Maximally supersymmetric backgrounds and partial $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ rigid supersymmetry breaking

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25 March, 2019

Outline

Supersymmetric backgrounds: Introductory comments

Supersymmetric backgrounds in supergravity

Supersymmetric solutions of supergravity

M. Duff & C. Pope (1982)
F. Englert, M. Rooman & P. Spindel (1983)
P. van Nieuwenhuizen & N. Warner (1984)
. . . .
M. Duff, B. Nilsson & C. Pope, *Kaluza-Klein Supergravity* (PR, 1986)

Concept of Killing spinors

One of the highlights:
All supersymmetric solutions of minimal (gauged) supergravity in 5D

J. Gauntlett, J. Gutowski, C. Hull, S. Pakis & H. Reall (2003)
J. Gauntlett & J. Gutowski (2003)

Superspace formalism to determine (super)symmetric backgrounds in off-shell supergravity

I. Buchbinder & SMK, *Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace*, IOP, 1995 + 1998

Superspace formalism to identify (super)symmetric backgrounds is universal since

- it is geometric;
- it may be extended to any off-shell supergravity theory formulated in superspace.

This formalism has been applied to construct off-shell rigid supersymmetric field theories in curved backgrounds:

- 5D supersymmetric field theories with $\mathcal{N} = 1$ AdS supersymmetry
  (Eight months before Pestun's work on $\mathcal{N} = 2$ SYM theories on S^4)
  SMK & G. Tartaglino-Mazzucchelli (2007)
- 4D supersymmetric field theories with $\mathcal{N} = 2$ AdS supersymmetry
  SMK & G. Tartaglino-Mazzucchelli (2008)
  (Most general nonlinear σ-models) D. Butter & SMK (2011)
  D. Butter, SMK, U. Lindström & G. Tartaglino-Mazzucchelli (2012)
- 3D supersymmetric field theories with (p, q) AdS supersymmetry
  SMK, & G. Tartaglino-Mazzucchelli (2012)
  SMK, U. Lindström & G. Tartaglino-Mazzucchelli (2012)
  D. Butter, SMK & G. Tartaglino-Mazzucchelli (2012)

Important developments in the last decade

Exact results (partition functions, Wilson loops etc.)
in rigid supersymmetric field theories on curved backgrounds
(e.g., S^3, S^4, $S^3 \times S^1$ etc.) using localisation techniques

V. Pestun (2007, 2009)
A. Kapustin, B. Willett & I. Yaakov (2010)
D. Jafferis (2010)
....

Necessary technical ingredients:

- Curved space $\mathcal{M}$ has to admit some unbroken rigid supersymmetry (supersymmetric background);
- Rigid supersymmetric field theory on $\mathcal{M}$ should be off-shell.

These developments have inspired much interest in the construction and classification of supersymmetric backgrounds that correspond to off-shell supergravity formulations.

Classification of supersymmetric backgrounds in off-shell supergravity

Component approaches

G. Festuccia and N. Seiberg (2011)
H. Samtleben and D. Tsimpis (2012)
C. Klare, A. Tomasiello and A. Zaffaroni (2012)
T. Dumitrescu, G. Festuccia and N. Seiberg (2012)
D. Cassani, C. Klare, D. Martelli, A. Tomasiello and A. Zaffaroni (2012)
T. Dumitrescu and G. Festuccia (2012)
A. Kehagias and J. Russo (2012)
....

These results naturally follow from the superspace formalism developed in the mid 1990s, as demonstrated in several publications:

4D $\mathcal{N} = 1$ — SMK (2012)
4D $\mathcal{N} = 2$ — D. Butter, G. Inverso & I. Lodato (2015)
3D $\mathcal{N} = 2$ — SMK, U. Lindström, M. Roček, I. Sachs & G. Tartaglino-Mazzucchelli (2013)
5D $\mathcal{N} = 1$ — SMK, J. Novak & G. Tartaglino-Mazzucchelli (2014)

Component approaches vs superspace formalism

- Both the component approaches and superspace formalisms can be used to derive supersymmetric backgrounds in off-shell supergravity. Practically all classification results have been obtained within the component settings.
- Superspace formalism is more useful in order to determine all (conformal) isometries of a given backgrounds.
- Superspace formalism is by far more powerful for constructing the most general rigid supersymmetric field theories on a given background.
- Superspace formalism is useful to describe partial breaking of rigid SUSY in curved maximally supersymmetric backgrounds.

Partial SUSY breaking: general comments

Is partial $\mathcal{N}=2$ SUSY breaking posible?

Impossible — E. Witten (1981)

Let's nevertheless try — J. Bagger & J. Wess (1984)

Possible — J. Hughes, J. Liu & J. Polchinski (1986)

- In principle, the formalism of nonlinear realisations may be used to derive models for partial $\mathcal{N}=2 \to \mathcal{N}=1$ SUSY breaking. In practice, not a single model was constructed in closed form using such an approach. J. Bagger & A. Galperin (1994)
- Superfield techniques were employed to construct four Goldstone multiplet actions for partial $\mathcal{N}=2 \to \mathcal{N}=1$ breaking of Poincaré SUSY, with manifest $\mathcal{N}=1$ SUSY. They make use of different $\mathcal{N}=1$ supermultiplets to which the Goldstino belongs, specifically:

  Vector multiplet — J. Bagger & A. Galperin (1997)

  Tensor multiplet — J. Bagger & A. Galperin (1997)

  Chiral scalar multiplet — J. Bagger & A. Galperin (1997)

  Complex linear (or non-minimal) scalar multiplet — F. Gonzalez-Ray, I. Park & M. Roček (1999)

Partial breaking of rigid SUSY in Minkowski space

- It was demonstrated that the vector and tensor Goldstone models for partially broken $\mathcal{N} = 2$ Poincaré SUSY can naturally be derived from constrained $\mathcal{N} = 2$ superfields.

  M. Roček & A. Tseytlin (1999)

- The non-minimal scalar Goldstone multiplet for partially broken $\mathcal{N} = 2$ Poincaré SUSY was also derived from constrained $\mathcal{N} = 2$ superfields (the so-called $\mathcal{O}(4)$ multiplet).

  F. Gonzalez-Ray, I. Park & M. Roček (1999)

- There is no direct way to read off the chiral Goldstone multiplet from constrained $\mathcal{N} = 2$ superfields (without intrinsic central charge).

  I. Antoniadis, H. Partouche & T. Taylor (1996)
  E. Ivanov & B. Zupnik (1998)

The above constructions correspond to the case of Poincaré SUSY.
Minkowski space is one of many supersymmetric spacetimes compatible with rigid $\mathcal{N} = 2$ SUSY (8 supercharges).
Do such spacetimes allow for models for partial SUSY breaking?

(Conformal) symmetries of curved spacetime

(Conformal) symmetries of curved spacetime

(Conformal) symmetries of a curved superspace may be defined similarly to those corresponding to a curved spacetime within the Weyl-invariant formulation for gravity (variation on a theme by Hermann Weyl).

S. Deser (1970)
B. Zumino (1970)
P. Dirac (1973)

Three formulations for gravity in d dimensions:

- Metric formulation;
- Vielbein formulation;
- Weyl-invariant formulation.

I briefly recall the metric and vielbein approaches and then concentrate in more detail of the Weyl-invariant formulation.

Metric and vielbein formulations for gravity

Metric formulation

Gauge field: metric $g_{mn}(x)$

Gauge transformation: $\delta g_{mn} = \nabla_m \xi_n + \nabla_n \xi_m$

$\xi = \xi^m(x)\partial_m$ a vector field generating an infinitesimal diffeomorphism.

