Deutsches Elektronen-Synchrotron
in der Helmholtz-Gemeinschaft

Solving Complex Projective Superspace

IPMU, February 2010

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Intro: QFT & Statistical Mechanics

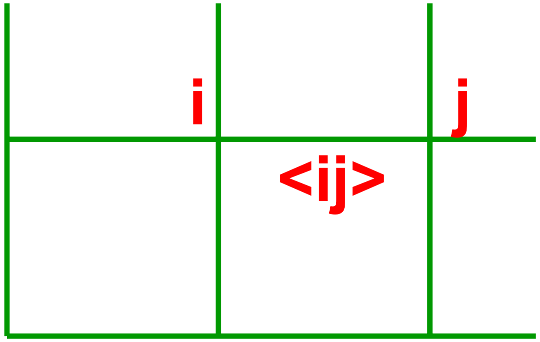
QFT in path integral formulation

System of statistical mechanics

$$S[\phi] \sim \frac{1}{g^2} \int d^D x (\partial_\mu \phi)^2$$

$$\text{lattice} \; \Big\downarrow$$

$$S_L(\{\phi_i\}) \sim \frac{L^D}{g^2} \sum_{\langle ij \rangle} \frac{(\phi_i - \phi_j)^2}{L^2}$$



$$\langle \prod_{a=1} \Psi_a(\{\phi_i\}) \rangle \sim \int \prod_i d\phi_i \; e^{-S} \prod_a \Psi_a$$

$$\langle \Psi(\phi_i)\Psi(\phi_j) \rangle \sim |x_i - x_j|^{-2\Delta_\Psi} + \dots$$

critical exponent

Intro: ST & Harmonic Analysis

Particle in S^1 with radius R: $f(x) = f(x+2\pi)$

$$\Delta f(x) = -R^{-2}\partial_x\partial_x f(x)$$

$$Z(q) = tr\left(e^{-\beta\Delta}\right) = \sum_n e^{-\beta\frac{n^2}{R^2}}$$

String on S^1: **$q = e^{-\beta}$**

momentum **winding**

$$\mathcal{Z}(q) \sim \sum_{n,w} \frac{e^{-\beta\left(\frac{n^2}{R^2}+\frac{R^2}{4}w^2\right)}}{\prod_{k=1}^{\infty}(1-e^{-\beta k})^2}$$

oscillations

T-duality: R → 2/R ; w ↔ n

Intro: Gauge/String duality

(SUSY) $U(N_c)$ Quantum gauge theories in 4D

↔ String theories in 5D Anti-deSitter geometry

[Polyakov] [Maldacena]

Observables ψ	↔	Stringy Harmonics
crit. exponents		Eigenvalues of Δ_S

Plan: 1 – AdS/CFT Correspondence

Gauge Theory, N=4 SYM, CFT; Strings in AdS, supercosets

2 – Stringy Harmonic Analysis

Calabi-Yau superspace $CP^{N-1|N}$; analytical & numerical res.

Quantum Gauge Theory

$$S^{YM}[A] \sim \frac{1}{g^2_{YM}} \int d^D x \, tr \left(F_{\mu\nu} F^{\mu\nu} \right)$$

$$F_{\mu\nu}(A) \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

N_c x N_c matrices

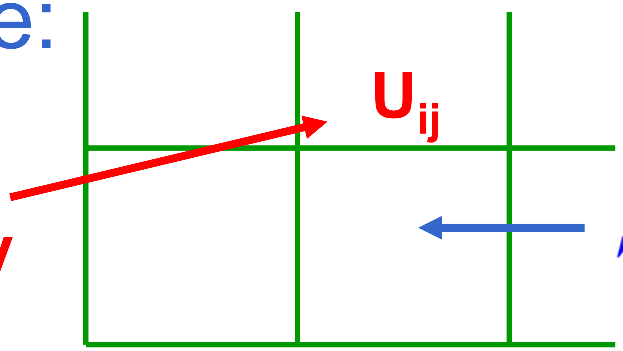
Observables are gauge invariants

$$\Psi(A) = \Psi(A^\Lambda)$$

$$A_\mu \rightarrow \Lambda^{-1} A_\mu \Lambda - \Lambda^{-1} \partial_\mu \Lambda$$

