Cardy formula of 4d N=1 SCFT

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Cardy Formula

- When we study a conformal field theory, it is natural to consider the partition function of the canonical ensemble:

$$Z(\beta) \equiv \mathrm{tr}_{\mathcal{H}}\left(e^{-\beta H}\right) = \sum_{i\in\mathcal{H}} e^{-\beta E_i}$$

weighted sum over energy eigenstates

- Temperature of the canonical ensemble controls the weight factor.

$$\beta = T^{-1}$$

At low temperature, all high energy states are suppressed by $e^{-\beta E} \ll 1$
At high temperature, $e^{-\beta E} \to 1$.

- High temperature asymptotics of the partition function becomes a good estimate for the number of microstates in a given CFT.

$$Z(\beta) \xrightarrow{\beta\to 0} \exp\left(\frac{\pi^2 c}{3\beta}\right)$$

for 2d CFT with central charge c.
known as Cardy's formula. [Cardy'86]

Cardy Formula

- We are interested in the similar, high 'temperature' asymptotics of the superconformal index, in 4d CFTs with N=1 supersymmetry.

  This observable enumerates all BPS microstates preserving some chosen supercharges.
  It is the Witten index, counting (# of bosonic states) - (# of fermionic states).

- The corresponding Cardy-like formula: [Di Pietro, Komargodski '14]

$$\mathcal{I}(\omega) \xrightarrow{\omega \to 0} \exp\left(\frac{16\pi^2}{3\omega}(c-a) + \cdots \right)$$

[Ardehali '15]
[Di Pietro, Honda '16]

- Notice that the asymptotic free energy captures much smaller d.o.f. than the d.o.f. counted by conformal anomalies, a and c.

  For example, in N=4 SYM theory, $a = c \sim |G|$ for any choice of gauge group G.

Deconfinement Phase Transition

- The high ‘temperature’ asymptotics seems boring, in contrast to the expected thermodynamic behavior of the large N gauge theory.

  Low-temperature phase can be thought as a gas of glueballs and mesons. $F \sim \mathcal{O}(N^0)$

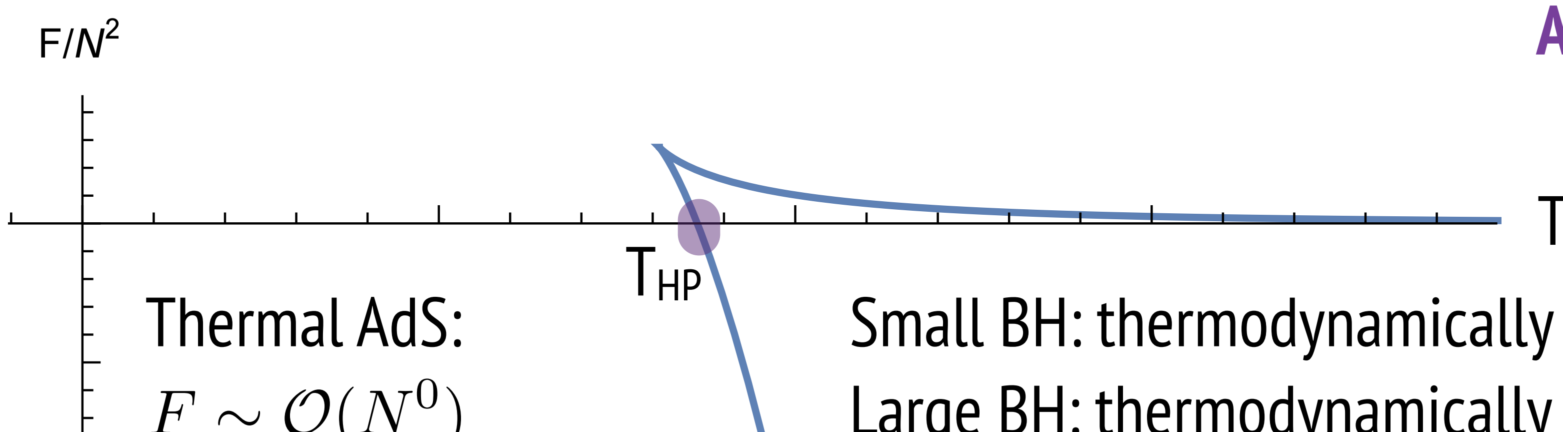
  High-temperature phase is a plasma of gluons and quarks. $F \sim \mathcal{O}(N^2)$

- Two phases are connected by the *deconfinement* phase transition, dual to the Hawking-Page transition of AdS black holes.