Vielbein formulation R. Utiyama (1956)

Gauge field: vielbein $e_m{}^a(x)$, $e := \det(e_m{}^a) \neq 0$

The metric is a composite field $g_{mn} = e_m{}^a e_n{}^b \eta_{ab}$

Gauge transformation: $\delta \nabla_a = [\xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}, \nabla_a]$

Gauge parameters: $\xi^a(x) = \xi^m e_m{}^a(x)$ and $K^{ab}(x) = -K^{ba}(x)$

Covariant derivatives (M_{bc} the Lorentz generators)

$$\nabla_a = e_a{}^m \partial_m + \frac{1}{2}\omega_a{}^{bc} M_{bc} \ , \qquad [\nabla_a, \nabla_b] = \frac{1}{2} R_{ab}{}^{cd} M_{cd}$$

$e_a{}^m$ the inverse vielbein, $e_a{}^m e_m{}^b = \delta_a{}^b$;
$\omega_a{}^{bc}$ the torsion-free Lorentz connection.

Weyl transformations

Weyl transformations

The torsion-free constraint

$$T_{ab}{}^{c} = 0 \quad \Longleftrightarrow \quad [\nabla_a, \nabla_b] \equiv T_{ab}{}^{c}\nabla_c + \frac{1}{2}R_{ab}{}^{cd}M_{cd} = \frac{1}{2}R_{ab}{}^{cd}M_{cd}$$

is invariant under a Weyl (local scale) transformation

$$\nabla_a \to \nabla'_a = \mathrm{e}^{\sigma}\Big(\nabla_a + (\nabla^b\sigma)M_{ba}\Big) ,$$

with the parameter $\sigma(x)$ being completely arbitrary. This transformation acts of the (inverse) vielbein and metric as

$$e_a{}^m \to \mathrm{e}^{\sigma} e_a{}^m \iff e_m{}^a \to \mathrm{e}^{-\sigma} e_m{}^a \iff g_{mn} \to \mathrm{e}^{-2\sigma} g_{mn}$$

Weyl transformations are gauge symmetries of conformal gravity. Einstein gravity possesses no Weyl invariance.

Weyl-invariant formulation for Einstein gravity

Weyl-invariant formulation for Einstein gravity

Gauge fields: vielbein $e_m{}^a(x)$, $e := \det(e_m{}^a) \neq 0$

& conformal compensator $\varphi(x)$, $\varphi \neq 0$

Gauge transformations ($\mathcal{K} := \xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}$)

$$\delta \nabla_a = [\xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}, \nabla_a] + \sigma \nabla_a + (\nabla^b \sigma) M_{ba} \equiv (\delta_{\mathcal{K}} + \delta_\sigma) \nabla_a \ ,$$

$$\delta \varphi = \xi^b \nabla_b \varphi + \frac{1}{2}(d-2)\sigma\varphi \equiv (\delta_{\mathcal{K}} + \delta_\sigma)\varphi$$

Gauge-invariant gravity action

$$S = \frac{1}{2} \int \mathrm{d}^d x \, e \left(\nabla^a \varphi \nabla_a \varphi + \frac{1}{4} \frac{d-2}{d-1} R \varphi^2 + \lambda \varphi^{2d/(d-2)} \right)$$

Imposing a Weyl gauge condition $\varphi = \frac{2}{\kappa}\sqrt{\frac{d-1}{d-2}} = \mathrm{const}$

reduces the action to

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^d x \, e \, R - \frac{\Lambda}{\kappa^2} \int \mathrm{d}^d x \, e$$

Conformal isometries

Conformal Killing vector fields

A vector field $\xi = \xi^m \partial_m = \xi^a e_a$, with $e_a := e_a{}^m \partial_m$, is conformal Killing if there exist local Lorentz, $K^{bc}[\xi]$, and Weyl, $\sigma[\xi]$, parameters such that

$$(\delta_{\mathcal{K}} + \delta_\sigma)\nabla_a = \Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] + \sigma[\xi]\nabla_a + (\nabla^b \sigma[\xi]) M_{ba} = 0$$

A short calculation gives

$$K^{bc}[\xi] = \frac{1}{2}(\nabla^b \xi^c - \nabla^c \xi^b) , \qquad \sigma[\xi] = \frac{1}{d} \nabla_b \xi^b$$

Conformal Killing equation

$$\nabla^a \xi^b + \nabla^b \xi^a = 2\eta^{ab} \sigma[\xi]$$

Equivalent spinor form in $d = 4$: $\qquad \nabla_{(\alpha}{}^{(\dot\alpha} \xi_{\beta)}{}^{\dot\beta)} = 0$

($\nabla_a \to \nabla_{\alpha\dot\alpha}$ and $\xi_a \to \xi_{\alpha\dot\alpha}$)

Equivalent spinor form in $d = 3$: $\qquad \nabla_{(\alpha\beta} \xi_{\gamma\delta)} = 0$

Conformal isometries

- Lie algebra of conformal Killing vector fields.
- Conformally related spacetimes (∇_a, φ) and $(\widetilde{\nabla}_a, \widetilde{\varphi})$

$$\widetilde{\nabla}_a = \mathrm{e}^{\rho}\Big(\nabla_a + (\nabla^b\rho)M_{ba}\Big) \ , \qquad \widetilde{\varphi} = \mathrm{e}^{\frac{1}{2}(d-2)\rho}\varphi$$

  have the same conformal Killing vector fields $\xi = \xi^a e_a = \tilde{\xi}^a \tilde{e}_a$.

  The parameters $K^{cd}[\tilde{\xi}]$ and $\sigma[\tilde{\xi}]$ are related to $K^{cd}[\xi]$ and $\sigma[\xi]$ as follows:

$$\mathcal{K}[\tilde{\xi}] := \tilde{\xi}^b\widetilde{\nabla}_b + \frac{1}{2}K^{cd}[\tilde{\xi}]M_{cd} = \mathcal{K}[\xi] \ ,$$
$$\sigma[\tilde{\xi}] = \sigma[\xi] - \xi\rho$$

- Rigid conformal field theories on curved backgrounds.

Isometries

Killing vector fields

Let $\xi = \xi^a e_a$ be a conformal Killing vector,

$$(\delta_{\mathcal{K}} + \delta_\sigma)\nabla_a = \left[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\right] + \sigma[\xi]\nabla_a + (\nabla^b \sigma[\xi]) M_{ba} = 0 \ .$$

It is called Killing if it leaves the compensator invariant,

$$(\delta_{\mathcal{K}} + \delta_\sigma)\varphi = \xi\varphi + \frac{1}{2}(d-2)\sigma[\xi]\varphi = 0 \ .$$

These Killing equations are Weyl invariant in the following sense:
Given a conformally related spacetime $(\widetilde{\nabla}_a, \widetilde{\varphi})$

$$\widetilde{\nabla}_a = \mathrm{e}^\rho \Big(\nabla_a + (\nabla^b \rho) M_{ba}\Big) \ , \qquad \widetilde{\varphi} = \mathrm{e}^{\frac{1}{2}(d-2)\rho}\varphi \ ,$$

the above Killing equations have the same functional form when rewritten in terms of $(\widetilde{\nabla}_a, \widetilde{\varphi})$, in particular

$$\xi\widetilde{\varphi} + \frac{1}{2}(d-2)\sigma[\tilde{\xi}]\widetilde{\varphi} = 0 \ .$$

Isometries

Because of Weyl invariance, we can work with a conformally related spacetime such that

$$\varphi = 1$$

Then the Killing equations turn into

$$\Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] = 0 \ , \qquad \sigma[\xi] = 0$$

