The R^*-method,

or how to simplify the calculation of anomalous dimensions of arbitrary local QFTs

Kavli IPMU, Tokyo

19th of April 2019

Franz Herzog

In collaboration with:
Ben Rujil, Takahiro Ueda, Jos Vermaseren, Andreas Vogt;
Giulio Falcioni, Konstantin Chetyrkin; Jordy de Vries

Overview

- **The R* method**
  - BPHZ
  - Euclidean IR divergences
  - Infrared Rearrangement
- **Results**
  - QCD beta function at 5 loops
  - Hadronic Higgs decay
  - First Mellin-Moments for QCD splitting functions at N4LO
- **Outlook**
  - Automisation
  - SM EFT

The R^*-method

History of BPHZ and R*

On the Multiplication of the causal function in the quantum theory of fields

ÜBER DIE MULTIPLIKATION DER KAUSALFUNKTIONEN IN DER QUANTENTHEORIE DER FELDER

1956

VON

N. N. BOGOLIUBOW und O. S. PARASIUK

in Moskau (N. N. B.) und in Kiev (O. S. P.)

Commun. math. Phys. 15, 208—234 (1969)

Convergence of Bogoliubov's Method of Renormalization in Momentum Space

W. ZIMMERMANN*

Institut des Hautes Etudes Scientifiques, Bures sur Yvette

and

Istituto di Fisica Teorica, Mostra d'Oltremare, Napoli

Proof of the Bogoliubov-Parasiuk Theorem on Renormalization

KLAUS HEPP* **

The Institute for Advanced Study, Princeton, New Jersey

Received February 7, 1966

INFRARED *R*-OPERATION AND ULTRAVIOLET COUNTERTERMS IN THE MS-SCHEME

K.G. CHETYRKIN and F.V. TKACHOV

Institute for Nuclear Research of the Academy of Sciences of the USSR, 60th October prospect 7a, Moscow 117312, USSR

Received 16 March 1982

Volume 144B, number 5,6 PHYSICS LETTERS 6 September 1984

R^*-OPERATION CORRECTED

K.G. CHETYRKIN

Institute for Nuclear Research of the Academy of Sciences of the USSR, Moscow, USSR

and

V.A. SMIRNOV

Nuclear Physics Institute of Moscow State University, Moscow, USSR

Bogoliubov's Recursion

$$R(\Gamma) = \sum_{S \subseteq \Gamma} Z(S) * \Gamma/S \qquad\qquad Z(S) = \prod_{\gamma \in S} Z(\gamma)$$

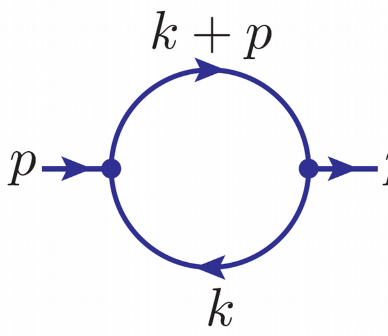
- S is a set of disjoint UV divergent 1PI subgraphs $S = \{\gamma \in \Gamma \,|\, \gamma_i \cap \gamma_j = \emptyset\}$
- Γ/S is obtained by contracting each $\gamma \in S$ into a point in Γ
- $Z(\gamma)$ Is the (local) UV counterterm associated to the 1PI graph γ
- The $*$ -symbol indicates insertion

The UV counterterm

$$Z(\Gamma) = -K\left(\sum_{S\subsetneq\Gamma}\prod_{\gamma\in S} Z(\gamma) * \Gamma/S\right)$$

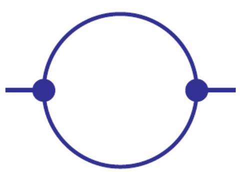
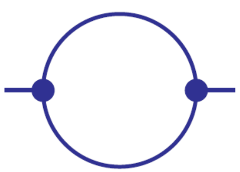
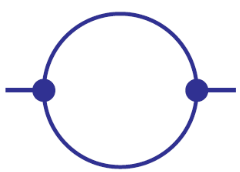
- The sum does not include Γ
- $Z(\Gamma)$ is defined recursively
- $Z(\Gamma)$ is a homogeneous polynomial in the external momenta of Γ
- K projects out the singular part and is renormalisation scheme dependent:
  - It projects onto poles in ϵ in MS
  - It Taylor expands around external momentum in MOM

Example at 1-loop

$\Gamma =$ 

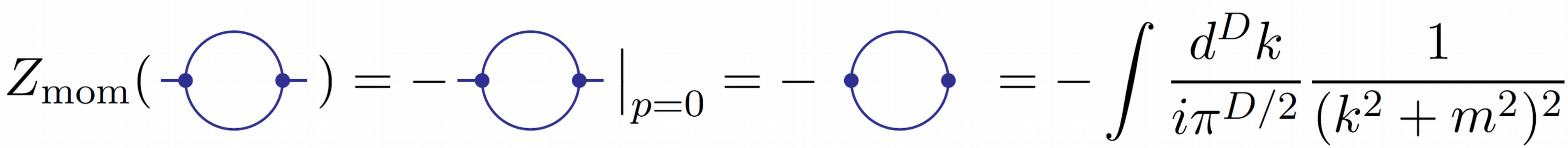
$= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2+m^2)((k+p)^2+m^2)}$

R produces the renormalised result:

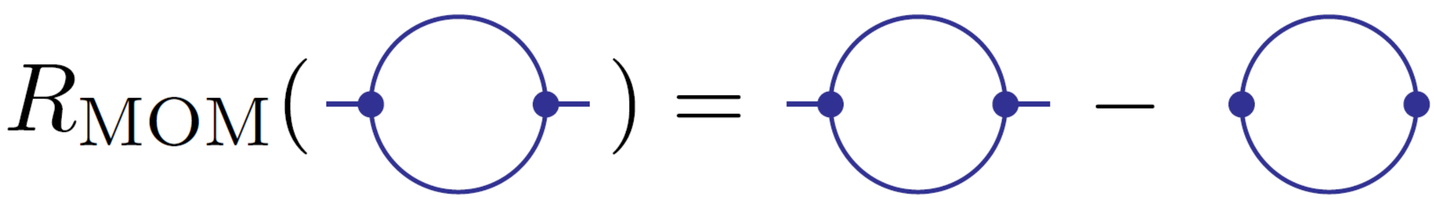
$R($  $) =$  $+ Z($  $) * 1$

Example at 1-loop: MOM scheme

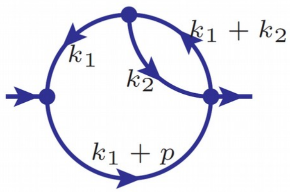
In Momentum subtraction the UV counterterm evaluates to:



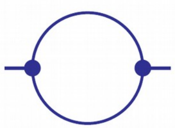
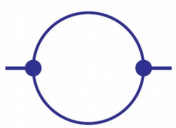
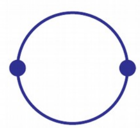
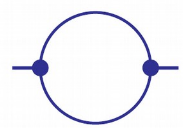
Such that the renormalised Feynman graph is:

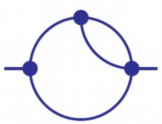
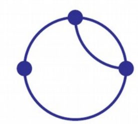
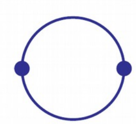
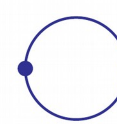


UV Divergences at 2 loops in MOM

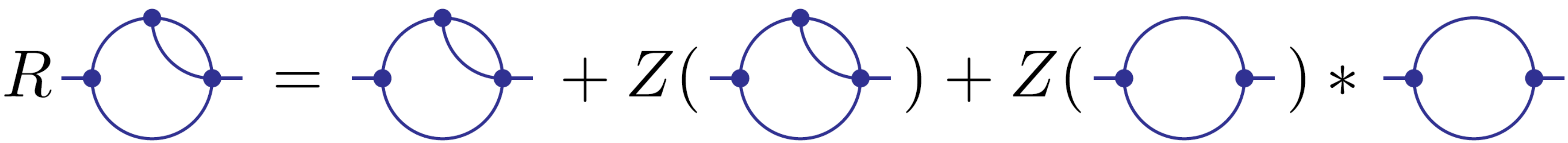
$\Gamma =$  $= \int \frac{d^D k_1}{i\pi^{D/2}} \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{(k_1^2+m^2)((k_1+k_2)^2+m^2)(k_2^2+m^2)((k_1+p)^2+m^2)}$

UV divergent (sub) graphs:

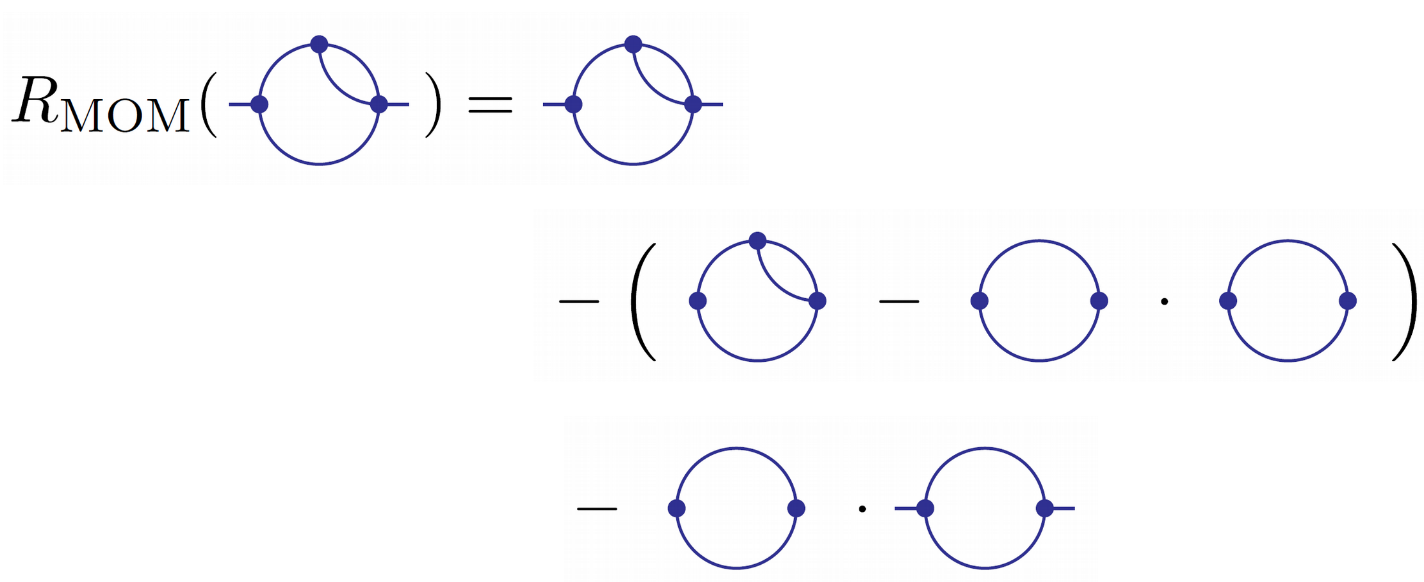
$\gamma =$  $,\qquad Z(\gamma)\cdot\Gamma/\gamma = -\gamma\big|_{p,k_1=0}\cdot$  $= -$  $\cdot$ 