Lattice:

lattice holonomy



Weight of plaquette:

$$S_P \sim tr(U_{ij} U_{jk} U_{kl} U_{li})$$

N=4 Super Yang-Mills Theory

$$S^{N=4} \sim S^{YM} + \int d^D x \ tr\left((D_\mu \Phi_n)^2 + ([\Phi_n, \Phi_m])^2 \right) + \dots$$

same on all length scales

6 matrix valued scalars

→ Conformal Quantum Field Theory

Symmetries: U(4) ~ SO(6) \`R-symmetry' and 4D conformal group SO(2,4)

Poincare, Dilations, Special Conformal

combine with 4 x 4 fermionic symmetries into

Lie Supergroup PSU(2,2|4)

Conformal Quantum Field Theory

Recall: $\langle\Psi(x)\Psi(y)\rangle = |x-y|^{-2\Delta_\Psi}$

Δ_ψ depends on g_{YM} & N_c: $\Delta_\psi = \Delta_\psi^0 + \delta_\psi(\lambda,N_c)$

$\lambda = g_{YM}^2 N_c$

‘t Hooft coupling

classical dim

anomalous dim

δ(λ) - Hamiltonian of (Heisenberg) spin chain

$$\delta(\lambda)_{N_c\to\infty} \sim \lambda \sum_{i=1}^{m} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} + O(\lambda^2)$$

Planar limit

Pauli matrices

long range int.

[Minahan,Zarembo]

[Beisert et al.]...

Is there systematic way of computing $\delta(\lambda)_{N_c\to\infty}$ **?**

Maldacena's AdS/CFT duality

Conjecture: [Maldacena] N=4 SYM is dual to

String theory on **AdS_5 x S^5**

$x = (x_0,..x_3)$

$$ds^2 = \overbrace{\frac{r^2}{R^2}dx^2 + \frac{R^2}{r^2}dr^2} + \overbrace{R^2 d\Omega_5}$$

line element on S^5

Parameters are related by

$$N_c = (R/l_s)^4 g_s^{-1} \qquad \lambda = (R/l_s)^4$$

string coupling

string length

ST in AdS_5 x S^5 & Supercosets

Symmetries of AdS_5 x S^5 : SO(2,4) x SO(6)

bosonic

same as in Gauge Theory!

Construction of superstring on AdS_5 x S^5 involves coset superspace

[Metsaev,Tseytlin]
[Berkovits]

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

bosonic base is AdS_5 x S^5

admits action of PSU(2,2|4)

+32 fermionic coordinates

Summary of first part

AdS/CFT correspondence [Maldacena]

Stat. Mech. Sys.		String Geometry
Observables ψ	⟷	Stringy Harmonics
crit. exponents		Eigenvalues of Δ_S

Compute stringy spectrum, e.g. function Z(q), for superspaces G/H as function of radius R

G = U(N|N) .... **very difficult!**

Outlook to second part

$$G/G^{Z_2}$$

$$\frac{U(N|N)}{U(N-n|N) \text{ x } U(n)}$$

$U(N|N$

$U(N-1|N)x$

- note: c^v (P(2S+2|2S))

- Sam 4 SYM

Cl [Witten]

- Com rspace

Continuous families of c = -2 CFTs

hard to solve (no Kac-Moody sym.)

The Sigma Model on CP$^{N-1|N}$

$$CP^{N-1|N} = \left\{ (Z_\alpha) = (z_\alpha, \eta_\alpha) | \varrho^2 = \bar{Z}_\alpha Z^\alpha = 1 \right\} / U(1)$$

complex bosonic **fermion** $z_\alpha, \eta_\alpha \rightarrow \varpi z_\alpha, \varpi \eta_\alpha$