Standard Killing equation

$$\nabla^a \xi^b + \nabla^b \xi^a = 0$$

- Lie algebra of Killing vector fields
- Rigid field theories in curved space

(Conformal) symmetries of curved superspace

(Conformal) symmetries of curved superspace

Weyl-invariant approach to spacetime symmetries has a natural superspace extension in all cases when supergravity is formulated as conformal supergravity coupled to certain conformal compensator(s) Ξ

$z^M = (x^m, \theta^\mu)$	local coordinates of curved superspace
$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha) = E_A + \Omega_A + \Phi_A$	superspace covariant derivatives
$E_A = E_A{}^M(z)\partial_M$	superspace inverse vielbein
$\Omega_A = \frac{1}{2}\Omega_A{}^{bc}(z)M_{bc}$	superspace Lorentz connection
$\Phi_A = \Phi_A{}^I(z)T_I$	superspace R-symmetry connection

Supergravity gauge transformation

$$\delta_\mathcal{K}\mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A] \ , \qquad \delta_\mathcal{K}\Xi = \mathcal{K}\Xi \ , \qquad \mathcal{K} := \xi^B\mathcal{D}_B + \frac{1}{2}K^{bc}M_{bc} + K^I T_I$$

Super-Weyl transformation

$$\delta_\sigma\mathcal{D}_a = \sigma\mathcal{D}_a + \cdots \ , \qquad \delta_\sigma\mathcal{D}_\alpha = \frac{1}{2}\sigma\mathcal{D}_\alpha + \cdots \ , \qquad \delta_\sigma\Xi = w_\Xi\sigma\Xi \ ,$$

with w_Ξ a non-zero super-Weyl weight

Conformal isometries of curved superspace

Let $\xi = \xi^B E_B$ be a real supervector field. It is called conformal Killing if

$$(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_A = 0 \ ,$$

for some Lorentz $K^{bc}[\xi]$, R-symmetry $K^I[\xi]$ and super-Weyl $\sigma[\xi]$ parameters.

- All parameters $K^{bc}[\xi]$, $K^I[\xi]$ and $\sigma[\xi]$ are uniquely determined in terms of ξ^B.
- The spinor component ξ^β is uniquely determined in terms of ξ^b.
- The vector component obeys an equation that contains all the information about the conformal Killing supervector field.
  Examples:

$$d = 3 \qquad \mathcal{D}^I_{(\alpha}\xi_{\beta\gamma)} = 0$$

$$d = 4 \qquad \mathcal{D}^I_{(\alpha}\xi_{\beta)\dot\beta} = 0$$

Isometries of curved superspace

Let $\xi = \xi^B E_B$ be a conformal Killing supervector field,

$$(\delta_{\mathcal{K}[\xi]} + \delta_{\sigma[\xi]})\mathcal{D}_A = 0 \ , \qquad (\star)$$

for uniquely determined parameters $K^{bc}[\xi]$, $K^I[\xi]$ and $\sigma[\xi]$.
It is called Killling if the compensators are invariant,

$$(\delta_{\mathcal{K}[\xi]} + w_\Xi \sigma[\xi])\Xi = 0 \ . \qquad (\star\star)$$

The Killing equations $(\star)$ and $(\star\star)$ are super-Weyl invariant in the sense that they hold for all conformally related superspace geometries.

Using the compensators Ξ we can always construct a scalar superfield $\phi = \phi(\Xi)$, which (i) is an algebraic function of Ξ; (ii) nowhere vanishing; and (iii) has a nonzero super-Weyl weight w_ϕ, $\delta_\sigma \phi = w_\phi \sigma \phi$.

$$(\delta_{\mathcal{K}[\xi]} + w_\Xi \sigma[\xi])\phi = 0 \ .$$

Super-Weyl invariance may be used to impose the gauge $\phi = 1$, and then

$$\sigma[\xi] = 0 \ .$$

(Conformal) supersymmetries of curved superspace

Of special interest are curved backgrounds which admit at least one (conformal) supersymmetry. Such a superspace must possess a conformal Killing supervector field ξ^A of the type

$$\xi^a| = 0 \ , \qquad \xi^\alpha| \neq 0$$

and describe a bosonic background with the property that all spinor components of the superspace torsion and curvature tensors

$$[\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}{}^C \mathcal{D}_C + \frac{1}{2} R_{AB}{}^{cd} M_{cd} + R_{AB}{}^I T_I$$

have vanishing bar-projections,

$$\varepsilon(T_{\dots}{}^{\dots}) = 1 \rightarrow T_{\dots}{}^{\dots}| = 0 \ , \qquad \varepsilon(R_{\dots}{}^{\dots}) = 1 \rightarrow R_{\dots}{}^{\dots}| = 0 \ .$$

These conditions are supersymmetric.
At the component level, all spinor fields may be gauged away.

Case study: Supersymmetric backgrounds in 3D $\mathcal{N} = 2$ supergravity

$\mathcal{N} = 2$ supergravity in three dimensions

3D $\mathcal{N} = 2$ curved superspace, $\mathcal{M}^{3|4}$, parametrised by local coordinates
$z^M = (x^m, \theta^\mu, \bar{\theta}_\mu)$, $m = 0, 1, 2$ and $\mu = 1, 2$
Superspace structure group $\mathrm{SO}(2,1) \times \mathrm{U}(1)_R$
Superspace covariant derivatives

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^\alpha) = E_A + \Omega_A + \mathrm{i}\,\Phi_A J\ .$$

Algebra of covariant derivatives

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4\bar{R}M_{\alpha\beta}\ , \qquad \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 4RM_{\alpha\beta}\ ,$$
$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = -2\mathrm{i}(\gamma^c)_{\alpha\beta}\mathcal{D}_c - 2\mathcal{C}_{\alpha\beta}J - 4\mathrm{i}\varepsilon_{\alpha\beta}\mathcal{S}J + 4\mathrm{i}\mathcal{S}M_{\alpha\beta} - 2\varepsilon_{\alpha\beta}\mathcal{C}^{\gamma\delta}M_{\gamma\delta}\ .$$

$M_{ab} = -M_{ba} \longleftrightarrow M_{\alpha\beta} = M_{\beta\alpha}$ Lorentz generators
Dimension-1 torsion superfields: (i) real scalar $\mathcal{S}$; (ii) complex scalar R such that $JR = -2R$; (iii) real vector $\mathcal{C}_a \longleftrightarrow \mathcal{C}_{\alpha\beta}$.
Bianchi Identities:

$$\bar{\mathcal{D}}_\alpha R = 0\ , \qquad (\bar{\mathcal{D}}^2 - 4R)\mathcal{S} = 0 \quad \ldots$$

Conformal isometries

The conformal Killing supervector fields obey the equation

$$(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_A = 0 \ ,$$

where

$$\delta_{\mathcal{K}}\mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A] \ , \qquad \mathcal{K} = \xi^C \mathcal{D}_C + \frac{1}{2}K^{cd}M_{cd} + \mathrm{i}\,\tau J$$

and

$$\delta_\sigma \mathcal{D}_\alpha = \frac{1}{2}\sigma\mathcal{D}_\alpha + (\mathcal{D}^\gamma\sigma)M_{\gamma\alpha} - (\mathcal{D}_\alpha\sigma)J \ , \quad \dots$$

It suffices to require $(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_\alpha = 0$, which implies

$$\xi^\alpha = -\frac{\mathrm{i}}{6}\bar{\mathcal{D}}_\beta\xi^{\beta\alpha} \ , \qquad K_{\alpha\beta} = \mathcal{D}_{(\alpha}\xi_{\beta)} - \bar{\mathcal{D}}_{(\alpha}\bar{\xi}_{\beta)} - 2\xi_{\alpha\beta}\mathcal{S}$$

$$\sigma = \frac{1}{2}\big(\mathcal{D}_\alpha\xi^\alpha + \bar{\mathcal{D}}^\alpha\bar{\xi}_\alpha\big) \ , \qquad \tau = -\frac{\mathrm{i}}{4}\big(\mathcal{D}_\alpha\xi^\alpha - \bar{\mathcal{D}}^\alpha\bar{\xi}_\alpha\big)$$

All parameters ξ^α, $K_{\alpha\beta}$, σ and τ are expressed in terms of ξ^a.

