classification of topological insulators and superconductors

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- what is a topological insulator/topological superconductor ?
    - what is an insulator ?
    - what is a topological insulator ?
    - examples of topological insulators

      quantum Hall effec (QHE), quantum spin Hall effect (QSHE),
      3-dimensional Z2 topological insulator

- classification of topological insulators/superconductors

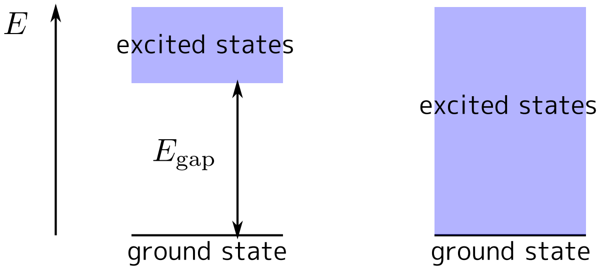
- boundary-to-bulk approach
    - non-linear sigma model (NLsM) approach to disordered systems

- topological superconductor

- summary

what is an insulator ?

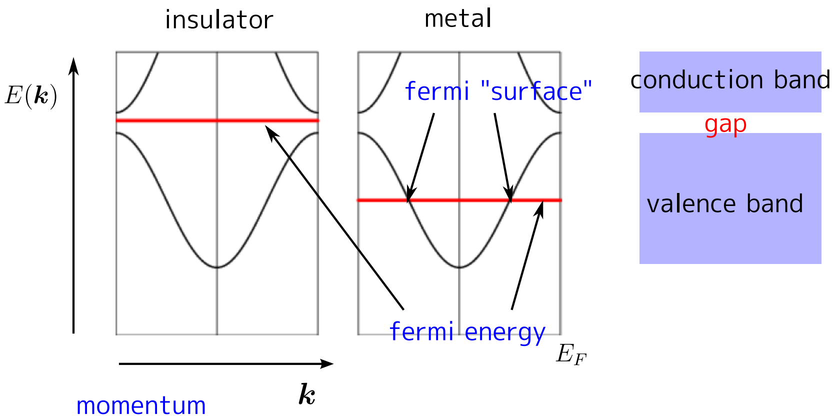
insulator	metal
vanishing dc conductivity	non-zero dc conductivity



band structure and band insulator

single particle energy spectrum of an electron in solids

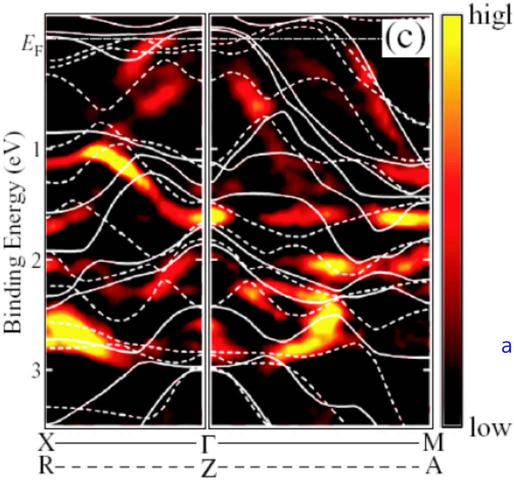
= "band structure"



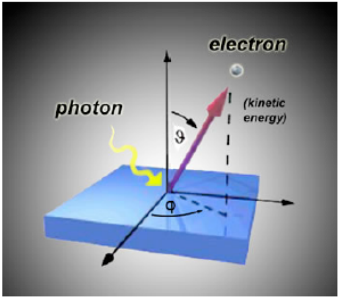
momentum
(or crystal momentum in
1st Brillouin Zone, BZ)

Wilson 1931

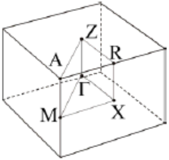
CeRhIn5



Fujimori et al
PRB (2003)



angle-resolved photoemission spectroscopy
= ARPES

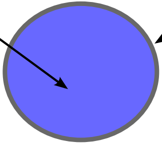


topological insulator: preview

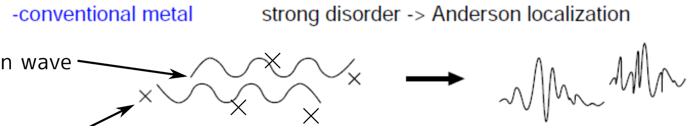
topological insulator = insulating in bulk, but
completely metalic at boundaries

topological insulators/SC

perfect metal



boundary completely
evades localization



electron wave

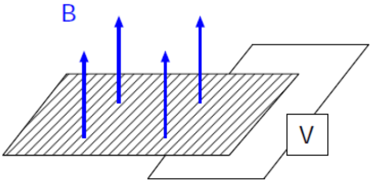
impurities

conducting

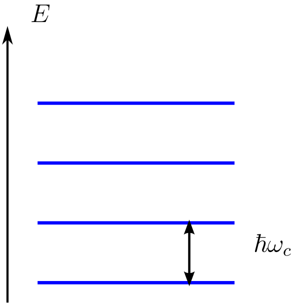
non-conducting

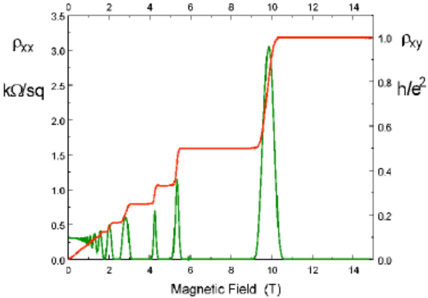
integer quantum Hall effect (IQHE)