Action of CP$^{N-1|N}$ Sigma Model given by:

here: No ws SUSY

$$\mathcal{S}_{\mathrm{CP}} = \frac{R^2}{2\pi} \int_{\varrho^2=1} d^2z \left(D\bar{Z}_\alpha \bar{D} Z^\alpha + D Z_\alpha \bar{D} \bar{Z}^\alpha \right)$$

$$+ \frac{i\theta}{2\pi} \int d^2z \left(D\bar{a} - \bar{D} a \right)$$

***D* covariant derivative**

a **non-dynamical gauge field**

***θ* - term**

Family$_{(R,\theta)}$ of interacting CFTs with c = -2

Introduction: Some Motivation

- $\mathrm{CP}^{N-1|N}$ simplest example of CY superspace

single Kähler parameter **[Sethi] [Schwarz] ....**

- 1-parameter family of interacting CFTs with cont. varying exponents

solvable; but no current algebras

- Coset with U(N|N) sym. ↔ Strings in AdS

$$CP^{N-1|N} = U(N|N)/U(N-1|N) \times U(1)$$

- Cont. limit of alternating U(N|N) spin chain

intersecting loop model, polymers

Main Results and Plan of Talk

Exact formula for the boundary partition fcts

complex Kähler modulus t

complex line bundles k_1 k_2

$$Z^{t;N=2}_{k_1k_2}(q)$$

modular parameter

of volume filling branes with bundles $\mathcal{O}(k)$

background field exp, cohomological reduction, lattice model

I Warmup: $CP^{0|1}$ - the bc ghost system

II $CP^{1|2}$ – continuum analysis & numerics

III Conclusions and some open problems

I.1 $CP^{0|1}$ and bc ghost system

Solve constraint $\rho^2 = 1$ in terms of fermions ξ^a

$$S \sim R^2 \int d^2z \left(\partial\xi^+\bar{\partial}\xi^- + \partial\xi^-\bar{\partial}\xi^+\right) + i\theta\left(\partial\xi^+\bar{\partial}\xi^- - \partial\xi^-\bar{\partial}\xi^+\right)$$

- Free field theory with c = - 2
- Dependence on *R*,*θ* is trivial
- Has affine psu(1|1) sym: $F^{\pm}(z) = \partial\xi^{\pm}(z,\bar{z})$

not true for N ≠ 1 !

I.2 $CP^{0|1}$ - The boundary theory

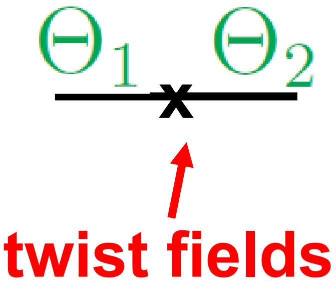
$$S \sim S_{\mathrm{bulk}} + \vartheta \int dx \xi^+ \partial_x \xi^-$$

boundary term

Implies twisted Neumann boundary conditions:

$$R^2 \partial_y \xi^\pm = \pm \Theta \partial_x \xi^\pm \qquad \Theta \sim \theta + \vartheta$$

Result: **[Creutzig,Quella,VS] [Creutzig,Roenne]**



Pair of ground states $\Delta^0_\lambda = \frac{\lambda(\lambda-1)}{2}$

Pair of excited states $\Delta^1_\lambda = \Delta^0_\lambda + \lambda$

$$\cos 2\pi\lambda_R(\Theta_1, \Theta_2) = \frac{(R^4 + \Theta_1\Theta_2)^2 - (\Theta_1 - \Theta_2)^2 R^4}{(R^4 + \Theta_1\Theta_2)^2 + (\Theta_1 - \Theta_2)^2 R^4}$$

I.3 The boundary partition function

- c/24 **ground states** $F^{+}_{-n-\lambda}$ $F^{-}_{-n-1+\lambda}$

$$Z^{t;N=1}_{\Theta_1\Theta_2} = q^{\frac{1}{12}+\Delta^0_\lambda}(1+x^{-1})\prod_{n=0}^{\infty}(1+xq^{\lambda+n})(1+x^{-1}q^{1-\lambda+n})$$

$$= q^{\Delta^0_\lambda}\sum_{m\in\mathbb{Z}}\frac{q^{\frac{1}{12}}}{\phi(q)}\,q^{\frac{m(m+1)}{2}+m\lambda}\,x^m(1+x^{-1}) = q^{\Delta^0_\lambda}\sum_{m\in\mathbb{Z}}\psi_m(q)\,q^{m\lambda}\chi_m(x)$$

