent of the orientation.

Representations of $\Pi_{\Bbbk}(Q)$ vs. $\Bbbk[\epsilon]Q$

- G algebraic $/\mathbb{F}_q$ and $\rho : \mathrm{G} \to \mathrm{GL}(\mathbb{V})$ finite dim. rep.
- let $X := \mathbb{V}(R)$ and $\mathbb{M} := \mathbb{V}(R)$ then $\mathrm{G}(R) \circlearrowright X$ and $\mathrm{G}(R) \circlearrowright \mathbb{M}$
- $\rightsquigarrow \mathcal{F} : \mathbb{C}[\mathbb{V}(R) \times \mathbb{V}(R)] \to \mathbb{C}[\mathbb{V}(R) \times \mathbb{V}(R)^\vee]$ is $\mathrm{G}(R)$-equivariant $\rightsquigarrow$ $|(\mathbb{V}(R) \times \mathbb{V}(R))/\mathrm{G}(R)| = |(\mathbb{V}(R) \times \mathbb{V}(R)^\vee)/\mathrm{G}(R)|$
- $\mathbb{V}(R) \times \mathbb{V}(R) \cong \mathbb{V}(R[\epsilon])$ and so acted on by $\mathrm{G}(R[\epsilon]) \to G(R)$
- $\mathrm{G}(R[\epsilon]) \cong \mathfrak{g}(R) \rtimes \mathrm{G}(R)$
- define $\mathrm{G}(R[\epsilon]) \circlearrowright \mathbb{C}[\mathbb{V}(R) \times \mathbb{V}(R)^\vee]$ by $((x,g)^{-1} \cdot f)(v,w) := \Psi\lambda\left(-\mu_R(v,w)(\mathrm{Ad}(g^{-1})(x))\right) f(g\cdot v, g\cdot w)$
- $\mathcal{F}$ is $\mathrm{G}(R[\epsilon])$ equivariant $\rightsquigarrow$

Theorem (HLV 2018)

$|\mathbb{V}(R[\epsilon])/\mathrm{G}(R[\epsilon])| = |\mu_R^{-1}(0)/\mathrm{G}(R)|$

Corollary

Q quiver then $|\mathrm{Rep}^\alpha(\Bbbk[\epsilon])/\mathrm{G}_\alpha(\Bbbk[\epsilon])| = |\mu^{-1}(0)/\mathrm{G}(\Bbbk)|$

Jordan quiver

- $Q = J$ the Jordan quiver, then Corollary says $|M_n(\mathbb{k}_2)/\mathrm{GL}_n(\mathbb{k}_2)| = |\{A, B \in M_n(\mathbb{k}) | AB = BA\}/G(\mathbb{k})|$
- due to (Jambor Plesken 2012) used it to compute $|\mathrm{GL}_6(\mathbb{Z}/4\mathbb{Z})/\mathrm{GL}_6(\mathbb{Z}/4\mathbb{Z})|$
- by assuming $a_n(J, \mathbb{F}_q[\epsilon])$ is a polynomial of degree $(n-1)$ one can interpolate for $4 < n < 9$ to get conjecture
-

n	$a_n(J, \mathbb{F}_q[\epsilon])$
1	1
2	$q + 1$
3	$q^2 + q + 2$
4	$q^3 + 2q^2 + 3q + 3$
5	$q^4 + 2q^3 + 5q^2 + 6q + 6$
6	$q^5 + 3q^4 + 9q^3 + 14q^2 + 14q + 9$
7	$q^6 + 3q^5 + 12q^4 + 25q^3 + 35q^2 + 29q + 18$
8	$q^7 + 4q^6 + 18q^5 + 47q^4 + 77q^3 + 85q^2 + 63q + 30$

Conjecture (HLV 2018)

$a_n(J, \mathbb{F}_q[\epsilon]) \in \mathbb{N}[q]$ and $a_n(J, \mathbb{F}_q[\epsilon])|_{q=0} = m_n$, where $m_n = \dim(FreeLie(x_1, x_2, \dots)_n)$