  [Witten ’98] [Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk]

AdS-Schwarzschild BH solution

[Hawking, Page]



Thermal AdS: $F \sim \mathcal{O}(N^0)$

Small BH: thermodynamically unfavored.

Large BH: thermodynamically preferred for above T_{HP}.

$$F \sim \mathcal{O}(G_N^{-1}) \sim \mathcal{O}(N^2)$$

Complex Fugacity

- Apparently, the deconfining (or black hole) phase seems invisible to the superconformal index, due to the boson/fermion cancelation.

  Witten index = (# of bosonic states) - (# of fermionic states).

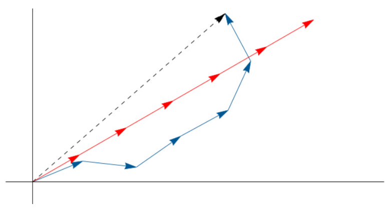
- But the index still counts *big enough* degeneracy of microstates!

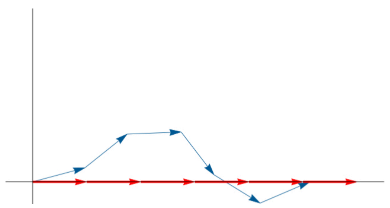
$$1 + 3x^2 - 2x^3 + 9x^4 - 6x^5 + 11x^6 - 6x^7 + 9x^8 + 14x^9 - 21x^{10} + 36x^{11} - 17x^{12} - 18x^{13} + 114x^{14} - 194x^{15}$$
$$+ 258x^{16} - 168x^{17} - 112x^{18} + 630x^{19} - 1089x^{20} + 1130x^{21} - 273x^{22} - 1632x^{23} + 4104x^{24} - 5364x^{25} + 3426x^{2}$$
$$+ 3152x^{27} - 13233x^{28} + 21336x^{29} - 18319x^{30} - 2994x^{31} + 40752x^{32} - 76884x^{33} + 78012x^{34} - 11808x^{35} + \cdots$$

for N=4 U(2) SYM

- When we naively study the asymptotic behavior of the index, those degeneracies add up with rapidly alternating signs.

- Such B/F cancelation might be avoided by introducing the relative phase factors.





Plan

- Introduction
- N=4 Superconformal Index Revisited
- Cardy Formula of 4d N=1 Superconformal Index
- Asymptotic Entropy and AdS_5 Black Hole

Superconformal Index

[Romelsberger] [Kinney, Maldacena, Minwalla, Raju]

- Enumerates all BPS states preserving supercharges: $\mathcal{Q}^{+++}_{--}$, $\mathcal{S}^{---}_{++}$
- Chemical potentials for SO(4) isometry and SO(6) R-symmetry.

$$\mathrm{Tr}\Big[e^{-\beta\mathcal{E}}\prod_{I=1}^{3}e^{-\Delta_I Q_I}\prod_{i=1}^{2}e^{-\omega_i J_i}\Big] \qquad \text{with} \qquad \mathcal{E}=\{\mathcal{Q}^{+++}_{--},\mathcal{S}^{---}_{++}\}$$

- Become a Witten index after imposing $\sum_{I=1}^{3}\Delta_I=\sum_{i=1}^{2}\omega_i+2\pi i$.

$$\{e^{-\Delta_I Q_I-\omega_i J_i},\mathcal{Q}^{+++}_{--}\}=0$$
$$\{e^{-\Delta_I Q_I-\omega_i J_i},\mathcal{S}^{---}_{++}\}=0$$

- Independent of the regulator β, so we formally take the limit β → 0.
- Can be evaluated from the free QFT calculus.