Jacobi triple id **branching functions** **pu(1|1) characters**

$R^2 \rightarrow \infty$ η_{-n} $\bar{\eta}_{-n}$

$$\longrightarrow q^{\frac{1}{12}}\phi(q)\underbrace{\int\frac{du}{u}\phi(q)}_{\text{U(1) gauging}}\underbrace{\lim_{s\to 1}(1-s^2)}_{\text{constraint}}\prod_{n=0}^{\infty}\frac{(1+x^{-\frac{1}{2}}u^{\frac{1}{2}}q^n)(1+x^{\frac{1}{2}}u^{-\frac{1}{2}}q^n)}{(1-x^{\frac{1}{2}}u^{\frac{1}{2}}q^n)(1-x^{-\frac{1}{2}}u^{-\frac{1}{2}}q^n)}$$

q^0 = s z_{-n} $\bar{z}_{-n}$

I.4 bc ghost system - Summary

$\Delta^0_\lambda = \frac{1}{2}\lambda(\lambda-1)$ with $\lambda = \lambda_R(\Theta_1,\Theta_2)$

moduli dependence

$$q^{\Delta^0_\lambda}\sum_{m\in\mathbb{Z}} \frac{q^{\frac{1}{12}}}{\phi(q)}\, q^{\frac{m(m+1)}{2}+m\lambda}\, x^m(1+x^{-1}) = q^{\Delta^0_\lambda}\sum_{m\in\mathbb{Z}}\psi_m(q)\, q^{m\lambda}\chi_m(x)$$

$$\longrightarrow\; q^{\frac{1}{12}}\phi(q)\int\frac{du}{u}\phi(q)\lim_{s\to1}(1-s^2)\prod_{n=0}^{\infty}\frac{(1+x^{-\frac{1}{2}}u^{\frac{1}{2}}q^n)(1+x^{\frac{1}{2}}u^{-\frac{1}{2}}q^n)}{(1-x^{\frac{1}{2}}u^{\frac{1}{2}}q^n)(1-x^{-\frac{1}{2}}u^{-\frac{1}{2}}q^n)}$$

- Moduli dependence only through function λ **twist fct**
- Branching fcts ψ_m can be computed at R = ∞ ← **2nd line**
- Exponent mλ depends on pu(1|1) label *m* and λ **linear**

II Spectrum of σ-Model on CP$^{1|2}$

[Candu,Mitev,Quella,VS,Saleur]

$$Z^{t}_{k_1,k_2} = q^{\frac{1}{2}\lambda_t(\lambda_t-1)} \sum \psi^{K}_{\Lambda}(q)\, q^{\frac{\lambda_t}{2|k_1-k_2|}\delta C^{(2)}(\Lambda)}\, \chi^{K}_{\Lambda}(x,y,z)$$

- **Obtained by summing all order perturbative expansion**

  possible because of target space SUSY
- **Tested through extensive numerical lattice simulations**

II Spectrum of σ-Model on $CP^{1|2}$

[Candu,Mitev,Quella,VS,Saleur]

$$Z^t_{k_1,k_2} = q^{\frac{1}{2}\lambda_t(\lambda_t-1)} \sum \psi^K_\Lambda(q)\, q^{\frac{\lambda_t}{2|k_1-k_2|}\delta C^{(2)}(\Lambda)}\, \chi^K_\Lambda(x,y,z)$$

Character $\chi_\Lambda = \chi_\Lambda(x,y,z)$ of representation Λ of pu(2|2)

k_1-k_2 determines value of central element in u(2|2)

II Spectrum of σ-Model on $\mathrm{CP}^{1|2}$

[Candu,Mitev,Quella,VS,Saleur]

$$Z^{t}_{k_1,k_2} = q^{\frac{1}{2}\lambda_t(\lambda_t-1)} \sum \psi^{K}_{\Lambda}(q)\, q^{\frac{\lambda_t}{2|k_1-k_2|}\delta C^{(2)}(\Lambda)}\, \chi^{K}_{\Lambda}(x,y,z)$$

Branching fcts at R = ∞ from decomposition of

$$Z^{R=\infty}_{k_1k_2} = q^{\frac{1}{12}}\phi(q) \oint \frac{du}{u^{|k_1-k_2|/2+1}}\phi(q) \lim_{s\to 1}(1-s^2) \prod_{n=0}^{\infty} \prod_{\alpha\beta=\pm\frac{1}{2}} \frac{1+y^{\alpha}(zu^{-1})^{\beta}q^{n}}{1-x^{\alpha}(zu)^{\beta}q^{n}}$$

explicitly known for N=2

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Value of Quadratic Casimir in representation of pu(2|2)