Conformal isometries

The remaining vector parameter ξ^a satisfies

$$\mathcal{D}_{(\alpha}\xi_{\beta\gamma)} = 0 \qquad\qquad (\star)$$

and its conjugate.
Implication: superfield analogue of the conformal Killing equation

$$\mathcal{D}_a\xi_b + \mathcal{D}_b\xi_a = \frac{2}{3}\eta_{ab}\mathcal{D}^c\xi_c \ .$$

Eq. $(\star)$ is fundamental in the sense that it implies $(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_A \equiv 0$ provided the parameters ξ^α, $K_{\alpha\beta}$, σ and τ are defined as above.
The conformal Killing supervector field is a real supervector field

$$\xi = \xi^A E_A \ , \qquad \xi^A \equiv (\xi^a, \xi^\alpha, \bar{\xi}_\alpha) = \Big(\xi^a, -\frac{\mathrm{i}}{6}\bar{\mathcal{D}}_\beta\xi^{\beta\alpha}, -\frac{\mathrm{i}}{6}\mathcal{D}^\beta\xi_{\beta\alpha}\Big)$$

which obeys the master equation $(\star)$.
If ξ_1 and ξ_2 are two conformal Killing supervector fields, their Lie bracket $[\xi_1, \xi_2]$ is a conformal Killing supervector field.

Conformal isometries

Equation $(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_\alpha = 0$ implies some additional results that have not been discussed above. Define

$$\Upsilon := (\xi^B, K^{\beta\gamma}, \tau)$$

It turns out that

- $\mathcal{D}_A \Upsilon$ is a linear combination of Υ, σ and $\mathcal{D}_C\sigma$;
- $\mathcal{D}_A\mathcal{D}_B\sigma$ can be represented as a linear combination of Υ, σ and $\mathcal{D}_C\sigma$.

The super Lie algebra of the conformal Killing vector fields on $\mathcal{M}^{3|4}$ is finite dimensional. The number of its even and odd generators cannot exceed those in the $\mathcal{N} = 2$ superconformal algebra $\mathfrak{osp}(2|4)$.

Charged conformal Killing spinors

Look for curved superspace backgrounds admitting at least one conformal supersymmetry. Such a superspace must possess a conformal Killing supervector field ξ^A with the property

$$\xi^a| = 0 \ , \qquad \epsilon^\alpha := \xi^\alpha| \neq 0 \ .$$

All other bosonic parameters are assumed to vanish, $\sigma| = \tau| = K_{\alpha\beta}| = 0$. Bosonic superspace backgrounds without covariant fermionic fields:

$$\mathcal{D}_\alpha \mathcal{S}| = 0 \ , \qquad \mathcal{D}_\alpha R| = 0 \ , \qquad \mathcal{D}_\alpha \mathcal{C}_{\beta\gamma}| = 0 \ .$$

These conditions mean that the gravitini can completely be gauged away

$$\mathcal{D}_a| = \mathbf{D}_a := e_a + \frac{1}{2}\omega_a{}^{bc} M_{bc} + \mathrm{i} b_a J \equiv \mathfrak{D}_a + \mathrm{i} b_a J \ , \qquad e_a := e_a{}^m \partial_m \ .$$

Introduce scalar and vector fields associated with the superfield torsion:

$$s := \mathcal{S}| \ , \qquad r := R| \ , \qquad c_a := \mathcal{C}_a| \ .$$

S-supersymmetry parameter: $\eta_\alpha := \mathcal{D}_\alpha \sigma|$.

Charged conformal Killing spinors

Q-supersymmetry parameter $\epsilon^\alpha := \xi^\alpha|$ obeys the equation

$$\mathbf{D}_a\epsilon^\alpha + \frac{\mathrm{i}}{2}(\tilde{\gamma}_a\bar{\eta})^\alpha + \mathrm{i}\varepsilon_{abc}\, c^b(\tilde{\gamma}^c\epsilon)^\alpha - s(\tilde{\gamma}_a\epsilon)^\alpha - \mathrm{i}r(\tilde{\gamma}_a\bar{\epsilon})^\alpha = 0 \ ,$$

which is equivalent to

$$\Big(\mathbf{D}_{(\alpha\beta} - \mathrm{i}c_{(\alpha\beta}\Big)\epsilon_{\gamma)} = \Big(\mathfrak{D}_{(\alpha\beta} - \mathrm{i}(b+c)_{(\alpha\beta}\Big)\epsilon_{\gamma)} = 0 \ ,$$

$$\bar{\eta}_\alpha = -\frac{2\mathrm{i}}{3}\Big((\gamma^a\mathbf{D}_a\epsilon)_\alpha + 2\mathrm{i}(\gamma^a\epsilon)_\alpha c_a + 3s\epsilon_\alpha + 3\mathrm{i}r\bar{\epsilon}_\alpha\Big) \ .$$

This follows by bar-projecting the equation ($\boldsymbol{C}_{\alpha\beta\gamma} = -\mathrm{i}\mathcal{D}_{(\alpha}\mathcal{C}_{\beta\gamma)}$)

$$0 = \mathcal{D}_a\xi_\alpha + \frac{\mathrm{i}}{2}(\gamma_a)_\alpha{}^\beta\bar{\mathcal{D}}_\beta\sigma - \mathrm{i}\varepsilon_{abc}(\gamma^b)_\alpha{}^\beta\mathcal{C}^c\xi_\beta - (\gamma_a)_\alpha{}^\beta(\xi_\beta\mathcal{S} + \bar{\xi}_\beta R)$$
$$-\frac{1}{2}\varepsilon_{abc}\xi^b(\gamma^c)^{\beta\gamma}\Big(\mathrm{i}\bar{\boldsymbol{C}}_{\alpha\beta\gamma} - \frac{4\mathrm{i}}{3}\varepsilon_{\alpha(\beta}\bar{\mathcal{D}}_{\gamma)}\mathcal{S} - \frac{2}{3}\varepsilon_{\alpha(\beta}\mathcal{D}_{\gamma)}R\Big) \ ,$$

which is one of the implications of $(\delta_\mathcal{K} + \delta_\sigma)\mathcal{D}_a = 0$.

Supersymmetric backgrounds

Rigid supersymmetry transformations (in super-Weyl gauge $\phi = 1$) are characterised by

$$\sigma[\xi] = 0 \quad \Longrightarrow \quad \eta_\alpha = 0 \ ,$$

The conformal Killing spinor equation turns into

$$\mathbf{D}_a \epsilon^\alpha = -\mathrm{i}\varepsilon_{abc} c^b (\tilde{\gamma}^c \epsilon)^\alpha + s(\tilde{\gamma}_a \epsilon)^\alpha + \mathrm{i} r(\tilde{\gamma}_a \bar{\epsilon})^\alpha \ .$$

$$\mathbf{D}_a = e_a + \frac{1}{2}\omega_a{}^{bc} M_{bc} + \mathrm{i} b_a J = \mathfrak{D}_a + \mathrm{i} b_a J \ .$$

$$[\mathbf{D}_a, \mathbf{D}_b] = \frac{1}{2}\mathcal{R}_{ab}{}^{cd} M_{cd} + \mathrm{i}\mathcal{F}_{ab} J = [\mathfrak{D}_a, \mathfrak{D}_b] + \mathrm{i}\mathcal{F}_{ab} J \ .$$

4D $\mathcal{N} = 1$ maximally symmetric backgrounds

4D $\mathcal{N} = 1$ maximally symmetric backgrounds

- There are only 5 maximally supersymmetric backgrounds in off-shell $\mathcal{N} = 1$ supergravity theories in four dimensions.