$\Gamma =$  $,\qquad Z(\Gamma) = -(\Gamma - \gamma\big|_{k_1=0}\cdot\Gamma/\gamma)\big|_{p=0} = -$  $+$  $\cdot$ 

UV Divergences at 2 loops in MOM



And substituting the expressions in MOM we obtain:



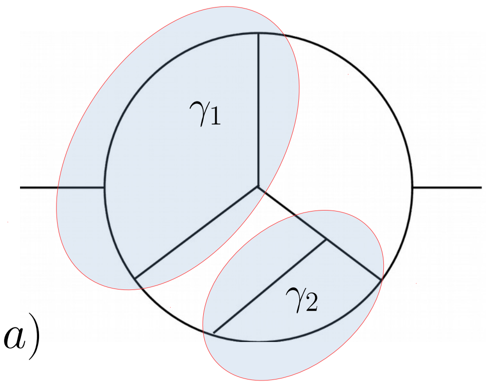
The Forest Formula

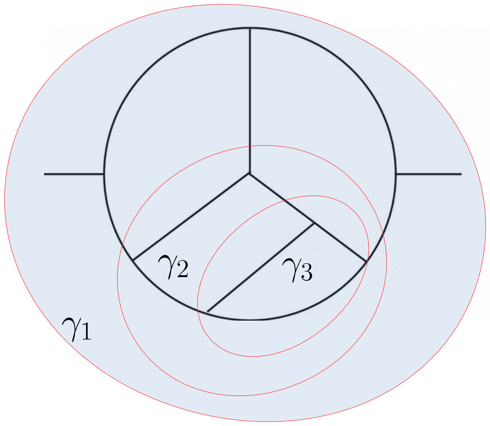
Bogoliubov’s recursion can be solved as the forest formula [Zimmermann]:

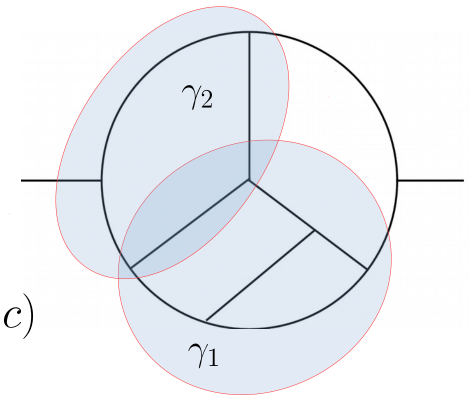
$$R(\Gamma) = \sum_{U \in \mathcal{U}_r(\Gamma)} \prod_{\gamma \in U} (-K_\gamma)\,\Gamma$$

A forest U is a set of subgraphs $\{\gamma_1, .., \gamma_n\}$ which are either **nested** $\gamma_i \subset \gamma_j$ or **disjoint** $\gamma_i \cap \gamma_j = 0$.

Forest Quiz







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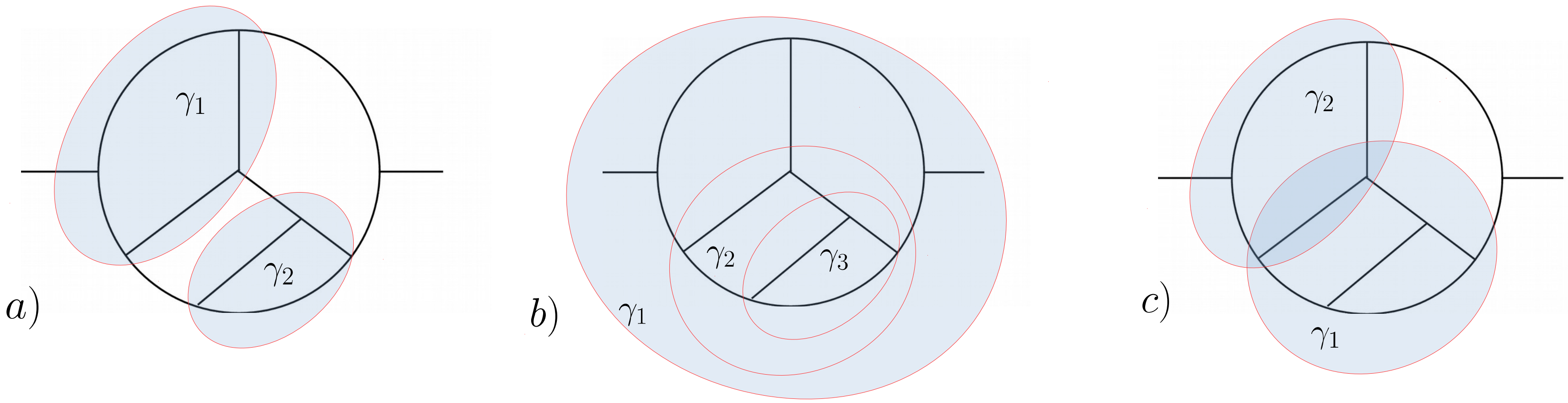

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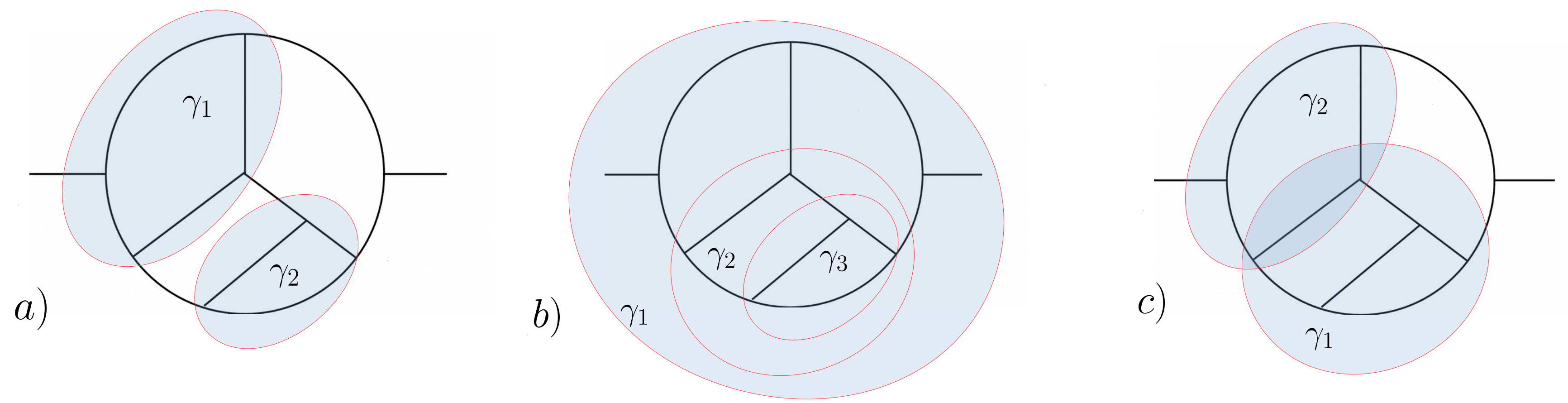
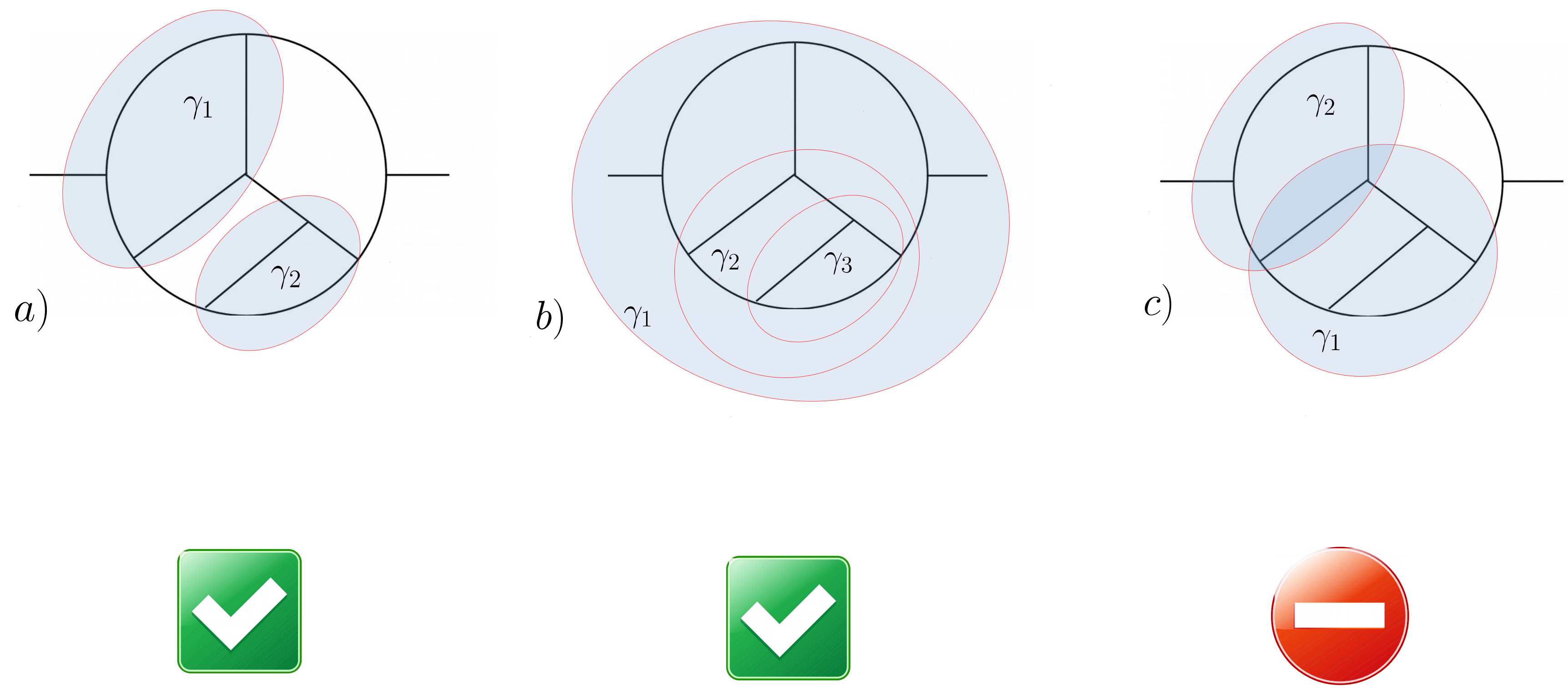