$$Z=\oint[d\alpha]\cdot\exp\Bigg[\sum_{a,b=1}^{N}\sum_{n=1}^{\infty}\frac{1}{n}\Bigg(1+\sum_{s_1,s_2,s_3=\pm1}\frac{s_1s_2s_3(-1)^{n-1}e^{\frac{ns_I\Delta_I}{2}}}{2\sinh\frac{n\omega_1}{2}\cdot2\sinh\frac{n\omega_2}{2}}\Bigg)e^{in\alpha_{ab}}\Bigg]$$

Saddle Point Analysis

- Introduce the large N eigenvalue distribution. $\rho(x) = \frac{1}{N}\sum_{a=1}^{N}\delta(x-\alpha_a)$
- The index can be expressed as:

$$Z = \int \prod_{n\neq 0} d\rho_n \, \exp\left(-\sum_{n=1}^{\infty}\frac{N^2}{n}\frac{\prod_I(1-e^{-n\Delta_I})}{\prod_i(1-e^{-n\omega_i})}\rho_n\rho_{-n}\right)$$

  - Polyakov loop: an order parameter for (de)confinement.

$$\rho_n = \rho^*_{-n} = \frac{1}{2\pi N}\sum_{a=1}^{N} e^{-in\alpha_a}$$

- Apply the saddle point approximation. [Kinney, Maldacena, Minwalla, Raju]
  - At *real-valued* chemical potentials, $\rho_n = 0$ is most dominant.
  - Gaussian integral → the super-graviton spectrum on $AdS_5 \times S^5$.

Saddle Point Analysis

- That saddle point can be unstable with complex chemical potentials.

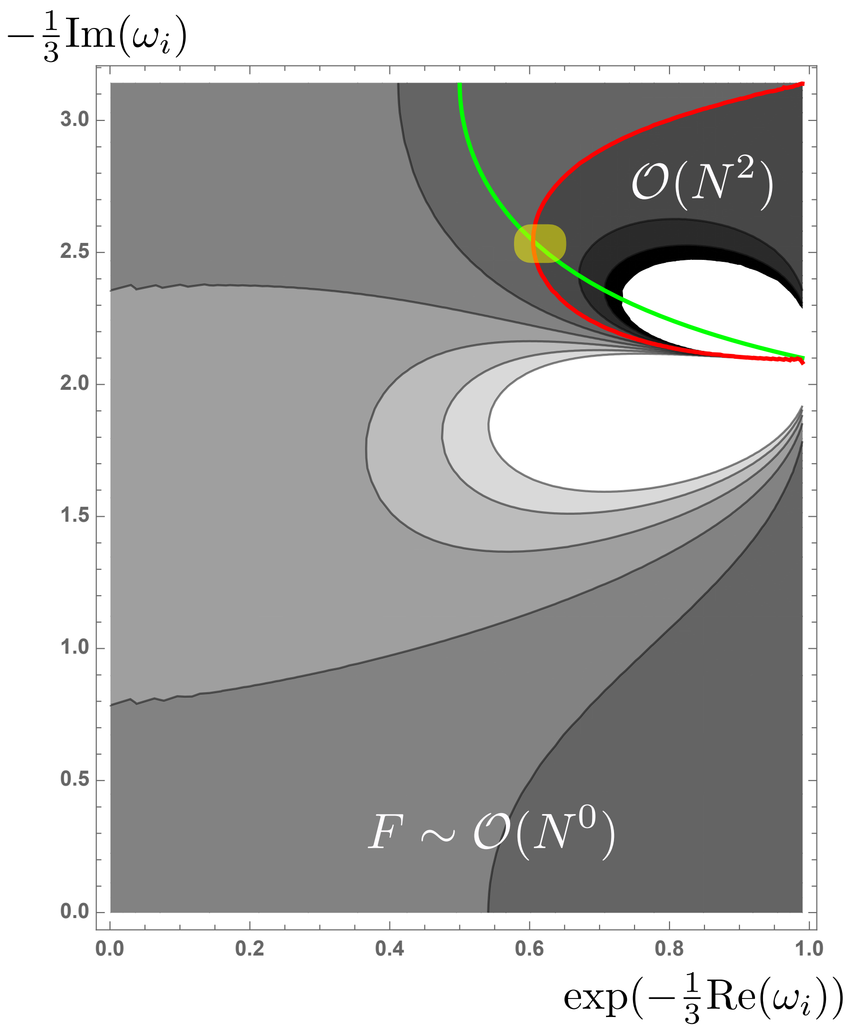
$$\mathrm{Re}\left[\frac{\prod_I (1-e^{-\Delta_I})}{\prod_i (1-e^{-\omega_i})}\right] < 0$$