in d=2 spatial dimensions, with strong T breaking by B



$$\sigma_{xy} = \frac{e^2}{h} \times (\text{integer})$$



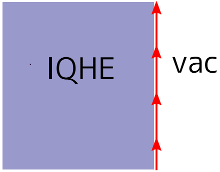


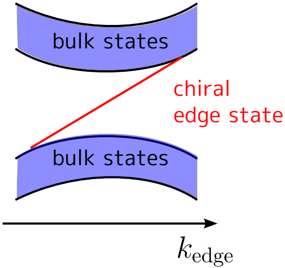
IQHE as a topological insulator

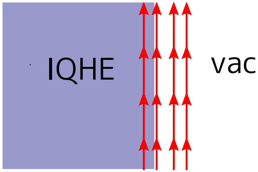
in d=2 spatial dimensions, with strong T breaking by B

- "boundary" point of view

stable edge modes,
immute to disorder



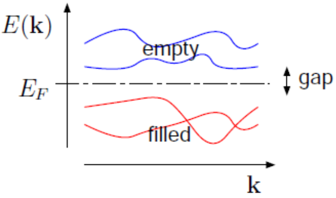




IQHE as a topological insulator

in d=2 spatial dimensions, with strong T breaking by B

- "bulk" point of view



Bloch wavefunction

$$\psi(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} u_{\boldsymbol{k}}(\boldsymbol{r})$$

projection operator:

$$Q(k) = 2 \sum_{a \in \text{filled}} |u_a(k)\rangle\langle u_a(k)| - 1$$

$$Q^2 = 1, \quad Q^\dagger = Q, \quad \mathrm{tr}\, Q = m - n$$

quantum ground state = map from Bz onto Grassmannian

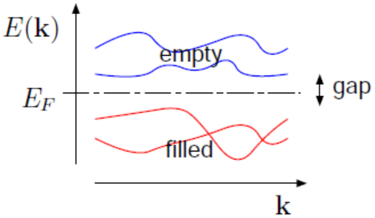
$$Q(k) : \mathrm{BZ} \longrightarrow U(m+n)/U(m) \times U(n)$$

$$\pi_2 \left[U(m+n)/U(m) \times U(n) \right] = \mathbb{Z}$$

IQHE as a topological insulator

in d=2 spatial dimensions, with strong T breaking by B

- "bulk" point of view



$$\pi_2\left[U(m+n)/U(m)\times U(n)\right]=\mathbb{Z}$$

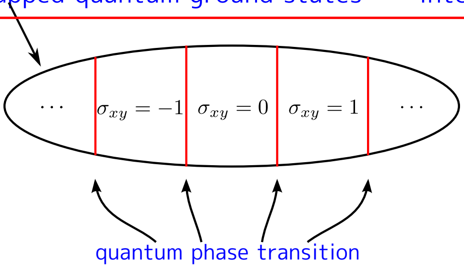
Thouless-Kohmoto-Nightingale-den Nijs (1982)

$$\sigma_{xy}=\frac{e^2}{h}\sum_{bands}\frac{i}{2\pi}\int d^2k\left(\left\langle\frac{\partial u(k)}{\partial k_x}\middle|\frac{\partial u(k)}{\partial k_y}\right\rangle-\left\langle\frac{\partial u(k)}{\partial k_y}\middle|\frac{\partial u(k)}{\partial k_x}\right\rangle\right)$$

topological invariant ! "Chern number"

IQHE as a topological insulator

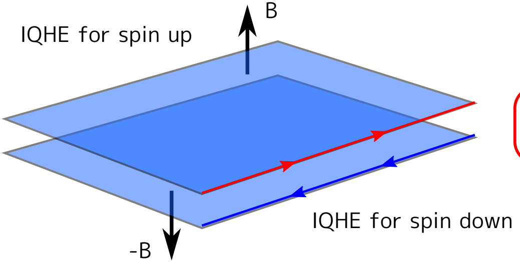
space of all gapped quantum ground states = integer $\mathbb{Z}$



quantum spin Hall effect (QSHE)

in d=2 spatial dimensions, with good T

- start from independent spin up and down

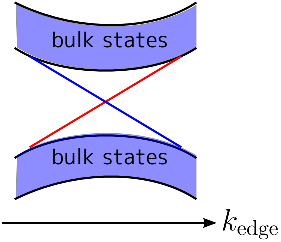


TRS

$$(i\sigma_y)\mathcal{H}^*(-i\sigma_y) = \mathcal{H}$$

S_z is conserved, classified by an integer (Z)