Forest Quiz



Extensions of BPHZ

- The Connes-Kreimer Hopf Algebra
  - The R-operation gives rise to a Hopf Algebra
  - R-operation becomes a twisted antipode acting on a coproduct

$$
\begin{tikzcd}
& H \otimes H \arrow[rr, "S \otimes \mathrm{id}"] & & H \otimes H \arrow[dr, "\nabla"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & K \arrow[rr, "\eta"] & & H \\
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- R* [Chetyrkin, Tkachov, Smirnov]
  - Generalises BPHZ to infra-red divergences in euclidean Feynman Graphs
  - Initially only developed for ϕ^4 theory
  - A global version can be formulated using the Hard Mass expansion [Chetyrkin]
  - Recently extended local R* formalism to arbitrary tensor/numerator Feynman[FH, Ben Rujil]

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Extensions of BPHZ

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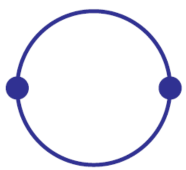
- Collinear Divergences [van Neerven]
  - BPHZ formulation of collinear divergences

What happens in the massless limit?

singularity localized on a plane surface. The complete similarity of patterns of UV and IR singularities in position and momentum space correspondingly forces us to conclude that *no reason on earth* can prevent IR divergences from being subtracted by counterterms proportional to δ-functions and their derivatives in momentum space in a way completely analogous to Bogoliubov's recipe. [Chetyrkin, Tkachov 1982]

Why are scaleless dimensionless integrals zero in dimension regularisation?

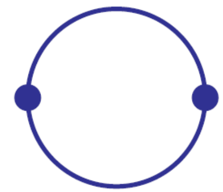


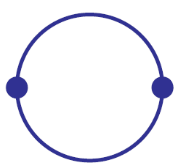
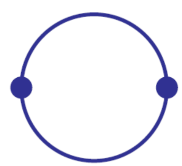
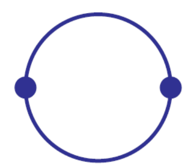
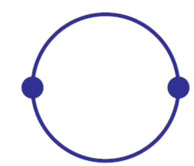
$$= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{[k^2]^2} = \frac{1}{\epsilon_{\mathrm{UV}}} - \frac{1}{\epsilon_{\mathrm{IR}}} = 0$$

$$k \to \infty \qquad k \to 0$$

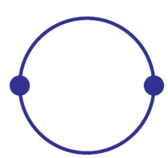
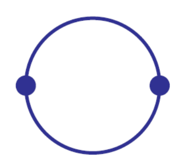
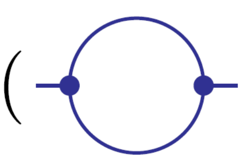
To see this just insert $$1 = \frac{k^2}{k^2 + m^2} + \frac{m^2}{k^2 + m^2}$$

Extracting IR poles from UV poles

R^*  $= 0$

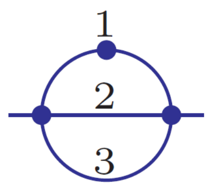
R^*  $=$  $+ Z($  $) + \tilde{Z}($  $) = 0$

UV counterterm IR counterterm

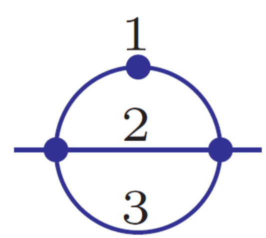
$\Rightarrow$ $Z($  $) = -\tilde{Z}($  $) = -K($  $) = \frac{1}{\epsilon}$

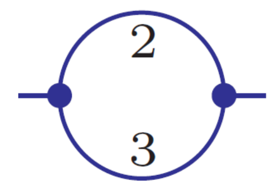
Infrared Rearrangement [Vladimirov 1980]

UV and IR Divergences at 2 loops

$\Gamma =$ 

$$= \int \frac{d^D k_1}{i\pi^{D/2}} \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{(k_1^2)^2 (k_2)^2 (k_1 + k_2 + P)^2}$$

UV divergences: $\{\varnothing, \{$ 

$\}, \{$ 

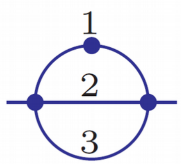
$\}\}$

IR divergences: $\{\varnothing,$ 

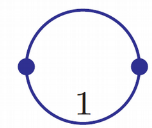
$\}$

IR Factorisation

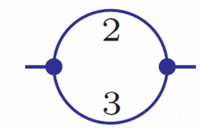
$$\lim_{k_1 \to 0}$$



$$= \int \frac{d^D k_1}{i\pi^{D/2}} \frac{1}{(k_1^2)^2} \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{(k_2)^2 (k_2 + P)^2} =$$



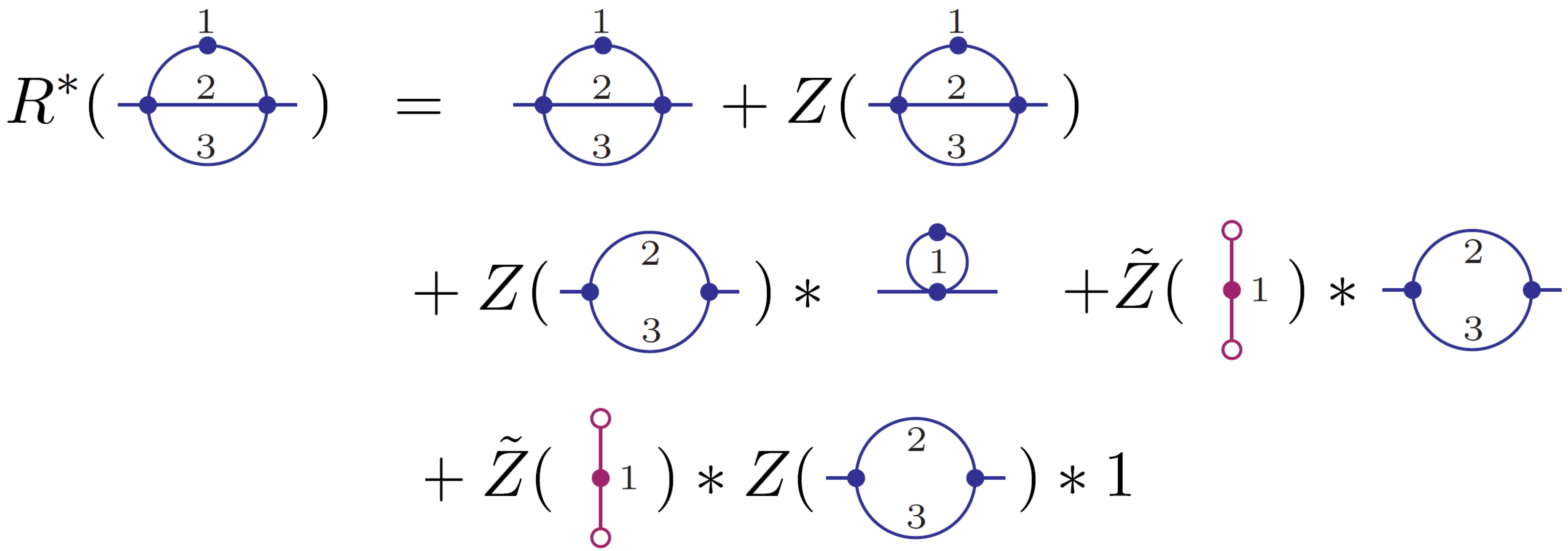
$$\cdot$$



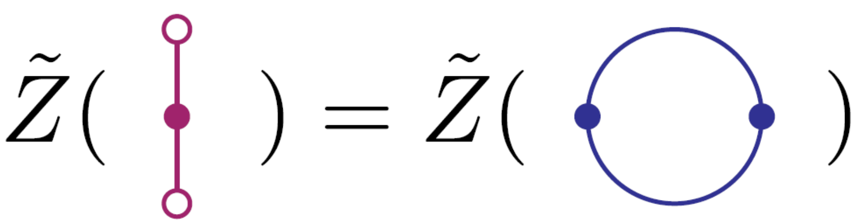
$$\lim_{1 \to 0} \Gamma = \tilde{\gamma} \cdot \Gamma \setminus \tilde{\gamma}$$

The remaining graph $\Gamma \setminus \tilde{\gamma}$ is constructed by deleting vertices and edges of $\tilde{\gamma}$ in Γ

Subtraction of IR and UV at 2 loops



"UV Rearrangement"



Subtraction of IR and UV at L loops

(for massive external legs)

$$R^*(\Gamma) = \sum_{\substack{S\subseteq\Gamma,\tilde{S}\subseteq\Gamma \\ S\cap\tilde{S}=\emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma/S \setminus \tilde{S}$$

- UV counterterm:

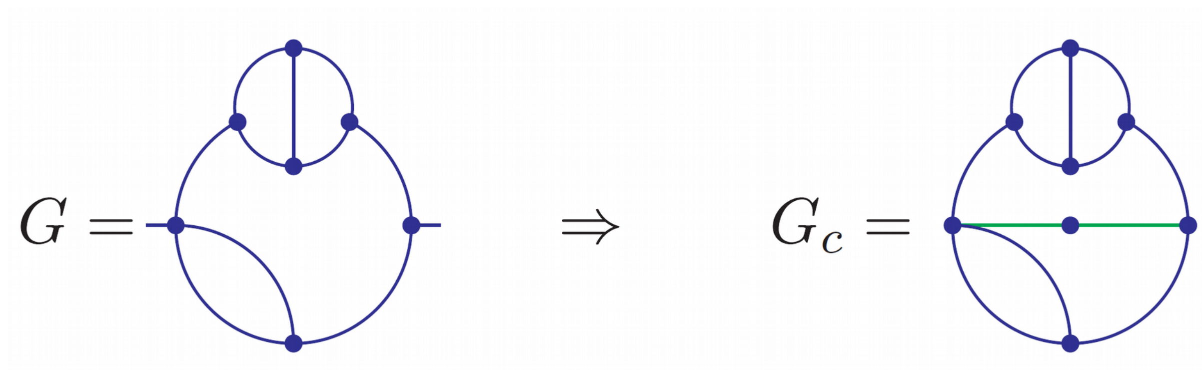
$$Z(\Gamma) = -K\Big(\sum_{\substack{S\subsetneq\Gamma,\tilde{S}\subseteq\Gamma \\ S\cap\tilde{S}=\emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma/S \setminus \tilde{S}\Big)$$

- IR counterterm:

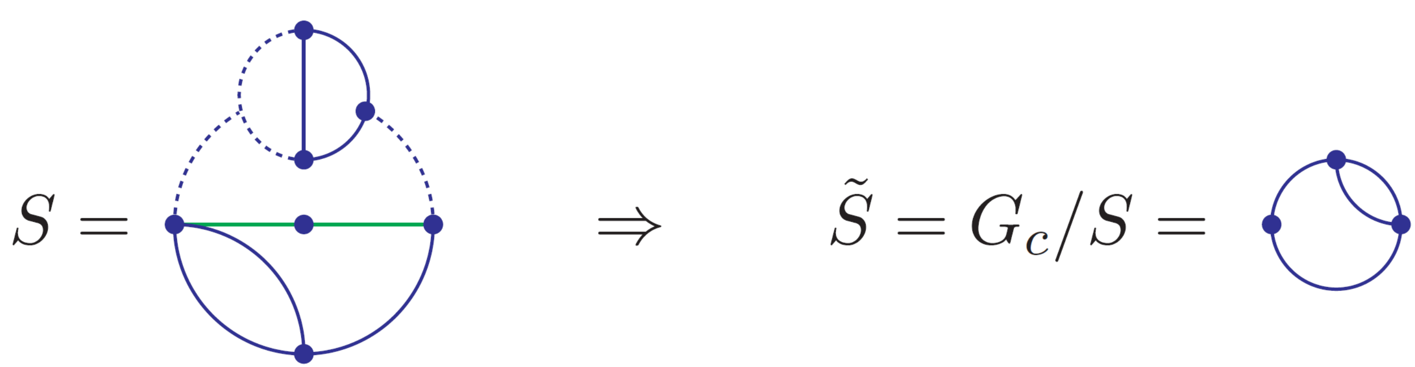
$$\tilde{Z}(\Gamma_0) = -K\Big(\sum_{\substack{S\subseteq\Gamma_0,\tilde{S}\subsetneq\Gamma_0 \\ S\cap\tilde{S}=\emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma_0/S \setminus \tilde{S}\Big)$$

IR subgraph search

- Wield external lines into point:



- Remaining graphs of UV Spinneys are IR spinneys:



Evaluating IR counterterms

IR counterterms can be extracted from UV counterterms from the relation:

$$R^*(\Gamma_0) = 0$$

where Γ_0 is an arbitrary scaleless logarithmically divergent vacuum graph

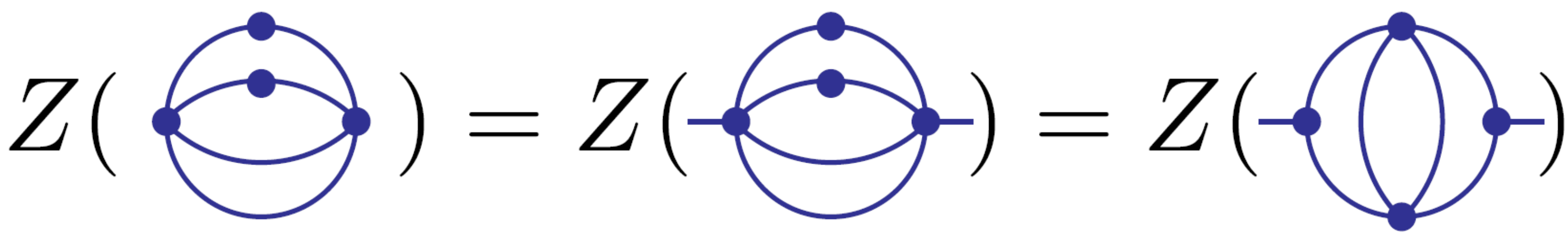
What is R* good for?

The idea of R* is not to somehow “renormalise away” IR divergences, but to construct an efficient algorithm for extracting UV anomalous dimensions from maximally simple 1-scale Feynman graphs via Infrared rearrangment (IRR).

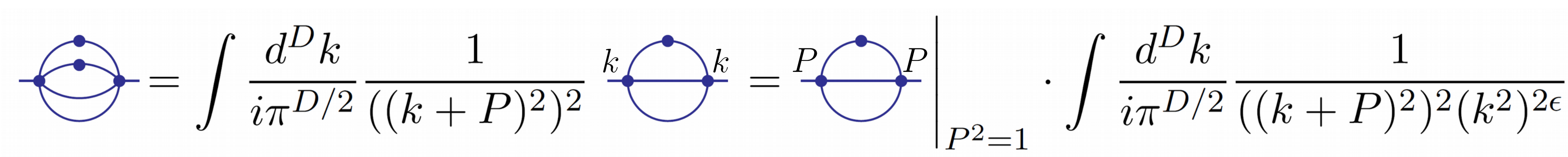
Theorem [Chetyrkin, Tkachov 1982]

IRR allows one to extract the renormalisation constants of arbitrary L-loop Feynman graphs or amplitudes from products of 1-scale propagator Feynman graphs of **lower** loops.

Infrared Rearrangement



Choosing an IRR which has the incoming lines connected by a single propagator always allows to play the following trick:

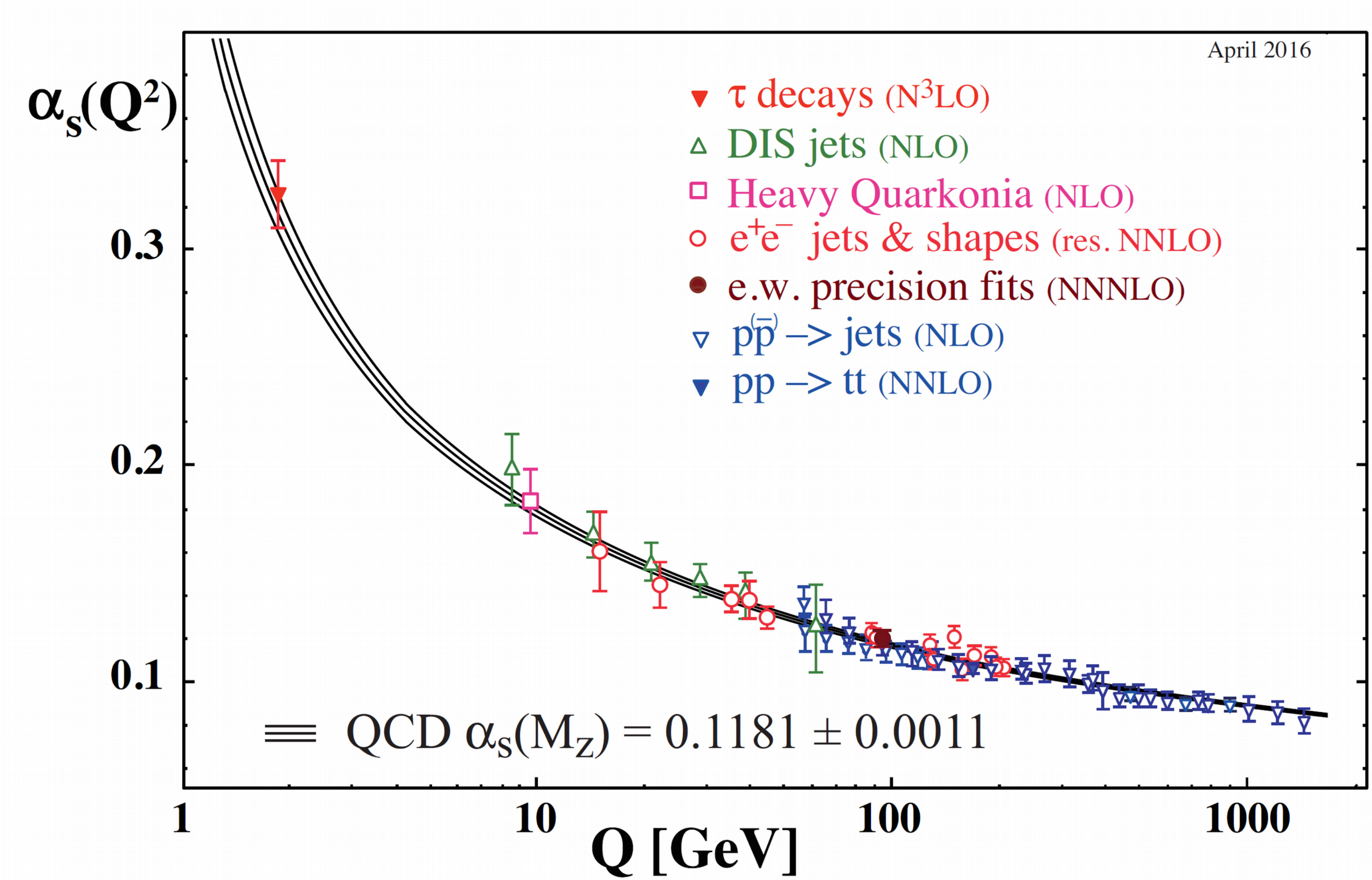


L=3

L=2

L=1 (with non integer power)

Results



[pdg2016]

$$\beta(\mu) = \frac{d\alpha_s(\mu)}{d\log\mu^2} = -\sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_s}{4\pi}\right)^{n+2}$$