- Let's inspect the above function at $\Delta_1 = \Delta_2 = \Delta_3$ and $\omega_1 = \omega_2$

Red line: where the function value becomes negative.

The instability arises only when imaginary value of chemical potentials is non-zero, within a certain range.

[Choi, **JK**, Seok Kim, Nahmgoong '18]



- Lesson learned:

  The Cardy formula can be improved by *complexifying* the chemical potentials.

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Superconformal Index

[Romelsberger] [Kinney, Maldacena, Minwalla, Raju]

- Consider the superconformal index for an arbitrary 4d N=1 SCFT.

$$\mathrm{Tr}_{\mathcal{H}}\left[e^{-\beta\mathcal{E}}e^{-\omega_1 J_1}e^{-\omega_2 J_2}e^{-\Delta R}\right] \quad \text{with} \quad \mathcal{E}=\{\mathcal{Q}_-,\mathcal{Q}^{\dagger -}\}$$

The relevant R-symmetry group is now only U(1).
There can be an extra flavor symmetry, for which we may introduce the chemical potentials.

- Become a Witten index only after imposing $\omega_1+\omega_2-2\Delta=2\pi i$.
- Independent of the regulator β, so we formally take the limit β → 0.

- After all, the index can be written in a slightly modified form:

$$\mathcal{I}=\mathrm{Tr}_{\mathcal{H}}\left[e^{\pi i R}\,e^{-\omega_1(J_1+R/2)}e^{-\omega_2(J_2+R/2)}\right]$$

The usual $(-1)^F$ is replaced with $(-1)^R$. Still qualifies as a Witten index since $[R,\ \mathcal{Q}_-]=\mathcal{Q}_-$

Superconformal Index

- Each multiplet's contribution is dressed by the phase factor $e^{i\pi r_{\mathcal{X}}}$.

  $r_{\mathcal{X}}$: R-charge of the top component in a given SUSY multiplet.

  This improves the high 'temperature' asymptotics of the superconformal index.

- For models with a flavor symmetry, $(-1)^R$ amounts to using $(-1)^F$ and shifting a flavor chemical potential by a suitable imaginary value.

- The index can be evaluated from the free QFT calculus.

$$\mathcal{Z}_V = \exp\left[\sum_{n\geq 1}\frac{1}{n}\left(1+\frac{(-1)^n\,2\sinh(n\Delta)}{2\sinh(n\omega_1/2)\,2\sinh(n\omega_2/2)}\right)\cdot\chi_{\mathbf{adj}}(n\alpha)\right]$$

$$\mathcal{Z}_{\mathcal{X}} = \exp\left[\sum_{n}\frac{(-1)^{n-1}}{n}\sum_{w\in\mathbf{R}}\left(\frac{t^{n(r_{\mathcal{X}}-1)}e^{inw(\alpha)}-t^{n(-r_{\mathcal{X}}+1)}e^{-inw(\alpha)}}{2\sinh(n\omega_1/2)\,2\sinh(n\omega_2/2)}\right)\right]$$

$$\longrightarrow \quad \mathcal{I} = \int [d\vec{\alpha}]\cdot\mathcal{Z}_V\prod_{\mathcal{X}}\mathcal{Z}_{\mathcal{X}}$$

Asymptotic Free Energy

- We study the asymptotics of the superconformal index in $|\omega_{1,2}| \ll 1$ in which microstates with *large angular momentum* are dominant.