$\sigma_{xy,\uparrow} - \sigma_{xy,\downarrow}$ is quantized

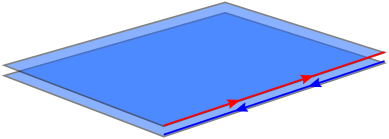


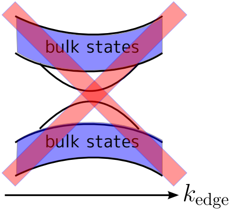
quantum spin Hall effect (QSHE)

in d=2 spatial dimensions, with good T

- mix spin up and down, but keep TRS

$$(i\sigma_y)\mathcal{H}^*(-i\sigma_y) = \mathcal{H}$$





- odd number of Kramers pairs at edge --> stable

  even number of Kramers pairs at edge --> unstable

quantum spin Hall effect (QSHE)

with TRS and without Sz conservation -> binary classification $\mathbb{Z}_2$

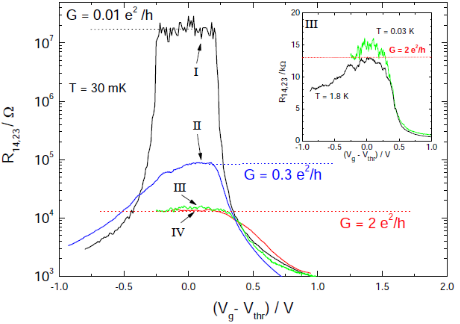
even # of Kramers pairs at edge	odd # of Kramers pairs at edge
($\simeq$ no edge mode)	($\simeq$ single Kramers pair)

quantum phase transition

experimental realization:
HgTe quantum well

strong spin-orbit interaction

Bernevig-Hughes-Zhang (2006)
M. Koenig et al. Science (2007)



Z2 topological insulator in d=3 spatial dimensions

Fu-Kane-Mele, Moore-Balents, Roy (06)

d=3 dimensions

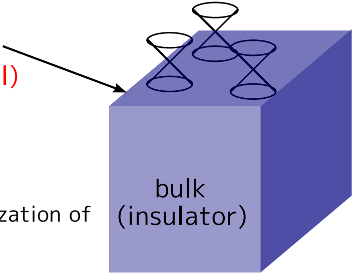
time-reversal invariant $i\sigma_y \mathcal{H}^* (-i\sigma_y) = \mathcal{H}$

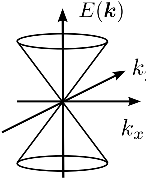
characterized by a Z2 quantity $\nu_0 = 0 \,\mathrm{or}\, 1$

trivial (0) non-trivial (1)

when $\nu_0 = 1$ surface states = odd number of Dirac fermions

surface
(perfect metal)





$$\mathcal{H} = v_F \left(\sigma_x p_x + \sigma_y p_y \right)$$

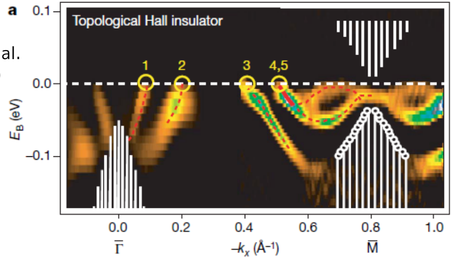
condensed matter realization of
domain-wall fermion

ARPES experiments on Z2 topological insulators

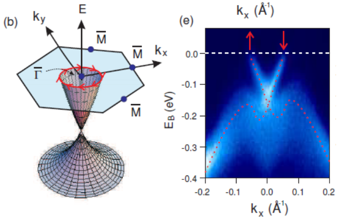
BiSb

5 Dirac cones !

D. Hsieh et al. Nature (08)

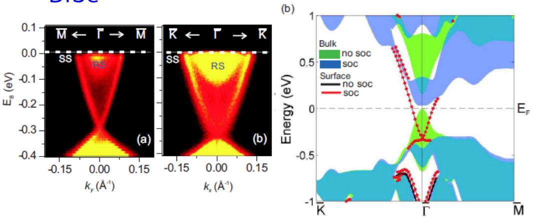


BiTe



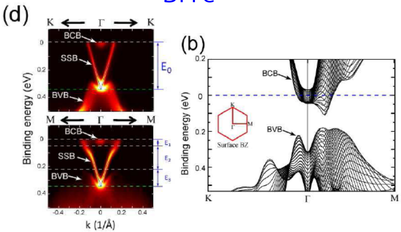
D. Hsieh et al. Science (2009)

BiSe



Y. Xia et al. Nature Phys. (2009)

BiTe



Y. L. Chen et al. Science (2009)

- IQHE GaAs

- QSHE HgTe

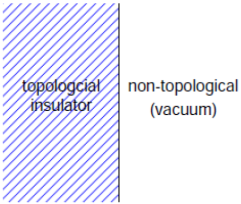
- 3D Z2 topological insulator BiTe, BiSe, BiSb

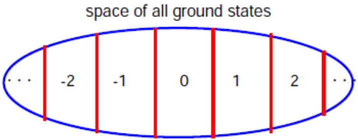
- chiral p-wave SC (topological superconductor) SrRuO
  nu=5/2 FQHE

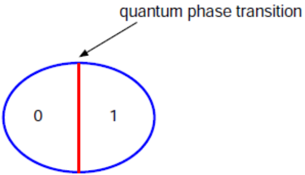
- more topological insulators/superconductors ?

topological:

- support stable gapless modes at boundaries, possibly in the presence of general discrete symmetries
- states with and without boundary modes are not adiabatically connected
- may be characterized by a bulk topological invariant of some sort







c.f. topological phase, topological field theory

How many different topoloigcal insulators and superconductors are there in nature ?