History of the QCD beta function

- β_0:
  - 1965 [Vanyashin, Terentev]; 1970 [Khriplovich];1972 [T'Hooft];
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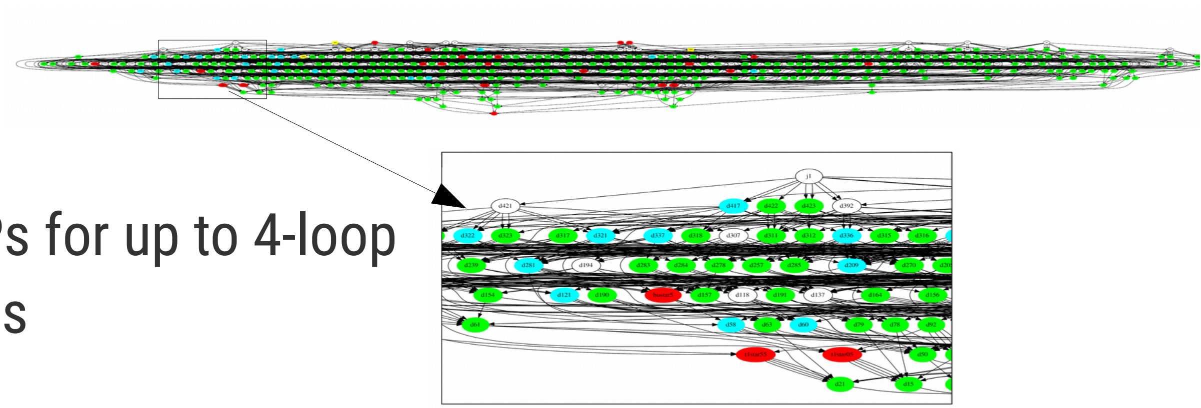
  - 2005 [Czakon]
- β_4:
  - 2016 [Baikov, Chetyrkin, Kühn]
  - 2016 partial results [Luthe, Maier, Marquard, Schroeder]
  - 2017 [FH, Ruijl, Ueda, Vermaseren, Vogt]

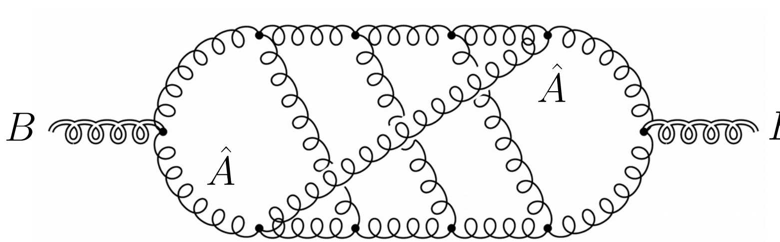
Calculation



- Forcer arXiv:1704.06650
  - Parameteric solution of IBPs for up to 4-loop massless self energy graphs
- Automated R* arXiv:1703.03776
  - for arbitrary tensor Feynman graphs
- Background field gauge:
  - Extract beta from background field self energy [Abbot 81]

$$Z_g \sqrt{Z_B} = 1$$

$\Pi_B(Q) \sim$



$+ \dots$

General gauge group

Besides C_A, C_F we only need the symmetric group invariant tensors

$$d_F^{abcd} = \frac{1}{6}\,\mathrm{Tr}(T^a T^b T^c T^d + \text{ five } bcd \text{ permutations })$$
$$d_A^{abcd} = \frac{1}{6}\,\mathrm{Tr}(C^a C^b C^c C^d + \text{ five } bcd \text{ permutations })$$

T^a are the generators of the fermionic representation.

C^a are the generators of the adoint representation.

Old Results

$$\beta_0 \; = \; \frac{11}{3}\, C_A \; - \; \frac{4}{3}\, T_F\, n_f \;\;,$$

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$$\beta_2 = \frac{2857}{54} C_A^3 - \frac{1415}{27} C_A^2 T_F n_f - \frac{205}{9} C_F C_A T_F n_f + 2 C_F^2 T_F n_f + \frac{44}{9} C_F T_F^2 n_f^2 + \frac{158}{27} C_A T_F^2 n_f^2 \ ,$$

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$$\begin{aligned}
\beta_3 = {} & C_A^4 \left(\frac{150653}{486} - \frac{44}{9} \zeta_3 \right) + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left(-\frac{80}{9} + \frac{704}{3} \zeta_3 \right) \\
& + C_A^3 T_F n_f \left(-\frac{39143}{81} + \frac{136}{3} \zeta_3 \right) + C_A^2 C_F T_F n_f \left(\frac{7073}{243} - \frac{656}{9} \zeta_3 \right) \\
& + C_A C_F^2 T_F n_f \left(-\frac{4204}{27} + \frac{352}{9} \zeta_3 \right) + \frac{d_F^{abcd} d_A^{abcd}}{N_A} n_f \left(\frac{512}{9} - \frac{1664}{3} \zeta_3 \right) \\
& + 46 C_F^3 T_F n_f + C_A^2 T_F^2 n_f^2 \left(\frac{7930}{81} + \frac{224}{9} \zeta_3 \right) + C_F^2 T_F^2 n_f^2 \left(\frac{1352}{27} - \frac{704}{9} \zeta_3 \right) \\
& + C_A C_F T_F^2 n_f^2 \left(\frac{17152}{243} + \frac{448}{9} \zeta_3 \right) + \frac{d_F^{abcd} d_F^{abcd}}{N_A} n_f^2 \left(-\frac{704}{9} + \frac{512}{3} \zeta_3 \right) \\
& + \frac{424}{243} C_A T_F^3 n_f^3 + \frac{1232}{243} C_F T_F^3 n_f^3 \ ,
\end{aligned}$$

5-loop result

[confirmed by Luthe, Maier, Marquard and Schroeder; Chetyrkin, Falcioni, FH, Vermaseren]

$$
\begin{aligned}
\beta_4 \;=\; & C_A^5 \left(\frac{8296235}{3888} - \frac{1630}{81}\zeta_3 + \frac{121}{6}\zeta_4 - \frac{1045}{9}\zeta_5\right) \\
& + \frac{d_A^{abcd}d_A^{abcd}}{N_A} C_A \left(-\frac{514}{3} + \frac{18716}{3}\zeta_3 - 968\,\zeta_4 - \frac{15400}{3}\zeta_5\right) \\
& + C_A^4\, T_F\, n_f \left(-\frac{5048959}{972} + \frac{10505}{81}\zeta_3 - \frac{583}{3}\zeta_4 + 1230\,\zeta_5\right) \\
& + C_A^3\, C_F\, T_F\, n_f \left(\frac{8141995}{1944} + 146\,\zeta_3 + \frac{902}{3}\zeta_4 - \frac{8720}{3}\zeta_5\right) \\
& + C_A^2\, C_F^2\, T_F\, n_f \left(-\frac{548732}{81} - \frac{50581}{27}\zeta_3 - \frac{484}{3}\zeta_4 + \frac{12820}{3}\zeta_5\right) \\
& + C_A\, C_F^3\, T_F\, n_f \left(3717 + \frac{5696}{3}\zeta_3 - \frac{7480}{3}\zeta_5\right) - C_F^4\, T_F\, n_f \left(\frac{4157}{6} + 128\,\zeta_3\right) \\
& + \frac{d_A^{abcd}d_A^{abcd}}{N_A} T_F\, n_f \left(\frac{904}{9} - \frac{20752}{9}\zeta_3 + 352\,\zeta_4 + \frac{4000}{9}\zeta_5\right) \\
& + \frac{d_F^{abcd}d_A^{abcd}}{N_A} C_A\, n_f \left(\frac{11312}{9} - \frac{127736}{9}\zeta_3 + 2288\,\zeta_4 + \frac{67520}{9}\zeta_5\right) \\
& + \frac{d_F^{abcd}d_A^{abcd}}{N_A} C_F\, n_f \left(-320 + \frac{1280}{3}\zeta_3 + \frac{6400}{3}\zeta_5\right) \\
& + C_A^3\, T_F^2\, n_f^2 \left(\frac{843067}{486} + \frac{18446}{27}\zeta_3 - \frac{104}{3}\zeta_4 - \frac{2200}{3}\zeta_5\right) \\
& + C_A^2\, C_F\, T_F^2\, n_f^2 \left(\frac{5701}{162} + \frac{26452}{27}\zeta_3 - \frac{944}{3}\zeta_4 + \frac{1600}{3}\zeta_5\right) \\
& + C_F^2\, C_A\, T_F^2\, n_f^2 \left(\frac{31583}{18} - \frac{28628}{27}\zeta_3 + \frac{1144}{3}\zeta_4 - \frac{4400}{3}\zeta_5\right) \\
& + C_F^3\, T_F^2\, n_f^2 \left(-\frac{5018}{9} - \frac{2144}{3}\zeta_3 + \frac{4640}{3}\zeta_5\right) \\
& + \frac{d_F^{abcd}d_A^{abcd}}{N_A} T_F\, n_f^2 \left(-\frac{3680}{9} + \frac{40160}{9}\zeta_3 - 832\,\zeta_4 - \frac{1280}{9}\zeta_5\right) \\
& + \frac{d_F^{abcd}d_F^{abcd}}{N_A} C_A\, n_f^2 \left(-\frac{7184}{3} + \frac{40336}{9}\zeta_3 - 704\,\zeta_4 + \frac{2240}{9}\zeta_5\right) \\
& + \frac{d_F^{abcd}d_F^{abcd}}{N_A} C_F\, n_f^2 \left(\frac{4160}{3} + \frac{5120}{3}\zeta_3 - \frac{12800}{3}\zeta_5\right) \\
& + C_A^2\, T_F^3\, n_f^3 \left(-\frac{2077}{27} - \frac{9736}{81}\zeta_3 + \frac{112}{3}\zeta_4 + \frac{320}{9}\zeta_5\right) \\
& + C_A\, C_F\, T_F^3\, n_f^3 \left(-\frac{736}{81} - \frac{5680}{27}\zeta_3 + \frac{224}{3}\zeta_4\right) \\
& + C_F^2\, T_F^3\, n_f^3 \left(-\frac{9922}{81} + \frac{7616}{27}\zeta_3 - \frac{352}{3}\zeta_4\right) \\
& + \frac{d_F^{abcd}d_F^{abcd}}{N_A} T_F\, n_f^3 \left(\frac{3520}{9} - \frac{2624}{3}\zeta_3 + 256\,\zeta_4 + \frac{1280}{3}\zeta_5\right) \\
& + C_A\, T_F^4\, n_f^4 \left(\frac{916}{243} - \frac{640}{81}\zeta_3\right) - C_F\, T_F^4\, n_f^4 \left(\frac{856}{243} + \frac{128}{27}\zeta_3\right)
\end{aligned}
$$