  Abuse of terminology: I will loosely call it the high 'temperature' limit, or the Cardy limit.
  Here the 'temperature' means the inverse of the chemical potential $\sim |\omega^{-1}|$

- In the Cardy limit, the contribution from short multiplets becomes

$$\mathcal{Z}_V \to \exp\left(+ \sum_{s=\pm} \frac{s}{\omega_1\omega_2} \sum_{\rho\in\Delta_G} \mathrm{Li}_3(-e^{s(\Delta+i\rho\cdot\alpha)}) \right)$$

$$\mathcal{Z}_{\mathcal{X}} \to \exp\left(- \sum_{s=\pm} \frac{s}{\omega_1\omega_2} \sum_{w\in\mathbf{R}} \mathrm{Li}_3(-e^{s(1-r_{\mathcal{X}})\Delta+isw(\alpha)}) \right)$$

- Quite different from the asymptotics without the phase factor $e^{i\pi r_{\mathcal{X}}}$!

$$\mathcal{Z}_V \to \exp\left(\mathcal{O}(\omega^0)\right), \quad \mathcal{Z}_{\mathcal{X}} \to \exp\left(\frac{\pi^2}{6}\frac{\omega_1+\omega_2}{\omega_1\omega_2}\cdot(1-r_{\mathcal{X}})|\mathbf{R}_{\mathcal{X}}| + \mathcal{O}(\omega^0)\right)$$

Saddle Point Analysis

- We apply the saddle point approximation to the holonomy integral.

$$\mathcal{I} = \int [d\alpha]\ \exp\left[\sum_{s=\pm} \frac{s}{\omega_1\omega_2}\Big(\sum_{\rho\in\Delta_G} \mathrm{Li}_3(-e^{s(\Delta+i\rho\cdot\alpha)}) - \sum_{w\in\mathbf{R}} \mathrm{Li}_3(-e^{s(1-r_{\mathcal{X}})\Delta+isw(\alpha)})\Big)\right]$$

Looking for the most dominant saddle points, especially for the high temperature regime!

See also [Benini, Milan '18] [Cabo-Bizet, Cassani, Martelli, Murthy '19]

- Intuitively, the most dominant saddle point should be at the origin.

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{|G|} = 0$$

When the holonomy variable gets a non-zero value, gauge symmetry is partially broken.

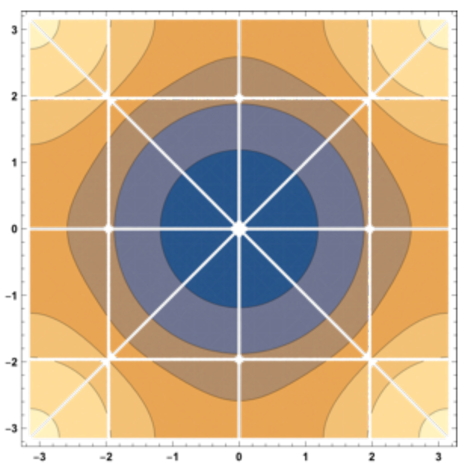
- High temperature behavior of an asymptotic-free gauge theory is deconfining (e.g., quark, gluon), rather than confining or Higgsed.

We expect the (modified) superconformal index to see the deconfining phase.

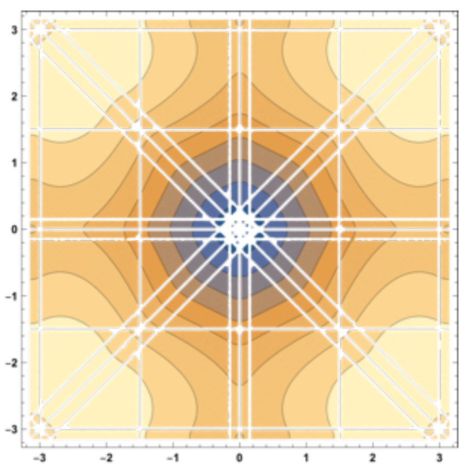
Saddle Point Analysis

- We numerically test our conjecture across various N=1 examples.