- what is a topological insulator/topological superconductor ?
    - what is an insulator ?
    - what is a topological insulator ?
    - examples of topological insulators

      quantum Hall effec (QHE), quantum spin Hall effect (QSHE),
      3-dimensional Z2 topological insulator

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    - non-linear sigma model (NLsM) approach to disordered systems

- topological superconductor

- summary

Discrete symmetries

two types of anti-unitary symmetries

Time-Reversal Symmetry (TRS)

$$\mathcal{T}\mathcal{H}^*\mathcal{T}^{-1} = \mathcal{H}$$

TRS =
- 0 no TRS
- +1 TRS with $\mathcal{T}^T = +\mathcal{T}$ ← integer spin particle
- -1 TRS with $\mathcal{T}^T = -\mathcal{T}$ ← half-odd integer spin particle

Particle-Hole Symmetry (PHS)

$$C\mathcal{H}^T C^{-1} = -\mathcal{H}$$

PHS =
- 0 no PHS
- +1 PHS with $C^T = +C$
- -1 PHS with $C^T = -C$

PHS + TRS = chiral symmetry

$$\left.\begin{array}{l} T\mathcal{H}^* T^{-1} = \mathcal{H} \\ C\mathcal{H}^* C^{-1} = -\mathcal{H} \end{array}\right\} \longrightarrow TC\mathcal{H}(TC)^{-1} = -\mathcal{H}$$

random matrix ensembles - "ten-fold way"

	sym. class	TRS	PHS	SLS	description
Wigner-Dyson	A	0	0	0	unitary
	AI	+1	0	0	orthogonal
	AII	-1	0	0	symplectic (spin-orbit
chiral	AIII	0	0	1	chiral unitary
	BDI	+1	+1	1	chiral orthogonal
	CII	-1	-1	1	chiral symplectic
BdG	D	0	+1	0	singlet/triplet SC
	C	0	-1	0	singlet SC
	DIII	-1	+1	1	singlet/triplet SC with T
	CI	+1	-1	1	singlet SC with TRS

- Wigner-Dyson (1951 -1963) : "three-fold way" — complex nuclei
- Verbaarschot (1992 -1993) — chiral phase transition in QCD
- Altland-Zirnbauer (1997) : "ten-fold way" — mesoscopic SC systems

claim: this is the exhaustive classification of discrete symmetries

classification of topological insulators and superconductors

AZ\\d	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

SR, Schnyder, Furusaki, Ludwig (for d=1,2,3, 2008)

Kitaev (all d and periodicity, 2009) "periodic table of topological insulators"

Alexei Kitaev's talk in Focus Week (Thu)

Qi, Hughes, Zhang (cases with one discrete symmetry, 2008)

Shoucheng Zhang's talk in Focus Week (Wed)

SR, Takayanagi (D-brane construction, 2010)

Tadashi Takayanagi's talk in Focus Week (Fri)

classification of topological insulators and superconductors

spatial dimensions

presence/absence
of topological band structure

AZ\ d	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

symmetry classes of quadratic fermionic
Hamiltonians (Altland-Zirnbauer)

$\mathbb{Z}$ integer classification

$\mathbb{Z}_2$ Z2 classification

0 no top. ins./SC

AZ\\d	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

IQHE

p+ip wave SC

polyacetylene

3He B

TMTSF

Z2 topological insulator

QSHE

d+id wave SC

some outcomes of classification:

- 3He B is newly identified as a topological SC (superfluid) in d=3.
- topological singlet SC in d=3 is predicted.

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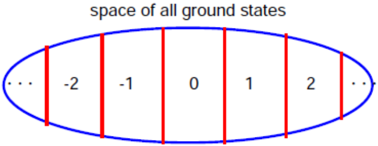
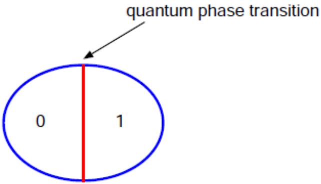
underlying strategies for classification