Toric case

- $\Bbbk := \mathbb{F}_q$, $Q = (V, E)$ quiver $\alpha = \mathbf{1}$ dimension vector
- define generating function $A(Q, q, T) := \sum_{d=1}^{\infty} a_{\mathbf{1}}(Q, \Bbbk_d) T^d$
- example for C_3 triangle of type $\hat{A}_2$ we have

$$A(C_3, q, T) = \frac{T(2qT+T+q+2)}{(1-T)^2(1-qT)}$$

Theorem (HLV 2018)

1. $a_{\mathbf{1}}(Q, \Bbbk_d) \in \mathbb{Z}[q]$ *polynomiality*
2. $A(Q, q, T) \in \mathbb{Z}(q, T)$ *rationality*
3. $A(Q, q^{-1}, T^{-1}) = (-1)^{|V|} A(Q, q, T)$ *functional equation*
4. $a_{\mathbf{1}}(Q, \Bbbk_d) \in \mathbb{N}[q]$ *positivity*

- (1) is straightforward
- (2) by combinatorial recursive formula
- (3) from graph Hopf algebra ($\rightsquigarrow A(Q, q, T)$ is like Igusa zeta)
- (4) from higher depth version of main theorem

Higher depth Fourier transform

- G algebraic $/\mathbb{k} = \mathbb{F}_q$ and $\rho : \mathrm{G} \to \mathrm{GL}(\mathbb{V})$ finite dim. rep.
- from $\mathbb{k} \hookrightarrow \mathbb{k}_d \twoheadrightarrow \mathbb{k}$ we get $\mathrm{G}(\mathbb{k}_d) \cong \mathrm{G}_d^1(\mathbb{k}) \rtimes \mathrm{G}(\mathbb{k})$
  $\mathbb{V}(\mathbb{k}_d) = \mathbb{V}(\mathbb{k}) \times \mathbb{V}_d^1(\mathbb{k})$
- $$\begin{array}{cccc} \mu_d : & \mathbb{V}(\mathbb{k}) \times \mathbb{V}_d^1(\mathbb{k})^* & \to & \mathfrak{g}_d^1(\mathbb{k})^\vee \\ & (v, w) & \mapsto & x \mapsto \langle \varrho(x) v, w \rangle \end{array}$$
  multi-moment map

Theorem (2018)

$$\#(\mathbb{V}(\mathbb{k}_d)/\mathrm{G}(\mathbb{k}_d)) = \#\left(\begin{array}{l} \{(v, w) \in \mathbb{V} \times \mathbb{V}_d^1(\mathbb{k})^\vee | \mu_d(v, w)(x) = 0 \\ \textit{for all } x \in \mathfrak{g}_d^1(\mathbb{k}) \textit{ s.t. } [x \cdot w]_1 = 0\}/\mathrm{G}(\mathbb{k}_d) \end{array} \right)$$

- in the toric case this implies the recursion:
  $a_{\mathbf{1}}(Q, \mathbb{k}_d) = \sum_{\mathcal{P} \in Part_Q} q^{(d-1) b_1(Q_{\mathcal{P}})} a_{\mathbf{1}}(Q_{\mathcal{P}}, \mathbb{k}) a_{\mathbf{1}}(Q_{/\mathcal{P}}, \mathbb{k}_{d-1})$
- example: $\mathrm{GL}_n \to \mathrm{GL}(M_{n\times n}(\mathbb{k}))$ Jordan quiver; $d = 3$
- $(v, w) = (A_0, t^{-1} B_1 + t^{-2} B_2) \in M_{n\times n}(\mathbb{k}) \times (M_{n\times n})_3^1(\mathbb{k})^\vee$ for all $x = (tX_1 + t^2 X_2) \in \mathfrak{g}_3^1(\mathbb{k})$ we have $[x \cdot w]_1 = t^{-1}[X_1, B_2]$
- thus RHS becomes $[A_0, B_2] = 0$ and if $[X_1, B_2] = 0$ then $tr(X_1[A_0, B_1]) = 0$