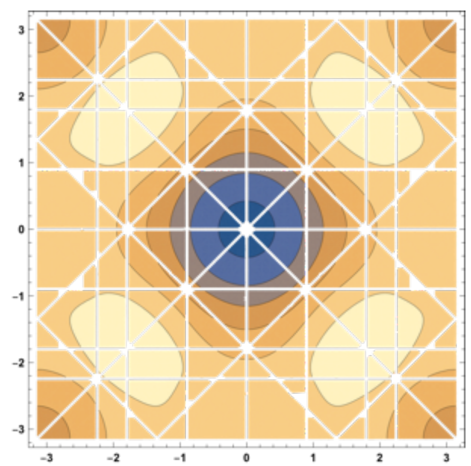
  SQCDs, N=4 SYM, N=1 with 2 adj., Argyres-Douglas theories, ISS, BCI, $SU(2)^3$ with trifund.



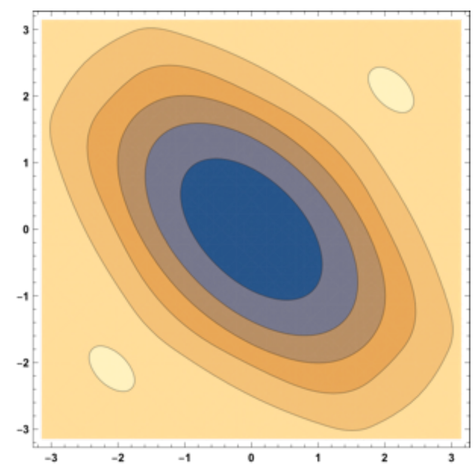
SO(5) Nf=8



(A_1, A_4) AD-theory



BCI Model



SU(3) Nf=7

- For all tested examples, the dominant saddle point is at the origin.

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{|G|} = 0$$

We assume this will be true also for other N=1 theories.

See also [Honda '19] [Ardehali '19]

Asymptotic Free Energy

- Inserting the saddle point value back to the integral, we find

$$\log(\mathcal{I}) = \sum_{s=\pm} \frac{s}{\omega_1\omega_2}\Big(|G|\,\mathrm{Li}_3(-e^{s\Delta}) - \sum_{\mathcal{X}} |\mathbf{R}_{\mathcal{X}}|\,\mathrm{Li}_3(-e^{s(1-r_{\mathcal{X}})\Delta})\Big)$$

$$\mathrm{Li}_3(-e^{x}) - \mathrm{Li}_3(-e^{-x}) = -\frac{x^3}{6} - \frac{\pi^2 x}{6}$$

$$\mathrm{Tr}\,R^3 = |G| + \sum_{\mathcal{X}}(r_{\mathcal{X}}-1)^3|\mathbf{R}_{\mathcal{X}}| = \tfrac{16}{9}(5a-3c)\ ,$$

$$\mathrm{Tr}\,R^1 = |G| + \sum_{\mathcal{X}}(r_{\mathcal{X}}-1)|\mathbf{R}_{\mathcal{X}}| = 16(a-c)\ ,$$

$$\log(\mathcal{I}) = \mathrm{Tr}\,R^3\,\frac{\Delta^3}{6\omega_1\omega_2} + \mathrm{Tr}\,R\,\frac{\pi^2\Delta}{6\omega_1\omega_2} = \frac{8(5a-3c)}{27\omega_1\omega_2}\Delta^3 + \frac{8\pi^2(a-c)}{3\omega_1\omega_2}\Delta.$$

[**JK**, Seok Kim, Song ‘19] [Cabo-Bizet, Cassani, Martelli, Murthy ’19]

Alternative approach to the Cardy free energy is the background field method on S^3.
The anomaly coefficient naturally appears here due to the ’t Hooft anomaly matching.