- discover a topological invariant

e.g. Hall conductivity

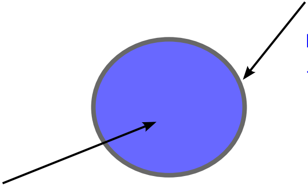




- bulk-boundary correspondence

Anderson delocalization

non-linear sigma model on G/H
+ (discrete) topological term

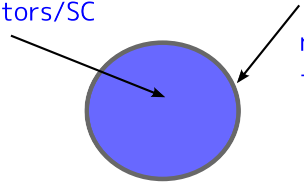


topological insulators/SC

bulk-boundary correspondence

topological insulators/SC

fully gapped,
no excitations

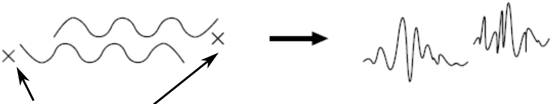


Anderson delocalization

non-linear sigma model on G/H
+ (discrete) topological term

-conventional metal

strong disorder -> Anderson localization



impurities

conducting

non-conducting

Anderson localization with time-reversal symmetry

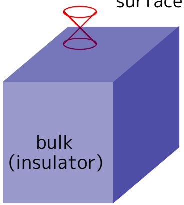
$$\mathcal{H} = v_F\,(\sigma_x p_x + \sigma_y p_y) + V(\mathbf{r})$$

$$V(\mathbf{r}) = \sum_i U(\mathbf{r} - \mathbf{R}_i)$$

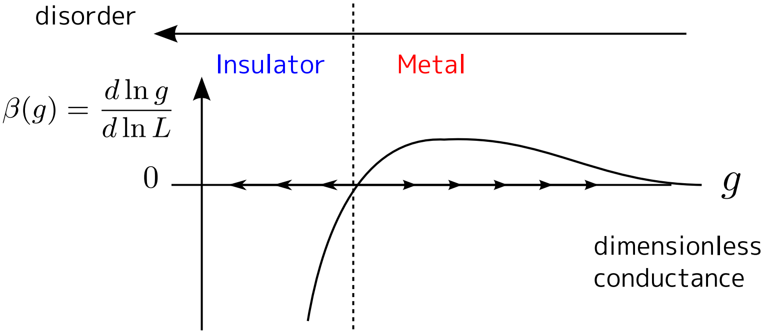
random potential (impurities)

time-reversal symmetry

$$i\sigma_y \mathcal{H}^* (-i\sigma_y) = \mathcal{H}$$



conventional theory:



"gang of four" (79)   Hikami-Larkin-Nagaoka (80)

effective field theory for Anderson localization

	AF quantum spin chain	symplectic Anderson localization
Nambu-Goldstone bosons	magnons	"Diffuson" and "Cooperons"
order parameter	Neel vector $\boldsymbol{n} \in \mathbb{R}^3$, $\boldsymbol{n} \cdot \boldsymbol{n} = 1$	matrix field Q
ordered phase	AF magnet	metal
disordered phase	para magnet	insulator
topological term	half-odd integer spin	odd number of Dirac cones
	integer spin	even number of Dirac cones (~ non relativistic)

Z2 topological term in symplectic symmetry class

SR, Mudry, Obuse, Furusaki (07)

microscopic model:

$$\mathcal{H} = v_F\left(\sigma_x p_x + \sigma_y p_y\right) + V(\mathbf{r})$$

$$i\sigma_y \mathcal{H}^* (-i\sigma_y) = \mathcal{H}$$

effective field theory: non-linear sigma model

(diffusive motion of electrons)

$$Q(\mathbf{r}) \in \mathrm{O}(4N)/[\mathrm{O}(2N)\times\mathrm{O}(2N)]$$

$$S = \sigma_{xx}\int d^2r\,\mathrm{tr}\left[\partial_\mu Q \partial_\mu Q\right] \qquad \pi_2(\mathrm{O}(4N)/\mathrm{O}(2N)\times\mathrm{O}(2N)) = \mathbb{Z}_2$$

even number of Dirac

$$Z = \int \mathcal{D}[Q] e^{-S}$$

odd number of Dirac
-> Z2 topological term

$$Z = \int \mathcal{D}[Q](-1)^{n[Q]} e^{-S} \qquad n[Q] = 0, 1$$

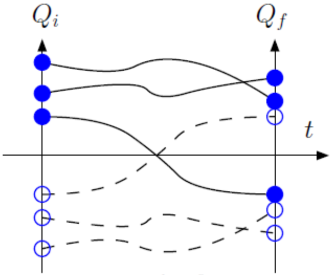
Z2 topological term in symplectic symmetry class

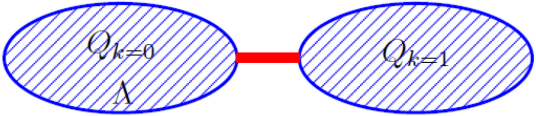
spectral flow

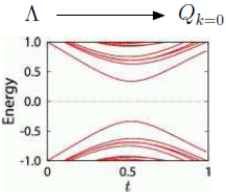
SR, Mudry, Obuse, Furusaki, PRL (07)