Background field expansion implies:

Casimir evolution of weights is typical for G/H with c^{V} (G) = 0

[Bershadsky, Zhukov, Vaintrob] ......
[Quella,VS,Creutzig] [Candu, Saleur]

II.2$_1$ Casimir Evolution

Free Boson:

In boundary theory

bulk more involved

$$Z_R(z,q) = \sum_{n\in\mathbb{Z}} z^n q^{\frac{n^2}{2R^2}} \eta^{-1}(q)$$

Prop.: Boundary spectra of CP$^{1|2}$ chiral field :

$$\Delta^R_\Phi = \Delta^0_\Phi + f(R) C^{(2)}_\Phi$$

← **quadratic Casimir**

Deformation of conf. weights is `quasi-abelian'

[Bershadsky et al] [Quella,VS,Creutzig] [Candu, Saleur]

e.g. (1+14+1) remains at Δ=0; 48, 80, ... are lifted

II.2_2 Casimir Evolution

Free Boson:

$$\Delta^R_\Phi = \Delta^0_\Phi + f(R) g^2_\Phi$$

at $R=R_0$ universal U(1) charge

In boundary theory

bulk more involved

<u>Prop.:</u> Boundary spectra of $CP^{1|2}$ chiral field :

$$\Delta^R_\Phi = \Delta^0_\Phi + f(R) C^{(2)}_\Phi$$

quadratic

Casimir

Deformation of conf. weights is `quasi-abelian’

[Bershadsky et al] [Quella,VS,Creutzig] [Candu, Saleur]

e.g. (1+14+1) remains at Δ=0; 48, 80, ... are lifted

II Spectrum of σ-Model on $CP^{1|2}$

[Candu,Mitev,Quella,VS,Saleur]

$$Z^t_{k_1,k_2} = q^{\frac{1}{2}\lambda_t(\lambda_t-1)} \sum \psi^K_\Lambda(q)\, q^{\frac{\lambda_t}{2|k_1-k_2|}\delta C^{(2)}(\Lambda)}\, \chi^K_\Lambda(x,y,z)$$

λ_t is universal (depends only on t, k_1,k_2)

$$\cos 2\pi\lambda_t(k_1,k_2) = \frac{(R^4+\Theta_1\Theta_2)^2-(\Theta_1-\Theta_2)^2R^4}{(R^4+\Theta_1\Theta_2)^2+(\Theta_1-\Theta_2)^2R^4}$$

$$t = R^2 + i\theta$$

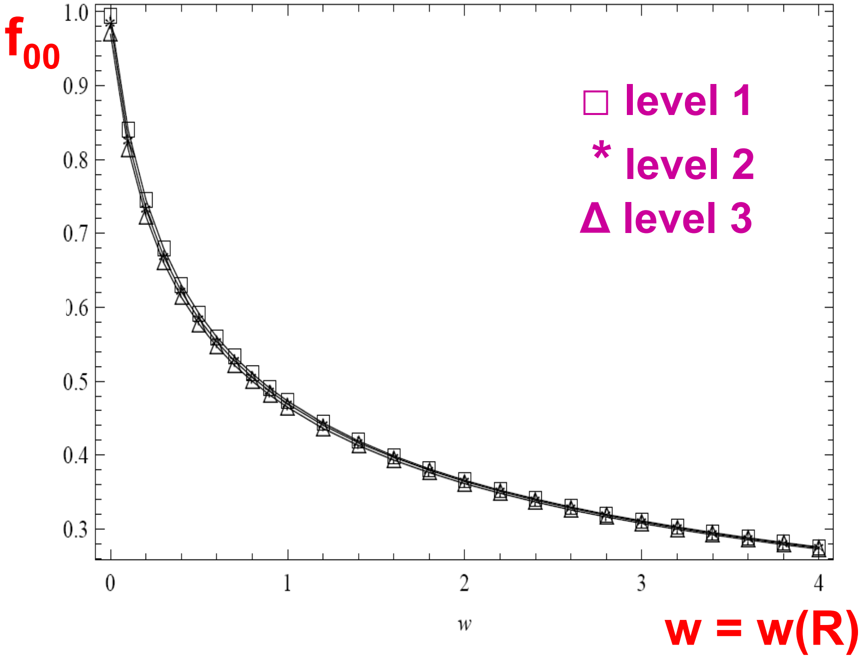
$$\Theta_i = 2k_i + \theta/\pi$$

← **Cohomological reduction: S^N = S^1 + Q F**

II Spectrum of σ-Model on $CP^{1|2}$

[Candu,Mitev,Quella,VS,Saleur]