[Banerjee et al.’12] [Jensen et al.’13] …. [Di Pietro, Komargodski ’14] … [Choi, **JK**, Seok Kim, Nahmgoong ’18]

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Asymptotic Entropy

- Given the superconformal index, the microstate degeneracy is extracted by taking an inverse Laplace transformation.

$$\Omega(R, J_i) = \int d\Delta \, d\omega_i \; \mathcal{I}(\Delta, \omega_i) \; \exp\left(\Delta R + \textstyle\sum_i \omega_i J_i\right)$$

- In particular, the asymptotic entropy (degeneracy) at large charges comes from the saddle point approximation of the above integral.
- This is the Legendre transformation of the Cardy free energy:

[Hosseini, Hristov, Zaffaroni '17]

$$S(R, J_i) = \frac{8(5a-3c)}{27\omega_1\omega_2}\Delta^3 + \frac{8\pi^2(a-c)}{3\omega_1\omega_2}\Delta + R\Delta + J_1\omega_1 + J_2\omega_2 \Bigg|_{\Delta^*, \omega^*}$$

$$(\text{with} \quad \omega_1 + \omega_2 - 2\Delta = 2\pi i)$$

Asymptotic Entropy

- For a 4d N=1 SCFT, the asymptotic entropy is given by

$$\mathrm{Re}(S) = +2^{1/3}\, 3^{1/2}\, (3c - 2a)^{1/3}\, \pi \cdot J^{2/3} + \mathcal{O}(J^{1/3}) > 0$$

Notice that an interacting N=1 SCFT satisfies the Hofman-Maldacena bound: $\frac{1}{2} < a/c < \frac{3}{2}$

- Does this asymptotic entropy saturate the upper bound, i.e., the true entropy that counts the BPS states without $(-1)^F$? No.

Compute the true BPS degeneracy for free chiral/vector theories, from their BPS partition function (as opposed to the Witten index).

$$S^{\mathrm{true}}(J_1, J_2) = \frac{7\zeta(3)}{4\omega_1\omega_2} + J_1\omega_1 + J_2\omega_2 \bigg|_{\omega_i = \omega_i^*} \simeq\ 4.467\,(J_1 J_2)^{1/3}.$$

Compare with the asymptotic entropy from the index:

$$\mathrm{Re}(S) = 2.995\, J^{2/3} \qquad \text{(a free chiral multiplet)}$$
$$\mathrm{Re}(S) = 0 \qquad \text{(a free vector multiplet)}$$

AdS_5 Black Hole

- Consider the asymptotic free energy of holographic SCFTs.

  Example: N=1 quiver theory engineered from N D3-branes on C^3 / Z_{2p} orbifold.

- Restoring flavor chemical potentials, the entropy function becomes

  N=4 SYM [Hosseini, Hristov, Zaffaroni '17] / N=2 quiver [Honda '19] / N=1 quiver [**JK**, Seok Kim, Song '19] [Amariti et al. '19]

$$S = pN^2 \cdot \frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2} + (R-F)\Delta_1 + (R-\tilde{B})\Delta_2 + (R+F+\tilde{B})\Delta_3 + \sum_{i=1}^{2} J_i\omega_i$$

- Imposing the charge relation $\mathrm{Im}(S) = 0$ of known black holes:

$$S = 2\pi\sqrt{3R^2 - F^2 - \tilde{B}^2 - \tilde{B}F - pN^2(J_1+J_2)} \sim \mathcal{O}(N^2)$$

This agrees with the Bekenstein-Hawking entropy of large black hole in AdS_5 X $Y^{p,p}$.

[Hosseini, Hristov, Zaffaroni '17]. See also [Cabo-Bizet et al.'18][Choi, **JK**, Seok Kim, Nahmgoong '18] [Benini, Milan '18], …

Conclusion

- We studied the Cardy-like asymptotics of the (modified) index.

$$\mathcal{I} \to \exp\left(\frac{8(5a-3c)}{27\omega_1\omega_2}\Delta^3 + \frac{8\pi^2(a-c)}{3\omega_1\omega_2}\Delta\right) \quad \text{with} \quad \omega_1+\omega_2-2\Delta = 2\pi i$$

- Imaginary chemical potentials obstructs the B/F cancelation, improving the high 'temperature' asymptotics of the index.
- Sometimes the asymptotic entropy can saturate the upper bound, accounting for the macroscopic entropy of AdS_5 black holes.

- Future problems:
  - Saddle point analysis beyond the Cardy limit.
  - Generalization to other dimensions.

[Benini, Milan '18]
[Cabo-Bizet et al.'19]