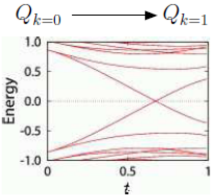
sign of Pfaffian ⟷ spectral flow

$$Q_t := (1-t)Q_i + tQ_f$$









$\mathrm{sign}(\Lambda) = \mathrm{sign}(Q_0)$

$\mathrm{sign}(Q_0) = -\mathrm{sign}(Q_1)$

$$\mathrm{Pf}\, D[Q] \equiv \prod_i^{\lambda_i > 0} \lambda_i$$

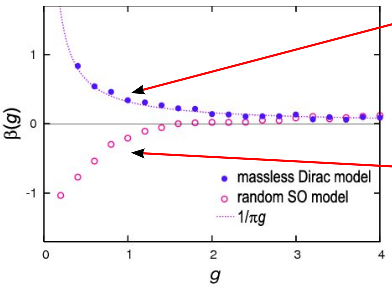
$$D[Q] := \sigma \cdot p - \Delta \sigma_z Q$$

Witten's su(2) global anomaly

numerical beta function

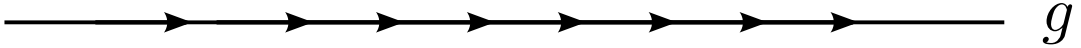
Nomura, Koshino, SR (07)

$$\mathcal{H} = v_F\left(\sigma_x p_x + \sigma_y p_y\right) + V(\mathbf{r})$$



odd number of Dirac cones

even number of Dirac cones
(~ non-relativistic case)



surface of 3D Z2 top. insulator = perfect metal !

"topological metal"

field theory : non-linear sigma model with Z2 topological term

Ken Nomura's talk in Focus week (Wed)

NLsM target spaces in Anderson localization

AZ class	bosonic replica NLσM target space	fermionic replica NLσM target space	supergroup NLσM target space
A	$U(N,N)/U(N)\times U(N)$	$U(2N)/U(N)\times U(N)$	$GL(2N\|2N)/GL(N\|N)\times GL(N\|N)$
AI	$O(N,N)/O(N)\times O(N)$	$Sp(2N)/Sp(N)\times Sp(N)$	$Osp(2N\|2N)/Osp(N\|N)\times Osp(N\|N)$
AII	$Sp(N,N)/Sp(N)\times Sp(N)$	$O(2N)/O(N)\times O(N)$	$Osp(2N\|2N)/Osp(N\|N)\times Osp(N\|N)$
AIII	$Gl(N,\mathbb{C})/U(N)$	$U(N)\times U(N)/U(N)$	$Gl(N\|N)$
BDI	$Gl(N,\mathbb{R})/O(N)$	$U(2N)/Sp(N)$	$Gl(2N\|2N)/Osp(2N\|2N)$
CII	$U^*(2N)/Sp(N)$	$U(N)/O(N)$	$Gl(2N\|2N)/Osp(2N\|2N)$
D	$Sp(N,\mathbb{R})/U(N)$	$O(2N)/U(N)$	$Osp(2N\|2N)/Gl(N\|N)$
C	$SO^*(2N)/U(N)$	$Sp(N)/U(N)$	$Osp(2N\|2N)/Gl(N\|N)$
DIII	$Sp(N,\mathbb{C})/Sp(N)$	$O(N)\times O(N)/O(N)$	$Osp(2N\|2N)$
CI	$SO(N,\mathbb{C})/SO(N)$	$Sp(N)\times Sp(N)/Sp(N)$	$Osp(2N\|2N)$

discrete symmetries --> NLsM target space = G/H

bosonic replica NLsM has a non-compact target space

fermionic replica NLsM has a compact target space

for the full treatment Andreson localization, we have both (SUSY)

NLsM topological terms

complex case:

	$G/H \setminus d$	$d=0$	$d=1$	$d=2$	$d=3$
A	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	$\mathrm{U}(N)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

real case:

	$G/H \setminus d$	$d=0$	$d=1$	$d=2$	$d=3$
AI	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)$	$\mathbb{Z}$	0	0	0
BDI	$\mathrm{U}(2N)/\mathrm{Sp}(N)$	0	$\mathbb{Z}$	0	0
D	$\mathrm{O}(2N)/\mathrm{U}(N)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
DIII	$\mathrm{O}(N)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
AII	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
CII	$\mathrm{U}(N)/\mathrm{O}(N)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	$\mathrm{Sp}(N)/\mathrm{U}(N)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
CI	$\mathrm{Sp}(N)$	0	0	0	$\mathbb{Z}$

compact NLsM target space = G/H

Z2 = existence of Z2 topological term
in d dimensions

Z = existence of WZW term
in (d-1) dimensions

from boundary to bulk

complex case:

AZ	G/H	$\bar{d}=0$	$\bar{d}=1$	$\bar{d}=2$	$\bar{d}=3$	$\bar{d}=4$	$\bar{d}=5$	$\bar{d}=6$	$\bar{d}=7$
A	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)$	$\mathbb{Z}$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	$\mathbf{0}$
AIII	$\mathrm{U}(N)$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$

real case:

AZ	G/H	$\bar{d}=0$	$\bar{d}=1$	$\bar{d}=2$	$\bar{d}=3$	$\bar{d}=4$	$\bar{d}=5$	$\bar{d}=6$	$\bar{d}=7$
AI	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)$	$\mathbb{Z}$	0	0	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$
BDI	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	0	0	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\mathrm{O}(2N)/\mathrm{U}(N)$	$\mathbb{Z}_2$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	0	0	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	$\mathbb{Z}_2$
DIII	$\mathrm{O}(N)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	0	0	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$
AII	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	0	0	$\mathbf{0}$
CII	$\mathrm{U}(N)/\mathrm{O}(N)$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	0	0
C	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	0	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	0
CI	$\mathrm{Sp}(2N)$	0	0	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0} \longleftarrow$	$\mathbb{Z}$

independent checks: for d=1 and d=2 Anderson localization,
many exact results are avaiable.