$$Z^t_{k_1,k_2} = q^{\frac{1}{2}\lambda_t(\lambda_t-1)} \sum \psi^K_\Lambda(q)\, q^{\frac{\lambda_t}{2|k_1-k_2|}\delta C^{(2)}(\Lambda)}\, \chi^K_\Lambda(x,y,z)$$



Value of Quadratic Casimir in representation of pu(2|2)

For θ = - π and $k_1 = 0 = k_2$

$$f_{00} = (h_l - h_0)/\delta C^{(2)}$$

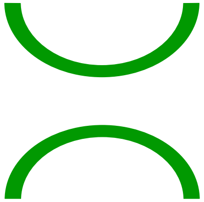
II.3$_1$ A discrete model for $CP^{N-1|N}$

U(N|N) spin chain on $V_f \otimes \bar{V}_f \otimes \cdots \otimes V_f \otimes \bar{V}_f$

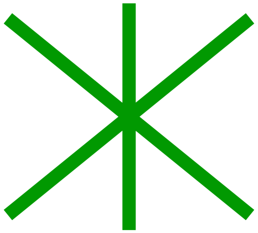
fundamental rep

$$H_w = -\sum_{j=1}^{2L} E_j - w \sum_{j=1}^{2L-1} P_{j,j+2}$$

$$E_j(v_j \otimes v_{j+1}) = (v_j, v_{j+1}) \sum_a e_j^a \otimes e_{j+1}^{\bar{a}}$$



$$P_{j,j+2}(v_j \otimes v_{j+1} \otimes v_{j+2}) = v_{j+2} \otimes v_{j+1} \otimes v_j$$



(Only) for N = 1 this spin chain is integrable

$$Z(q) = \lim_{\left(\substack{L\to\infty \\ N\to\infty}\right)} tr\, e^{-NH} \xrightarrow{N=1}$$

$N/L = \beta$

$q = e^{-\beta}$

$\varepsilon(p) = v_f(w) \sin p$

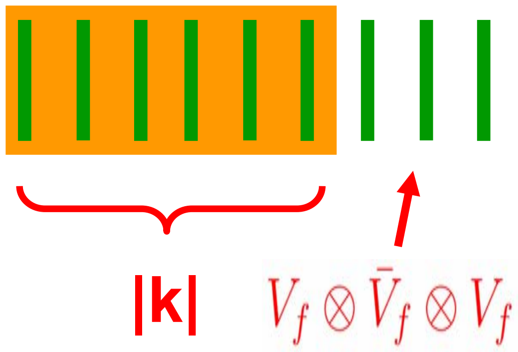
$p = \frac{\pi s}{L}; 0 \leq s \leq L$

Partition function of bc ghosts

II.3$_2$ Boundary Conditions

BC of continuum theory ↔ line bundle $\mathcal{O}(k)$

Idea: Introduce boundary layer



$$V_f^k = \begin{cases} V_f^k & \text{for k} \geq 0 \\ \bar{V}_f^{-k} & \text{for k} < 0 \end{cases}$$

$$H_b^L = \begin{cases} w_b \sum_{a=1}^{|k|-1} P_{a,a+1} + w_b P_{|k|,|k|+1} \\ w_b \sum_{a=1}^{|k|-1} P_{a,a+1} + w_b P_{|k|,|k|+2} \end{cases}$$

III Open Problems & Directions

Integrable CFT: $CP^{N-1|N}$ provides a first step

Boundary spectrum known **Numerical evaluation possible**

What about the bulk spectrum ?

Is there Gepner/WZ point in moduli space ?

~ supersphere – GN duality [Candu et al][Mitev et al]

$Z(R_0)$ given by characters of affine psu(2|2) ?

Extension to $CP^{N-1|N}$ with N=2 ws SUSY ?

↔ N=4 SYM / twistor string [Witten]

Derived category / stability ?

Extension to non-compact target ? **↔ AdS ?**