Result of computing ~150.000 five-loop Feynman diagrams

QCD

[agrees with result of Baikov, Chetyrkin and Kühn]

$$\beta_0 = 11 - \frac{2}{3} n_f \,, \qquad \beta_1 = 102 - \frac{38}{3} n_f \,,$$

$$\beta_2 = \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \,,$$

$$\begin{aligned}\beta_3 = {} & \frac{149753}{6} + 3564\,\zeta_3 + n_f \left(-\frac{1078361}{162} - \frac{6508}{27}\zeta_3\right) \\ & + n_f^2 \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right) + \frac{1093}{729} n_f^3\end{aligned}$$

$$\begin{aligned}\beta_4 = {} & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\,\zeta_5 \\ & + n_f \left(-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\right) \\ & + n_f^2 \left(\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5\right) \\ & + n_f^3 \left(-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5\right) + n_f^4 \left(\frac{1205}{2916} - \frac{152}{81}\zeta_3\right)\end{aligned}$$

QED

[agrees with result of Baikov, Chetyrkin, Kühn and Rittinger]

$$\beta_0 = \frac{4}{3} n_f \ , \quad \beta_1 = 4\, n_f \ , \quad \beta_2 = -\, 2\, n_f - \frac{44}{9} n_f^2 \ ,$$

$$\beta_3 = -\, 46\, n_f + n_f^2 \left(\frac{760}{27} - \frac{832}{9} \zeta_3 \right) - \frac{1232}{243} n_f^3$$

$$\beta_4 = n_f \left(\frac{4157}{6} + 128\, \zeta_3 \right) + n_f^2 \left(-\frac{7462}{9} - 992\, \zeta_3 + 2720\, \zeta_5 \right)$$
$$+ \ n_f^3 \left(-\frac{21758}{81} + \frac{16000}{27} \zeta_3 - \frac{416}{3} \zeta_4 - \frac{1280}{3} \zeta_5 \right) + n_f^4 \left(\frac{856}{243} + \frac{128}{27} \zeta_3 \right)$$

n_f-dependence

$$\widetilde{\beta} \equiv -\beta(a_\mathrm{s})/(a_\mathrm{s}^2\beta_0)$$

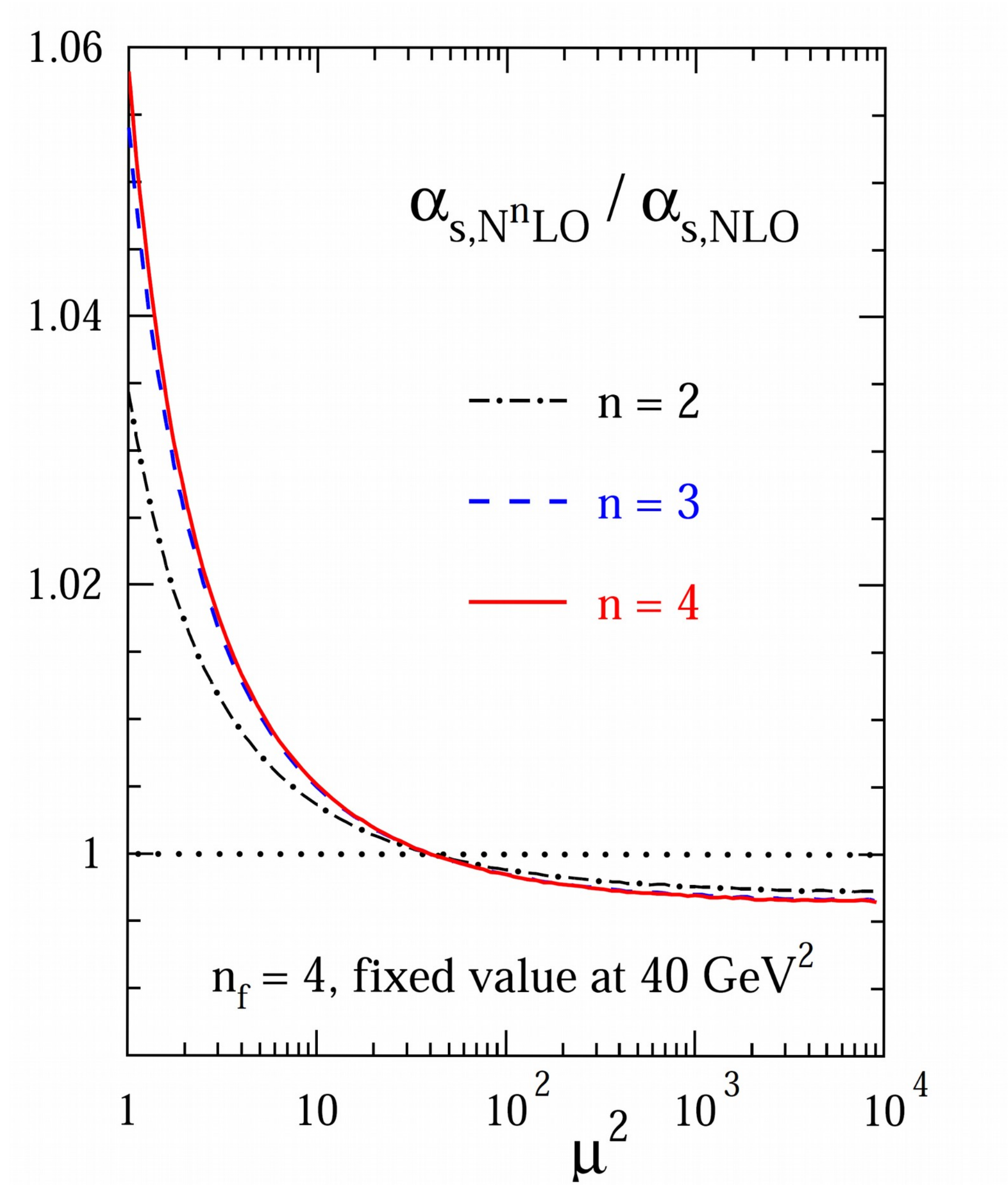
$$
\begin{aligned}
\widetilde{\beta}(\alpha_\mathrm{s}, n_f\!=\!3) &= 1 + 0.565884\,\alpha_\mathrm{s} + 0.453014\,\alpha_\mathrm{s}^2 + 0.676967\,\alpha_\mathrm{s}^3 + 0.580928\,\alpha_\mathrm{s}^4 \\
\widetilde{\beta}(\alpha_\mathrm{s}, n_f\!=\!4) &= 1 + 0.490197\,\alpha_\mathrm{s} + 0.308790\,\alpha_\mathrm{s}^2 + 0.485901\,\alpha_\mathrm{s}^3 + 0.280601\,\alpha_\mathrm{s}^4 \\
\widetilde{\beta}(\alpha_\mathrm{s}, n_f\!=\!5) &= 1 + 0.401347\,\alpha_\mathrm{s} + 0.149427\,\alpha_\mathrm{s}^2 + 0.317223\,\alpha_\mathrm{s}^3 + 0.080921\,\alpha_\mathrm{s}^4 \\
\widetilde{\beta}(\alpha_\mathrm{s}, n_f\!=\!6) &= 1 + 0.295573\,\alpha_\mathrm{s} - 0.029401\,\alpha_\mathrm{s}^2 + 0.177980\,\alpha_\mathrm{s}^3 + 0.001555\,\alpha_\mathrm{s}^4
\end{aligned}
$$

Convergence enhanced for larger n_f

Scale Evolution at low scales

Let us hypothetically fix

$$\alpha_s(6.3GeV) = 0.2$$



Same computational method applies to decay rates too!

On Higgs decays to hadrons and the R-ratio at N^{4}LO

F. Herzog[a], B. Ruijl[a,b], T. Ueda[a], J.A.M. Vermaseren[a] and A. Vogt[c]

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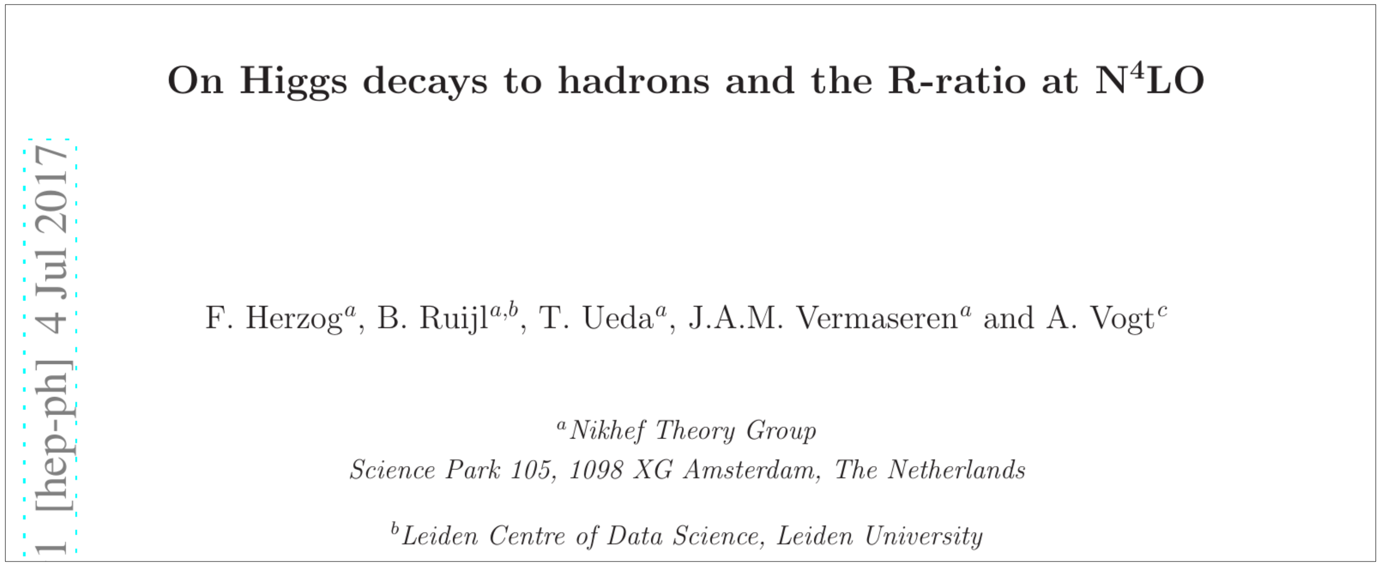