from d=2 boundary to d=3 bulk

		TRS	PHS	SLS	fermionic replica NLsM	
Wigner-Dyson (standard)	A	0	0	0	$U(2N)/U(N)\times U(N)$	Pruisken
	AI	+1	0	0	$Sp(4N)/Sp(2N)\times Sp(2N)$	
	AII	-1	0	0	$O(2N)/O(2N)\times O(2N)$	$\mathbb{Z}_2$
chiral (sublattice)	AIII	0	0	1	$U(N)$	WZW
	BDI	+1	+1	1	$U(2N)/Sp(N)$	
	CII	-1	-1	1	$U(N)/O(N)$	$\mathbb{Z}_2$
BdG	D	0	+1	0	$O(2N)/U(N)$	Pruisken
	C	0	-1	0	$Sp(N)/U(N)$	Pruisken
	DIII	-1	+1	1	$O(N)$	WZW
	CI	+1	-1	1	$Sp(N)$	WZW

newly derived !

- Bernard-Le Clair: 13-fold symmetry classifcation of 2d Dirac fermions
- AIII, CI, DIII; exact results
- "abnormal terms" in NLsM

WZW type $$Z=\int \mathcal{D}[g]e^{2\pi i\nu\Gamma_{\mathrm{WZW}}}e^{-S[g]} \qquad \Gamma_{\mathrm{WZW}}=\frac{1}{24\pi^2}\int_{\mathcal{M}^3}\mathrm{tr}\left[(g^{-1}dg)^3\right]$$

Z2 type $$Z=\int \mathcal{D}[Q](-1)^{N[Q]}e^{-S[Q]}$$

SR, Mudry, Obuse Furusaki (07)

- what is a topological insulator/topological superconductor ?
  - what is an insulator ?
  - what is a topological insulator ?
  - examples of topological insulators

    quantum Hall effec (QHE), quantum spin Hall effect (QSHE),
    3-dimensional Z2 topological insulator

- classification of topological insulators/superconductors

- boundary-to-bulk approach
  - non-linear sigma model (NLsM) approach to disordered systems

- topological superconductor

- summary

chiral p-wave SC in d=2 - a topological SC

topological SC = BdG quasi-particles are topologically non-trivial

$$H = \frac{1}{2}\int \Psi^\dagger \begin{pmatrix} \xi & \Delta \\ -\Delta^* & -\xi^T \end{pmatrix} \Psi$$

$$\Psi^\dagger = \begin{pmatrix} \psi^\dagger, & \psi \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \psi^\dagger_\uparrow, & \psi^\dagger_\downarrow, & \psi_\uparrow, & \psi_\downarrow \end{pmatrix}$$ etc.

px + i py SC order parameter:

$$\xi(k) = \frac{k^2}{2m} - \mu \quad \Delta(k) = |\Delta|\,(k_x + ik_y)$$

Read-Green (2000)

strong pairing (BEC) weak pairing (BCS)

$\longrightarrow \mu$

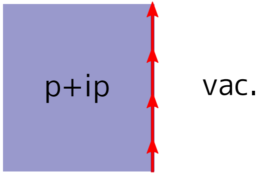
$$\Psi(\{\boldsymbol{r}_i\}) \sim \text{short ranged}$$

$$\Psi(\{\boldsymbol{r}_i\}) \sim \text{Pf}\left(\frac{1}{z_i - z_j}\right)$$

Many-body GS wavefunction

chiral p-wave SC in d=2 - a topological SC

stable boundary Majorana fermion in the weak pairing phase



$$S = \int dx d\tau \bar{\psi} \bar{\partial} \psi$$

quantized thermal Hall conductivity

with inclusion of the dynamics of Cooper pair:

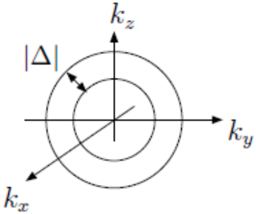
- non-trivial ground state degeneracy — topologically protected q-bit
- non-Abelian statistics of vortices
- vortex supports an isolated Majorana mode