G. Festuccia & N. Seiberg (2011)

This classification was stated in the component setting.

Superspace formalism to determine (super)symmetric backgrounds in off-shell supergravity:
I. Buchbinder & SMK, *Walk Through Superspace* (1995, RE: 1998)

This formalism was used to provide a rigorous derivation of the Festuccia-Seiberg statements.

SMK (2012)

4D $\mathcal{N} = 1$ maximally symmetric superspaces

- Minkowski superspace $\mathbb{M}^{4|4}$ — V. Akulov & D. Volkov (1973), A. Salam & J. Strathdee (1974)
- Anti-de Sitter superspace $\mathrm{AdS}^{4|4}$ — B. Keck (1975), B. Zumino (1977), E. Ivanov & A. Sorin (1980)

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\} = -2\mathrm{i}\mathcal{D}_{\alpha\dot\beta} \ ,$$
$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4\bar{R}M_{\alpha\beta} \ , \quad \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 4R\bar{M}_{\dot\alpha\dot\beta} \ ,$$
$$[\mathcal{D}_a, \mathcal{D}_\beta] = -\tfrac{\mathrm{i}}{2}\bar{R}(\sigma_a)_{\beta\dot\gamma}\bar{\mathcal{D}}^{\dot\gamma} \ , \qquad [\mathcal{D}_a, \bar{\mathcal{D}}_{\dot\beta}] = \tfrac{\mathrm{i}}{2}R(\sigma_a)_{\gamma\dot\beta}\mathcal{D}^\gamma \ ,$$
$$[\mathcal{D}_a, \mathcal{D}_b] = -|R|^2 M_{ab} \ ,$$

with $R = \mathrm{const}$.

- Superspaces $\mathbb{M}^{4|4}_T$, $\mathbb{M}^{4|4}_S$ and $\mathbb{M}^{4|4}_N$ ($G^2 < 0$, $G^2 > 0$ and $G^2 = 0$)

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 0 \ , \quad \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 0 \ , \quad \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\} = -2\mathrm{i}\mathcal{D}_{\alpha\dot\beta} \ ,$$
$$[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot\beta}] = \mathrm{i}\varepsilon_{\alpha\beta}G^\gamma{}_{\dot\beta}\mathcal{D}_\gamma \ , \quad [\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] = -\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}G_\beta{}^{\dot\gamma}\bar{\mathcal{D}}_{\dot\gamma} \ ,$$
$$[\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] = -\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}G_\beta{}^{\dot\gamma}\mathcal{D}_{\alpha\dot\gamma} + \mathrm{i}\varepsilon_{\alpha\beta}G^\gamma{}_{\dot\beta}\mathcal{D}_{\gamma\dot\alpha} \ ,$$

where G_b is covariantly constant, $\mathcal{D}_A G_b = 0$

- Minkowski superspace $\mathbb{M}^{4|4}$ is a unique maximally supersymmetric solution of supergravity without a cosmological term.

$$S = -\frac{3}{\kappa^2} \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E$$

- AdS superspace $\mathrm{AdS}^{4|4}$ is a unique maximally supersymmetric solution of AdS supergravity.

$$S_{\mathrm{SUGRA}} = -\frac{3}{\kappa^2} \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E + \left\{ \frac{\mu}{\kappa^2} \int \mathrm{d}^4x \mathrm{d}^2\theta\, \mathcal{E} + \mathrm{c.c.} \right\}$$

- Superspaces $\mathbb{M}_T^{4|4}$, $\mathbb{M}_S^{4|4}$ and $\mathbb{M}_N^{4|4}$ are maximally supersymmetric solutions of scale-invariant R^2 supergravity.

SMK, arXiv:1606.00654

$$S = \alpha \int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, R\bar{R} + \beta \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathcal{E}\, R^3 + \mathrm{c.c.}$$

Scale-invariant R^2 supergravity theories were studied by several groups:

C. Kounnas, D. Lüst & N. Toumbas (2015)

A. Kehagias, C. Kounnas, D. Lüst & A. Riotto (2015)

L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Lüst & A. Riotto (2015)

S. Ferrara, A. Kehagias & M. Porrati (2015)

Conformal supergravity in U(1) superspace

U(1) superspace

P. Howe (1982)

Structure group: $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{U}(1)_R$

Algebra of supergravity covariant derivatives

$$
\begin{aligned}
\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} &= -2\mathrm{i}\mathcal{D}_{\alpha\dot\alpha}\ , \\
\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} &= -4\bar{R}M_{\alpha\beta}\ , \qquad \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 4R\bar{M}_{\dot\alpha\dot\beta}\ , \\
\Big[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot\beta}\Big] &= \mathrm{i}\varepsilon_{\alpha\beta}\Big(\bar{R}\,\bar{\mathcal{D}}_{\dot\beta} + G^\gamma{}_{\dot\beta}\mathcal{D}_\gamma - (\mathcal{D}^\gamma G^\delta{}_{\dot\beta})M_{\gamma\delta} + 2\bar{W}_{\dot\beta}{}^{\dot\gamma\dot\delta}\bar{M}_{\dot\gamma\dot\delta}\Big) \\
&\quad + \mathrm{i}(\bar{\mathcal{D}}_{\dot\beta}\bar{R})M_{\alpha\beta} - \frac{\mathrm{i}}{3}\varepsilon_{\alpha\beta}\bar{X}^{\dot\gamma}\bar{M}_{\dot\gamma\dot\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\bar{X}_{\dot\beta}J\ .
\end{aligned}
$$

The superfields R, $G_{\alpha\dot\alpha}$, $W_{\alpha\beta\gamma}$, and X_α obey the Bianchi identities:

$$
\begin{aligned}
&\bar{\mathcal{D}}_{\dot\alpha}R = 0\ , \qquad \bar{\mathcal{D}}_{\dot\alpha}X_\alpha = 0\ , \qquad \bar{\mathcal{D}}_{\dot\alpha}W_{\alpha\beta\gamma} = 0\ ; \\
&X_\alpha = \mathcal{D}_\alpha R - \bar{\mathcal{D}}^{\dot\alpha}G_{\alpha\dot\alpha}\ , \qquad \mathcal{D}^\alpha X_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar{X}^{\dot\alpha}\ .
\end{aligned}
$$

The $\mathrm{U}(1)_R$ generator J is normalised by

$$
[J, \mathcal{D}_\alpha] = \mathcal{D}_\alpha\ , \qquad [J, \bar{\mathcal{D}}_{\dot\alpha}] = -\bar{\mathcal{D}}_{\dot\alpha}\ .
$$

Grimm-Wess-Zumino superspace geometry

R. Grimm, J. Wess & B. Zumino (1978, 1979)

Structure group: $SL(2, \mathbb{C})$

This superspace geometry is obtained from U(1) superspace by setting

$$X_\alpha = 0$$

Conformal superspace

D. Butter (2010)

Maximally supersymmetric spacetimes

For any (off-shell) supergravity theory in d dimensions, all maximally supersymmetric spacetimes correspond to those supergravity backgrounds which are characterised by the following properties:

- all Grassmann-odd components of the superspace torsion and curvature tensors vanish; and
- all Grassmann-even components of the torsion and curvature tensors are annihilated by the spinor derivatives.