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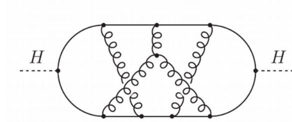
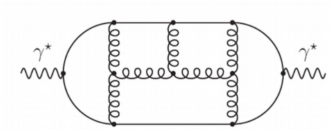
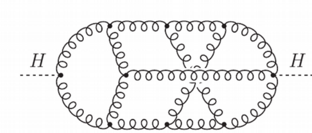
- $H \to bb$ at N4LO



- Hadronic R-ratio at N4LO



- $H \to gg$ at N4LO



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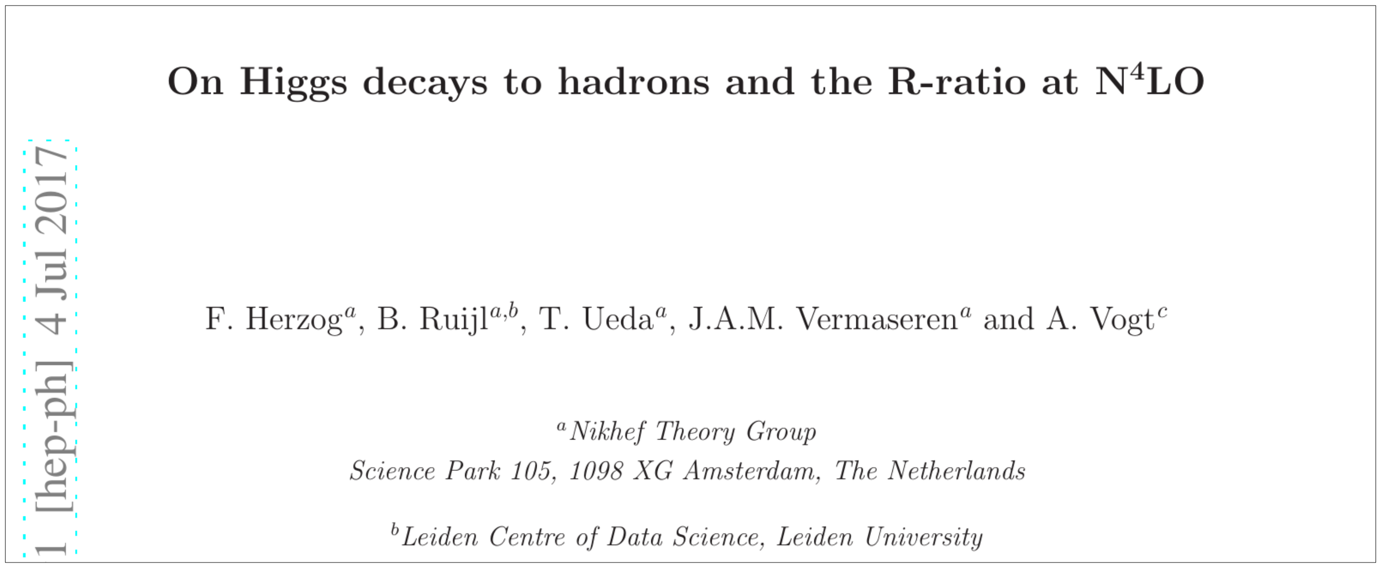

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- $H \to bb$ at N4LO
- Hadronic R-ratio at N4LO
- $H \to gg$ at N4LO

First independent confirmation of results by Baikov, Chetyrkin and Kuehn

NEW!

Unitarity

Unitarity allows to obtain the decay rate from the imaginary part of the corresponding self energy

$$\Gamma_{H\to gg} = \frac{\sqrt{2}\,G_{\mathrm{F}}}{M_{\mathrm{H}}}\,|C_1|^2\,\mathrm{Im}\,\Pi^{GG}(-M_{\mathrm{H}}^2 - i\delta)$$

Analytic continuation leads to a prefactor proportional to ϵ

$$\mathrm{Im}(-p^2 - i\delta)^{-L\epsilon} = L\pi\epsilon\left(1 - \frac{(L\pi\epsilon)^2}{3!} + \ldots\right)$$

Upshot: The decay rate can be extracted from the UV poles of the self energy, making it possible to use the R*-method.

Analytic result

$$\frac{4\pi}{N_A q^4}\,\mathrm{Im}\,\Pi^{GG}(q^2) \;\equiv\; G(q^2) \;=\; 1+\sum_{n=1} g_n a_s^n$$

$$g_1 = \frac{73}{3}C_A - \frac{14}{3}n_f \;,$$

$$\begin{aligned} g_2 = {} & C_A^2\left[\frac{37631}{54} - \frac{242}{3}\zeta_2 - 110\,\zeta_3\right] - C_A n_f\left[\frac{6665}{27} - \frac{88}{3}\zeta_2 + 4\,\zeta_3\right] \\ & - C_F n_f\left[\frac{131}{3} - 24\,\zeta_3\right] + n_f^2\left[\frac{508}{27} - \frac{8}{3}\zeta_2\right] \;, \end{aligned}$$

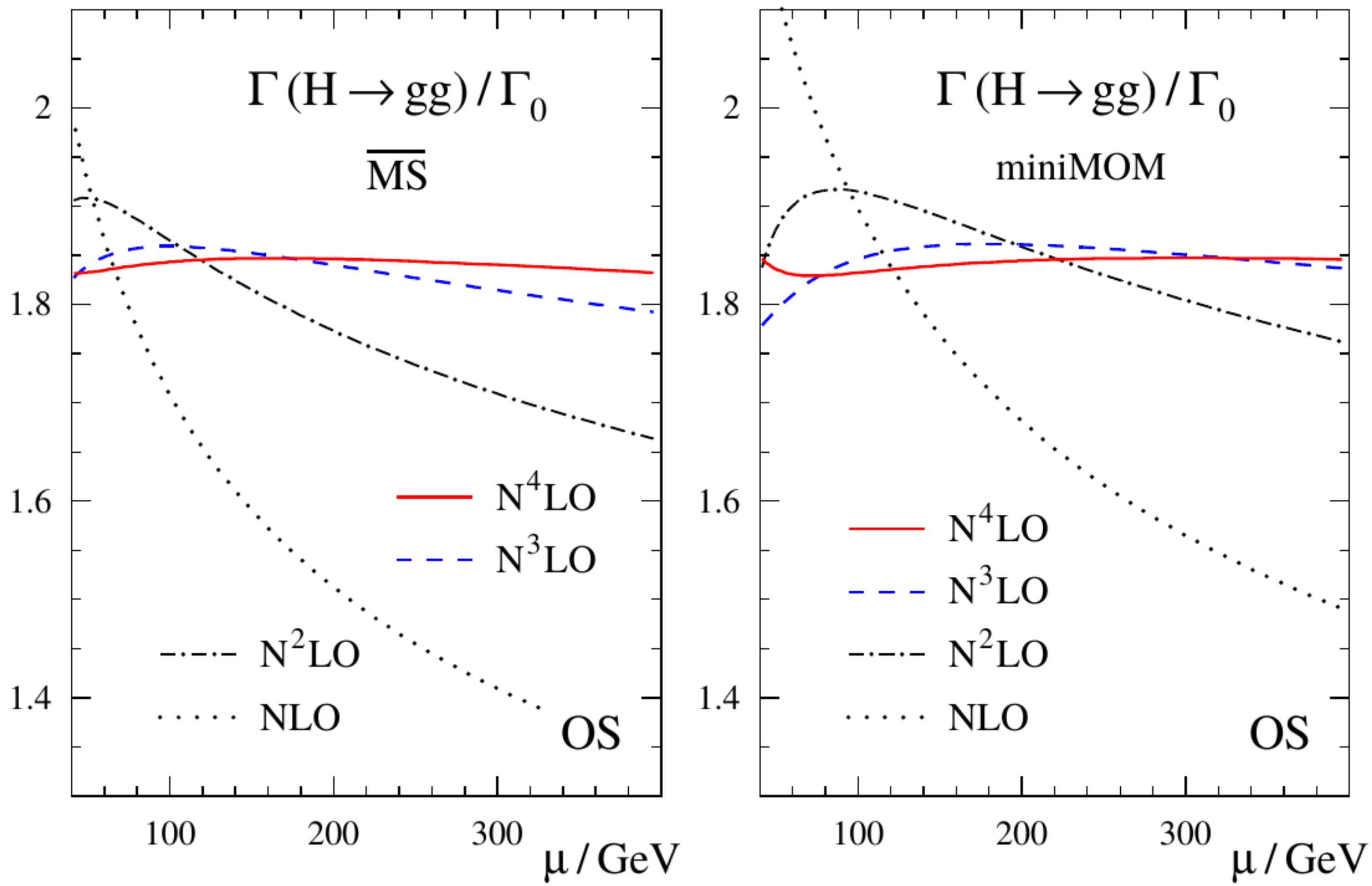
$$\begin{aligned} g_3 = {} & C_A^3\left[\frac{15420961}{729} - \frac{45056}{9}\zeta_2 - \frac{178156}{27}\zeta_3 + \frac{3080}{3}\zeta_5\right] \\ & - C_A^2 n_f\left[\frac{2670508}{243} - \frac{8084}{3}\zeta_2 - \frac{9772}{9}\zeta_3 + \frac{80}{3}\zeta_5\right] \\ & - C_F C_A n_f\left[\frac{23221}{9} - \frac{572}{3}\zeta_2 - 1364\,\zeta_3 - 160\,\zeta_5\right] \\ & + C_F^2 n_f\left[\frac{221}{3} + 192\,\zeta_3 - 320\,\zeta_5\right] + C_A n_f^2\left[\frac{413308}{243} - \frac{1384}{3}\zeta_2 + \frac{56}{9}\zeta_3\right] \\ & + C_F n_f^2\left[440 - \frac{104}{3}\zeta_2 - 240\,\zeta_3\right] - n_f^3\left[\frac{57016}{729} - \frac{224}{9}\zeta_2 - \frac{64}{27}\zeta_3\right] \end{aligned}$$