$\nu = 5/2$ FQHE Moore-Read state

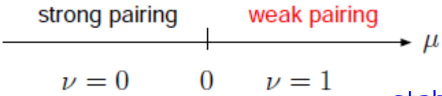
3He B is a topological "superconductor" in class DIII

$$H = \frac{1}{2}\int d^3 r \Psi^\dagger \mathcal{H} \Psi \quad \mathcal{H} = \begin{pmatrix} \xi & \Delta \\ \Delta^\dagger & -\xi \end{pmatrix}$$

$$\xi_{\boldsymbol{k}} = \frac{\boldsymbol{k}^2}{2m} - \mu \qquad \Delta_{\boldsymbol{k}} = |\Delta| i\sigma_y \boldsymbol{k}\cdot\boldsymbol{\sigma}$$



isotropic gap



stable surface Majorana fermion

3d analogue of Moore-Read state

$$\Psi(\{\boldsymbol{r}_i\},\{\sigma_i\}) \sim \mathrm{Pf}\left(\frac{[(\boldsymbol{r}_i - \boldsymbol{r}_j)\cdot i\boldsymbol{\sigma}\sigma_y]_{\sigma_i\sigma_j}}{|\boldsymbol{r}_i - \boldsymbol{r}_i|^3}\right)$$

summary

topological insulators and superconductors:
expanding the quantum Hall paradigm to much wider context

higher-dimensions, symmetry protected topological states

complete classification of topological phases in free fermion systems
in all dimensions and symmetry classes

some predictions:

- surface of 3d Z2 topological insulator: perfect metal

- 3He B is identified as a topological SC:

  stable gapless Majorana surface mode

- there are topological singlet SC with good T and in d=3 spatial dimensions

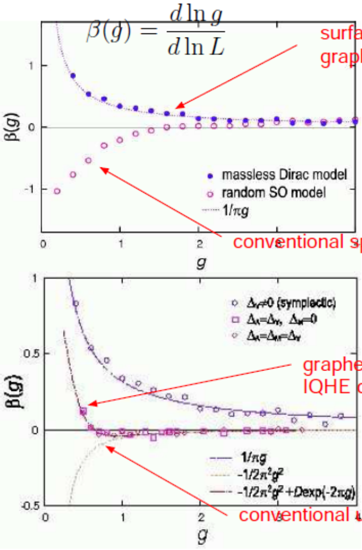
classification of D-branes

	D(−1)	D0	D1	D2	D3	D4	D5	D6	D7	D8	D9
type IIB	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$O9^-$ (type I)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$O9^+$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

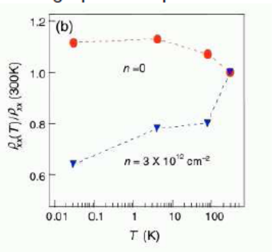
TABLE II: Dp-brane charges from K-theory, classified by $K(\mathbb{S}^{9-p})$, $KO(\mathbb{S}^{9-p})$ and $KSp(\mathbb{S}^{9-p})$ [11]. A $\mathbb{Z}_2$ charged Dp-brane with p even or p odd represents a non-BPS Dp-brane or a bound state of a Dp and an anti-Dp brane, respectively [12].

Tadashi Takayanagi's talk in Focus week (Fri)

simulation of surface topological metal by graphene



graphene experiment



Tan, Zhang, Stormer, Kim,
Eur. Phys. J (2007)

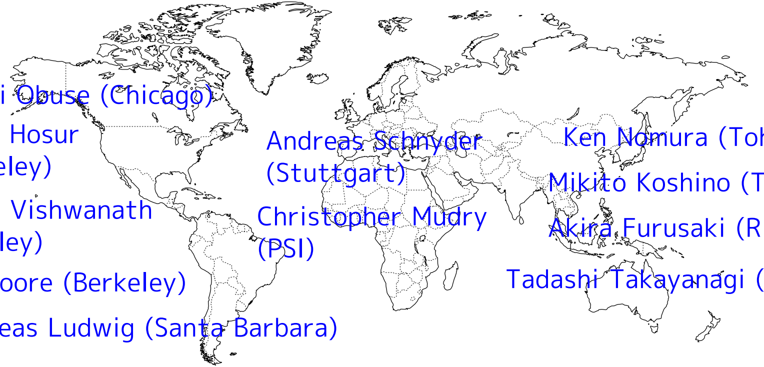
$$\mathcal{H} = -i\sigma_\mu \partial_\mu + V(r)\sigma_0$$

charge impurity

$$i\sigma_y \mathcal{H}^* (-i\sigma_y) = \mathcal{H}$$

"effective time reversal symmetry"

collaborators



- Hideaki Obuse (Chicago)
- Pavan Hosur (Berkeley)
- Ashvin Vishwanath (Berkeley)
- Joel Moore (Berkeley)
- Andreas Ludwig (Santa Barbara)
- Andreas Schnyder (Stuttgart)
- Christopher Mudry (PSI)
- Ken Nomura (Tohoku)
- Mikito Koshino (Titech)
- Akira Furusaki (RIKEN)
- Tadashi Takayanagi (IPMU)

refs:

- surface of topological insulators: SR, Obuse, Mudry, Furusaki, PRL (2007)
- numerical beta function: Nomura, Koshino, SR, PRL (2007)
- classification: Schnyder, SR, Furusaki, Ludwig, PRB (2008)
- many-body topological insulator: SR, PRB (2009)
- 3D topological singlet SC: Schnyder, SR, Ludwig, PRL (2009)
- topological insulators and duality: Hosur, SR, Vishwanath, PRB (2009)
- topological insulators and D-branes: SR, Takayanagi, arxiv (2010)