SMK, J. Novak and G. Tartaglino-Mazzucchelli (2014)
SMK (2015)

Maximally supersymmetric backgrounds in 4D $\mathcal{N} = 1$ supergravity

$$X_\alpha = 0 \ , \qquad W_{\alpha\beta\gamma} = 0 \ ;$$
$$\mathcal{D}_\alpha R = 0 \longrightarrow \mathcal{D}_A R = 0 \ ;$$
$$\mathcal{D}_\alpha G_{\beta\dot\beta} = \bar{\mathcal{D}}_{\dot\alpha} G_{\beta\dot\beta} \longrightarrow \mathcal{D}_A G_{\beta\dot\beta} = 0 \ .$$

Integrability condition

$$0 = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} G_{\gamma\dot\gamma} = 2R(\varepsilon_{\dot\gamma\dot\alpha} G_{\gamma\dot\beta} + \varepsilon_{\dot\gamma\dot\beta} G_{\gamma\dot\alpha}) \quad \Longrightarrow \quad R\, G_a = 0 \ .$$

Isometries of curved superspace

Consider a background superspace $(\mathcal{M}^{4|4}, \mathcal{D})$. A supervector field $\xi = \xi^B E_B = \xi^b E_b + \xi^\beta E_\beta + \bar{\xi}_{\dot\beta} \bar{E}^{\dot\beta}$ on $(\mathcal{M}^{d|\delta}, \mathcal{D})$ is called Killing if

$$\delta_\mathcal{K} \mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A] = 0 \ , \qquad \mathcal{K} := \xi^B(z)\mathcal{D}_B + \frac{1}{2} K^{bc}(z) M_{bc} + \mathrm{i}\tau(z)\mathbb{J} \ ,$$

for some Lorentz (K^{bc}) and R-symmetry (τ) parameters.
All parameters ξ^β, K^{bc}, τ are determined in terms of ξ^b, in particular

$$\xi^B = \left(\xi^b, \xi^\beta, \bar{\xi}_{\dot\beta}\right) = \left(\xi^b, -\frac{\mathrm{i}}{8}\bar{\mathcal{D}}_{\dot\gamma}\xi^{\dot\gamma\beta}, -\frac{\mathrm{i}}{8}\mathcal{D}^\gamma \xi_{\gamma\dot\beta}\right) \ .$$

Vector component $\xi_{\beta\dot\beta}$ obeys the equation (supersymmetric analogue of the conformal Killing equation $\nabla_a \zeta_b + \nabla_b \zeta_a = \frac{2}{d}\eta_{ab}\nabla_c \zeta^c$)

$$\mathcal{D}_{(\alpha}\xi_{\beta)\dot\beta} = 0 \quad \Longrightarrow \quad (\mathcal{D}^2 + 2\bar{R})\xi_{\alpha\dot\alpha} = 0 \ .$$

I. Buchbinder & SMK, *Walk Through Superspace* (1995)

Let $(\mathcal{M}^{4|4}, \mathcal{D})$ be a maximally supersymmetric background. Then G^b is covariantly constant, $\mathcal{D}_\alpha G_{\beta\dot\beta} = 0$, hence $\xi^b = G^b$ defines a Killing (super)vector field, hence $R\, G_a = 0$.

4D $\mathcal{N} = 1$ maximally symmetric superspaces

Spacetimes supported by the superspaces $\mathbb{M}_T^{4|4}$, $\mathbb{M}_S^{4|4}$ and $\mathbb{M}_N^{4|4}$ are:

- $\mathbb{R} \times S^3$;
- $\mathrm{AdS}_3 \times S^1$ or its covering $\mathrm{AdS}_3 \times \mathbb{R}$;
- a pp-wave spacetime isometric to the Nappi-Witten group.

- $\mathbb{M}_T^{4|4}$ is the universal covering of superspace $\mathcal{M}^{4|4} = \mathsf{SU}(2|1)$.
  The bosonic body of $\mathcal{M}^{4|4}$ is $\mathsf{U}(2) = (S^1 \times S^3)/\mathbb{Z}_2$.
  The isometry group of $\mathcal{M}^{4|4}$ is $\mathsf{SU}(2|1) \times \mathsf{U}(2)$.
- $\mathbb{M}_S^{4|4}$ is the universal covering of superspace $\widetilde{\mathcal{M}}^{4|4} = \mathsf{SU}(1,1|1)$.
  The bosonic body of $\widetilde{\mathcal{M}}^{4|4}$ is $\mathsf{U}(1,1) = (\mathrm{AdS}_3 \times S^1)/\mathbb{Z}_2$.
  The isometry group of $\widetilde{\mathcal{M}}^{4|4}$ is $\mathsf{SU}(1,1|1) \times \mathsf{U}(2)$.
- $\mathbb{M}_N^{4|4}$. The spacetime metric looks like

$$\mathrm{d}s^2 = \mathrm{e}^u(2\mathrm{d}u\,\mathrm{d}v + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j) \ , \qquad i,j = 1,2 \ ,$$

with $G^a e_a \propto \partial/\partial v$.

C. Nappi & E. Witten (1993)

Maxwell Goldstone multiplet for partially broken SUSY

Maxwell Goldstone multiplet for partially broken SUSY

SMK & G. Tartaglino-Mazzucchelli (2016)

Let $\mathbb{M}^{4|4}$ any of the superspaces $\mathbb{M}^{4|4}_T$, $\mathbb{M}^{4|4}_S$ and $\mathbb{M}^{4|4}_N$.
$\mathbb{M}^{4|4}$ allows the existence of covariantly constant spinors,

$$\mathcal{D}_A \epsilon_\beta = 0 \ .$$

Consider the $\mathcal{N} = 1$ supersymmetric BI theory on $\mathbb{M}^{4|4}$ with action

$$S_{\rm SBI}[W, \bar{W}] = \frac{1}{4} \int {\rm d}^4 x \, {\rm d}^2\theta \, \mathcal{E} \, X + {\rm c.c.} \ ,$$

where chiral superfield X is a unique solution of the constraint

$$X + \frac{1}{4} X \bar{\mathcal{D}}^2 \bar{X} = W^2 \ .$$

W_α is the field strength of an Abelian vector multiplet

$$\bar{\mathcal{D}}_{\dot\alpha} W_\alpha = 0 \ , \qquad \mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot\alpha} \bar{W}^{\dot\alpha} \ .$$

Maxwell Goldstone multiplet for partially broken SUSY

The action is invariant under a second supersymmetry given by

$$\delta_\epsilon X = 2\epsilon^\beta W_\beta \ , \qquad \mathcal{D}_A \epsilon_\beta = 0 \ .$$

Of course, this transformation should be induced by that of W_α. The correct supersymmetry transformation of W_α proves to be

$$\delta_\epsilon W_\alpha = \epsilon_\alpha + \frac{1}{4}\epsilon_\alpha \bar{\mathcal{D}}^2 \bar{X} + \mathrm{i}\bar{\epsilon}^{\dot\beta} \mathcal{D}_{\alpha\dot\beta} X - \bar{\epsilon}^{\dot\beta} G_{\alpha\dot\beta} X \ .$$

It has the correct flat superspace limit, $G_{\alpha\dot\alpha} \to 0$,

J. Bagger & A. Galperin (1997)

and respects the Bianchi identity

$$\mathcal{D}^\alpha \delta_\epsilon W_\alpha = \bar{\mathcal{D}}_{\dot\alpha} \delta_\epsilon \bar{W}^{\dot\alpha} \ .$$

Nonlinear self-duality and partial SUSY breaking

Supersymmetric Born-Infeld action in curved superspace

$$S_{\rm SBI}[W,\bar W] = \frac{1}{4}\int {\rm d}^4x\,{\rm d}^2\theta\,\mathcal{E}\,X + {\rm c.c.}\ ,$$
$$X + \frac{1}{4}X(\bar{\mathcal{D}}^2 - 4R)\bar X = W^2$$

is an example of U(1) duality invariant models for supersymmetric nonlinear electrodynamics.

$$\mathrm{Im}\Big(W\cdot W \ + \ M\cdot M\Big) = 0\ , \qquad {\rm i}\,M_\alpha := 2\,\frac{\delta}{\delta W^\alpha}\,S[W,\bar W]\ .$$

SMK & S. Theisen (2000)
SMK & S. McCarthy (2003)

Nonlinear self-duality fixes a unique locally $\mathcal{N}=1$ supersymmetric extension of the BI action.
Otherwise there exist a one-parameter family of models:

S. Cecotti and S. Ferrara (1987)

Nilpotent chiral superfield in $\mathcal{N} = 2$ supergravity

Nilpotent chiral superfield in $\mathcal{N} = 2$ supergravity

SMK & G. Tartaglino-Mazzucchelli (2016)

$$\begin{aligned} \bar{\mathcal{D}}^i_{\dot\alpha} \mathcal{Z} &= 0 , \\ \big(\mathcal{D}^{ij} + 4S^{ij}\big)\mathcal{Z} - \big(\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij}\big)\bar{\mathcal{Z}} &= 4\mathrm{i}\, G^{ij} , \\ \mathcal{Z}^2 &= 0 , \end{aligned}$$

where $\mathcal{D}^{ij} := \mathcal{D}^{\alpha(i}\mathcal{D}^{j)}_\alpha$, $\bar{\mathcal{D}}^{ij} := \bar{\mathcal{D}}^{(i}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha j)}$.
G^{ij} is a linear multiplet constrained by $G^{ij}G_{ij} \neq 0$.

SU(2) × U(1) superspace — P. Howe (1982)

$$\begin{aligned} \{\mathcal{D}^i_\alpha, \mathcal{D}^j_\beta\} &= 4S^{ij}M_{\alpha\beta} + 2\varepsilon^{ij}\varepsilon_{\alpha\beta}Y^{\gamma\delta}M_{\gamma\delta} + 2\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar{W}^{\dot\gamma\dot\delta}\bar{M}_{\dot\gamma\dot\delta} \\ &\quad +2\varepsilon_{\alpha\beta}\varepsilon^{ij}S^{kl}J_{kl} + 4Y_{\alpha\beta}J^{ij} , \\ \{\mathcal{D}^i_\alpha, \bar{\mathcal{D}}^{\dot\beta}_j\} &= -2\mathrm{i}\delta^i_j(\sigma^c)_\alpha{}^{\dot\beta}\mathcal{D}_c + 4\big(\delta^i_j G^{\delta\dot\beta} + \mathrm{i}G^{\delta\dot\beta i}{}_j\big)M_{\alpha\delta} \\ &\quad +4\big(\delta^i_j G_{\alpha\dot\gamma} + \mathrm{i}G_{\alpha\dot\gamma}{}^i{}_j\big)\bar{M}^{\dot\gamma\dot\beta} + 8G_\alpha{}^{\dot\beta}J^i{}_j \\ &\quad -4\mathrm{i}\delta^i_j G_\alpha{}^{\dot\beta kl}J_{kl} - 2\big(\delta^i_j G_\alpha{}^{\dot\beta} + \mathrm{i}G_\alpha{}^{\dot\beta i}{}_j\big)\mathbb{J} . \end{aligned}$$

Comment for superspace practitioners

SU(2) superspace R. Grimm (1980)
is obtained from SU(2) $\times$ U(1) superspace by setting

$$G_{\alpha\dot{\alpha}}^{ij} = 0 \ .$$

SU(2) superspace is suitable to describe general off-shell matter couplings in $\mathcal{N} = 2$ supergravity.
SMK, U. Lindström, M. Roček & G. Tartaglino-Mazzucchelli (2008)

$\mathcal{N}=2$ linear multiplet is described in curved superspace by a real SU(2) triplet G^{ij} subject to the covariant constraints

P. Breitenlohner & M. Sohnius (1980)

$$\mathcal{D}_\alpha^{(i}G^{jk)} = \bar{\mathcal{D}}_{\dot\alpha}^{(i}G^{jk)} = 0 \ .$$

These constraints are solved in terms of a chiral prepotential Ψ

P. Howe, K. Stelle & P. Townsend (1981)

$$G^{ij} = \frac{1}{4}\Big(\mathcal{D}^{ij} + 4S^{ij}\Big)\Psi + \frac{1}{4}\Big(\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij}\Big)\bar{\Psi} \ , \qquad \bar{\mathcal{D}}^i_{\dot\alpha}\Psi = 0 \ ,$$

which is invariant under Abelian gauge transformations

$$\delta_\Lambda \Psi = \mathrm{i}\Lambda \ ,$$

with Λ a reduced chiral superfield,

$$\bar{\mathcal{D}}^i_{\dot\alpha}\Lambda = 0 \ , \qquad \big(\mathcal{D}^{ij} + 4S^{ij}\big)\Lambda - \big(\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij}\big)\bar{\Lambda} = 0 \ .$$

Field strength of every U(1) vector multiplet is a reduced chiral superfield.

R. Grimm, M. Sohnius and J. Wess (1978)

Deformed reduced chiral superfield in $\mathcal{N} = 2$ supergravity

Constraints

$$\bar{\mathcal{D}}^i_{\dot\alpha} \mathcal{Z} = 0 \ , \qquad \big(\mathcal{D}^{ij} + 4S^{ij}\big)\mathcal{Z} - \big(\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij}\big)\bar{\mathcal{Z}} = 4\mathrm{i}\, G^{ij}$$

define a deformed reduced chiral superfield. It may be introduced as follows.

- Start with the massless (improved) tensor multiplet

  B. de Wit, R. Philippe & A. Van Proeyen (1983)

$$S_{\rm TM} = -\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\, \Psi \mathbb{W} + \mathrm{c.c.} \ , \quad \mathbb{W} := -\frac{G}{8}(\bar{\mathcal{D}}_{ij} + 4\bar{S}_{ij})\left(\frac{G^{ij}}{G^2}\right)$$

- Introduce a massive deformation SMK (2009)

$$S_{\rm massive} = -\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E} \left\{ \Psi \mathbb{W} + \frac{1}{4}\mu(\mu + \mathrm{i}e)\Psi^2 \right\} + \mathrm{c.c.} \ ,$$

with μ and e real parameters, $\mu \neq 0$ (mass $m = \sqrt{\mu^2 + e^2}$).

Deformed reduced chiral superfield in $\mathcal{N} = 2$ supergravity

- Consider a Stückelberg-type extension of the massive model

$$\widetilde{S}_{\rm massive} = -\int {\rm d}^4x\, {\rm d}^4\theta\, \mathcal{E}\left\{\Psi\mathbb{W} + \frac{1}{4}\mu(\mu + {\rm i}e)(\Psi - {\rm i}W)^2\right\} + {\rm c.c.}\ ,$$

  where W is the field strength of a vector multiplet.
  $\widetilde{S}_{\rm massive}$ is gauge invariant provided $\delta_\Lambda W = \Lambda$.
- Chiral superfield

$$\mathcal{Z} := W + {\rm i}\Psi$$

  obeys the constraint

$$\left(\mathcal{D}^{ij} + 4S^{ij}\right)\mathcal{Z} - \left(\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij}\right)\bar{\mathcal{Z}} = 4{\rm i}\, G^{ij}$$

- Constraints

$$
\begin{aligned}
\bar{\mathcal{D}}^i_{\dot\alpha} \mathcal{Z} &= 0 \ , \\
\big(\mathcal{D}^{ij} + 4 S^{ij}\big)\mathcal{Z} - \big(\bar{\mathcal{D}}^{ij} + 4 \bar{S}^{ij}\big)\bar{\mathcal{Z}} &= 4\mathrm{i}\, G^{ij} \ , \\
\mathcal{Z}^2 &= 0
\end{aligned}
$$

imply that, for certain supergravity backgrounds, the degrees of freedom described by $\mathcal{Z}$ are in a one-to-one correspondence with those of an Abelian $\mathcal{N}=1$ vector multiplet.

- The specific feature of such $\mathcal{N}=2$ supergrvaity backgrounds is that they possess an $\mathcal{N}=1$ subspace of the full $\mathcal{N}=2$ superspace.
- This property is not universal. In particular, there exist maximally $\mathcal{N}=2$ supersymmetric backgrounds with no truncation to $\mathcal{N}=1$.
  D. Butter, G. Inverso and I. Lodato (2015)
- Action

$$
S = -\frac{\mathrm{i}}{4}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\Psi\mathcal{Z} + \mathrm{c.c.}
$$

describes partial $\mathcal{N}=2 \to \mathcal{N}=1$ SUSY breaking on certain maximally supersymmetric backgrounds (with Ψ being background).

Maximally $\mathcal{N} = 2$ SUSY backgrounds & partial SUSY breaking

Consider a maximally supersymmetric background $\mathbb{M}^{4|8}$ with the property that the chiral prepotential Ψ for G^{ij} may be chosen such that

- Complex linear multiplet

$$G_+^{ij} := \frac{1}{4}\Big(\mathcal{D}^{ij} + 4S^{ij}\Big)\Psi$$

  is covariantly constant and null,

$$\mathcal{D}_\mathcal{A} G_+^{ij} = 0 \ , \qquad G_+^{ij} G_{+ij} = 0 \ .$$

- Ψ may be chosen to be nilpotent,

$$\Psi^2 = 0 \ .$$

$G^{ij} = G_+^{ij} + G_-^{ij}$ is covariantly constant, $\mathcal{D}_\mathcal{A} G^{ij} = 0$.
Then, action

$$S = -\frac{\mathrm{i}}{4}\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\, \Psi \mathcal{Z} + \mathrm{c.c.}$$

is $\mathcal{N} = 2$ supersymmetric.

Example: The super-Poincaré case

Example: The super-Poincaré case

$\mathcal{N} = 2$ Minkowski superspace, $\mathbb{R}^{4|8}$, is the simplest maximally supersymmetric background.

in $\mathbb{R}^{4|8}$ every constant real SU(2) triplet G^{ij} is covariantly constant,

$$D_{\mathcal{A}} G^{ij} = 0 \ ,$$

where $D_{\mathcal{A}} = (\partial_a, D_\alpha^i, \bar{D}_i^{\dot{\alpha}})$ are the flat superspace covariant derivatives.
Let Ψ be a chiral prepotential for G^{ij}, $\bar{D}^i_{\dot{\alpha}}\Psi = 0$. We represent

$$G^{ij} = G^{ij}_+ + G^{ij}_- \ , \qquad G^{ij}_+ = \frac{1}{4} D^{ij}\Psi \ , \qquad G^{ij}_- = \frac{1}{4}\bar{D}^{ij}\bar{\Psi} \ .$$

In $\mathbb{R}^{4|8}$ the constraints on $\mathcal{Z}$ turn into

$$\bar{D}^i_{\dot{\alpha}}\mathcal{Z} = 0 \ , \qquad D^{ij}\mathcal{Z} - \bar{D}^{ij}\bar{\mathcal{Z}} = 4\mathrm{i}\, G^{ij} \ , \qquad \mathcal{Z}^2 = 0 \ .$$

The $\mathcal{N} = 2$ supersymmetric action becomes

$$S = -\frac{\mathrm{i}}{4}\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{Z}\Psi \ + \ \mathrm{c.c.}$$

Example: The super-Poincaré case

Grassmann coordinates of $\mathcal{N}=2$ Minkowski superspace

$$\theta^\alpha_i \ , \qquad \bar\theta^j_{\dot\alpha} \ , \qquad i,j=\underline{1},\underline{2}$$

Without loss of generality we can choose

$$G_+^{ij} = -\mathrm{i}\delta^i_{\underline{2}}\delta^j_{\underline{2}} \equiv q^i q^j \ , \qquad \Psi = \mathrm{i}\theta^\alpha_{\underline{2}}\theta_{\alpha\underline{2}}$$

such that $G_+^{ij}G_{+ij}=0$ and $\Psi^2=0$.

$\mathcal{N}=2 \to \mathcal{N}=1$ superspace reduction
Given a superfield $U(x,\theta_i,\bar\theta^i)$ on $\mathcal{N}=2$ Minkowski superspace, we introduce its bar-projection

$$U| := U(x,\theta_i,\bar\theta^i)|_{\theta_{\underline{2}}=\bar\theta^{\underline{2}}=0} \ ,$$

which is a superfield on $\mathcal{N}=1$ Minkowski superspace with Grassmann coordinates $\theta^\alpha=\theta^\alpha_{\underline{1}}$ and $\bar\theta_{\dot\alpha}=\bar\theta^{\underline{1}}_{\dot\alpha}$
& spinor covariant derivatives $D_\alpha = D^{\underline{1}}_\alpha$ and $\bar D^{\dot\alpha}=\bar D^{\dot\alpha}_{\underline{1}}$.

Example: The super-Poincaré case

$\mathcal{N} = 1$ components of $\mathcal{Z}$:

$$X := \mathcal{Z}| \ , \qquad W_\alpha := -\frac{\mathrm{i}}{2} D^2_\alpha \mathcal{Z}| \ .$$

The original $\mathcal{N} = 2$ constraints, $\bar{D}^i_{\dot\alpha} \mathcal{Z} = 0$ & $D^{ij}\mathcal{Z} - \bar{D}^{ij}\bar{\mathcal{Z}} = 4\mathrm{i}\, G^{ij}$, imply

$$\bar{D}_{\dot\alpha} X = 0 \ , \qquad \bar{D}_{\dot\alpha} W_\alpha = 0 \ , \qquad D^\alpha W_\alpha = \bar{D}_{\dot\alpha} \bar{W}^{\dot\alpha} \ .$$

W_α is the field strength of U(1) vector multiplet.
Taking into account the nilpotent constraint $\mathcal{Z}^2 = 0$ gives

$$X + \frac{1}{4} X \bar{D}^2 \bar{X} = W^2 \ , \qquad W^2 := W^\alpha W_\alpha \ .$$

The original $\mathcal{N} = 2$ action, $S = -\frac{\mathrm{i}}{4} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{Z}\Psi + \mathrm{c.c.}$, turns into

$$S = \frac{1}{4} \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, X + \frac{1}{4} \int \mathrm{d}^4x\, \mathrm{d}^2\bar\theta\, \bar{X} \ .$$

Bagger-Galperin solution

The Bagger-Galperin solution to the constraint

$$X + \frac{1}{4} X \bar{D}^2 \bar{X} = W^2$$

is as follows:

$$X = W^2 - \frac{1}{2}\bar{D}^2 \frac{W^2 \bar{W}^2}{\left(1 + \frac{1}{2}A + \sqrt{1 + A + \frac{1}{4}B^2}\right)} ,$$

$$A = \frac{1}{2}\left(D^2 W^2 + \bar{D}^2 \bar{W}^2\right) , \qquad B = \frac{1}{2}\left(D^2 W^2 - \bar{D}^2 \bar{W}^2\right) .$$