Analytic result at 5 loops

$$
\begin{aligned}
g_4 = \; & C_A^4\left[\frac{5974862279}{8748} - \frac{58922654}{243}\zeta_2 - \frac{25166402}{81}\zeta_3 + \frac{292556}{45}\zeta_2^2 + \frac{266200}{9}\zeta_2\zeta_3\right.\\
& \left. + \frac{1817200}{27}\zeta_5 + \frac{121000}{9}\zeta_3^2 - \frac{96250}{9}\zeta_7\right]\\
& - \frac{d_A^{abcd}d_A^{abcd}}{N_A}\left[\frac{6416}{27} - \frac{54160}{9}\zeta_3 - \frac{1408}{5}\zeta_2^2 + \frac{13760}{3}\zeta_5 - \frac{19360}{3}\zeta_3^2 + \frac{6160}{3}\zeta_7\right]\\
& - C_A^3 n_f\left[\frac{1025827736}{2187} - \frac{41587004}{243}\zeta_2 - \frac{8812352}{81}\zeta_3 + \frac{211736}{45}\zeta_2^2 + 9680\,\zeta_2\zeta_3\right.\\
& \left. + \frac{109220}{9}\zeta_5 - \frac{8800}{9}\zeta_3^2 + \frac{3500}{9}\zeta_7\right]\\
& - C_A^2 C_F n_f\left[\frac{348948545}{2916} - 22340\,\zeta_2 - \frac{1869710}{27}\zeta_3 + \frac{656}{15}\zeta_2^2 + \frac{19360}{3}\zeta_2\zeta_3\right.\\
& \left. - \frac{35540}{3}\zeta_5 + \frac{17600}{3}\zeta_3^2 - \frac{7000}{3}\zeta_7\right]\\
& + C_A C_F^2 n_f\left[\frac{609521}{162} - \frac{484}{3}\zeta_2 + \frac{450374}{27}\zeta_3 + \frac{352}{15}\zeta_2^2 - \frac{63040}{3}\zeta_5 - 5600\,\zeta_7\right]\\
& + C_F^3 n_f\left[\frac{1034}{3} - 388\,\zeta_3 - 4560\,\zeta_5 + 5600\,\zeta_7\right]\\
& + \frac{d_F^{abcd}d_A^{abcd}}{N_A} n_f\left[\frac{44864}{27} - \frac{140128}{9}\zeta_3 - \frac{3328}{5}\zeta_2^2 + \frac{20800}{3}\zeta_5 - \frac{14080}{3}\zeta_3^2 + \frac{2240}{3}\zeta_7\right]\\
& + C_A^2 n_f^2\left[\frac{26855351}{243} - \frac{3479386}{81}\zeta_2 - \frac{83536}{9}\zeta_3 + \frac{19472}{15}\zeta_2^2 + \frac{1760}{3}\zeta_2\zeta_3 - \frac{1240}{9}\zeta_5 + \frac{160}{9}\zeta_3^2\right]\\
& + C_F C_A n_f^2\left[\frac{29816212}{729} - \frac{71888}{9}\zeta_2 - \frac{563948}{27}\zeta_3 + \frac{224}{15}\zeta_2^2 + \frac{7040}{3}\zeta_2\zeta_3 - \frac{7000}{3}\zeta_5 - \frac{640}{3}\zeta_3^2\right]\\
& + C_F^2 n_f^2\left[\frac{90491}{81} - \frac{200}{3}\zeta_2 - \frac{138968}{27}\zeta_3 - \frac{352}{15}\zeta_2^2 + 4400\,\zeta_5 + 640\,\zeta_3^2\right]\\
& - \frac{d_F^{abcd}d_F^{abcd}}{N_A} n_f^2\left[\frac{68096}{27} - \frac{39424}{9}\zeta_3 - \frac{1024}{5}\zeta_2^2 + 1280\,\zeta_5 - \frac{2560}{3}\zeta_3^2\right]\\
& - C_A n_f^3\left[\frac{46491973}{4374} - \frac{1099028}{243}\zeta_2 - \frac{23720}{81}\zeta_3 + \frac{1408}{9}\zeta_2^2 - \frac{320}{9}\zeta_2\zeta_3 - \frac{800}{27}\zeta_5\right]\\
& - C_F n_f^3\left[\frac{2282351}{729} - \frac{6224}{9}\zeta_2 - \frac{5200}{3}\zeta_3 + \frac{640}{3}\zeta_2\zeta_3\right]\\
& + n_f^4\left[\frac{773024}{2187} - \frac{40640}{243}\zeta_2 - \frac{2240}{81}\zeta_3 + \frac{64}{9}\zeta_2^2\right].
\end{aligned}
$$

In contrast to the beta function here also weight 6,7 ζ-terms are present

Scale and scheme dependence of $\Gamma(H \to gg)$



$$\Gamma_{\mathrm{N^4LO}}(H \to gg) \;=\; \Gamma_0 \left(1.844 \;\pm\; 0.011_{\,\mathrm{series}} \;\pm\; 0.045_{\,\alpha_{\mathrm{s}}(M_{\mathrm{Z}}),1\%}\right)$$

First moments of N4LO Splitting functions

- Splitting functions are the (non-local) anomalous dimensions of the parton distribution functions:

$$\frac{d}{d\log\mu^2} f_i(x,\mu) = \sum_j \int_x^1 \frac{dy}{y} P_{ij}(y) f_j(x/y,\mu)$$

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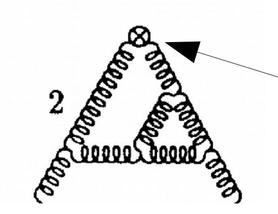

$$O_g^{\mu_1..\mu_n} = \frac{1}{2} F_{\alpha}^{a\mu_1} D^{\mu_2} ... D^{\mu_{n-1}} F^{a\alpha\mu_n}$$

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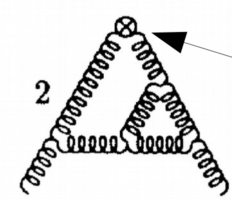

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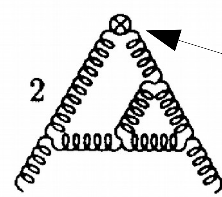

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- With enough Mellin-moments one can approximate (or even “bootstrap”) the complete splitting functions

Outlook

Automating R^* for general QFTs

- Input: Lagrangian, 1PI correlator
- Output: UV counterterm

$$\Gamma = \langle \Phi_1^{s_1}(x_1) ... \Phi_n^{s_n}(x_n) \rangle_{\mathrm{1PI}}$$

$$\mathcal{L} = \sum_i c_i O_i$$

local R^*

$$Z(\Gamma) = \sum_i Z_i O_i^{s_1 .. s_n}$$

Renormalising the SM EFT at two loops (D=6)

$$\mathcal{L}_{\mathrm{SMEFT}} = \mathcal{L}_{\mathrm{SM}} + \mathcal{L}^{(5)} + \mathcal{L}^{(6)} + \mathcal{L}^{(7)} + \ldots \qquad \mathcal{L}^{(d)} = \sum_{i=1}^{n_d} \frac{c_i^{(d)}}{\Lambda^{d-4}} O_i^{(d)}$$

- Many operators and they mix!
  already 84(59) operators at D=6, 993 at D=8,…
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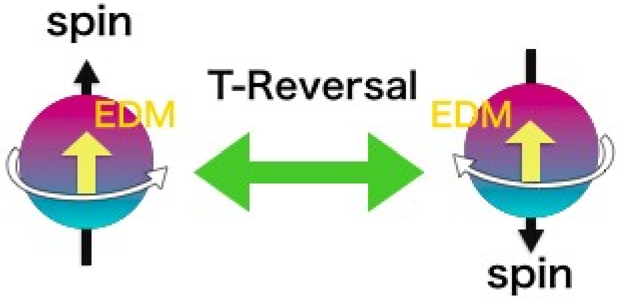
- Full AD mixing matrix so far only known at 1-loop
  [Jenkins, Manohar, Trott; ...]
- 2-loop corrections to ADs:
  - Improve accuracy of SM EFT predictions
  - Allow to run Wilson coefficients up to higher scales to address impact on: Vacuum stability, Inflation, ..

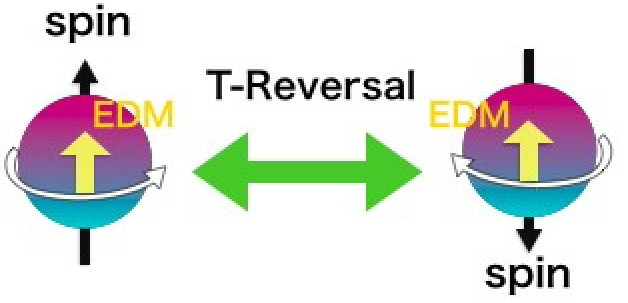
SMEFT constraints from EDMs

- loop QCD effects are important for extracting Wilson coefficients from low energy experiments, such as Neutron EDM measurements



SMEFT constraints from EDMs

- loop QCD effects are important for extracting Wilson coefficients from low energy experiments, such as Neutron EDM measurements
- The dominant operator relevant for Neutron EDMs is the operator



$$O_W = \frac{d_W}{6} f^{abc} \varepsilon^{\mu\nu\alpha\beta} G^a_{\alpha\beta} G^b_{\mu\rho} G^{c\rho}_{\nu}$$

[Weinberg 1989]

Calculation of 2-loop AD of Weinberg’s operator

Method: Use R*-method with background field method

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Pros:

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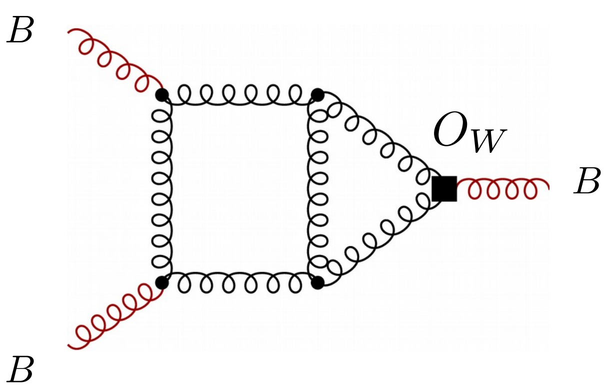
- $\varepsilon^{\mu\nu\rho\sigma}$ not well defined in dimreg. We used both `T Hooft-Veltman and Larin scheme check.



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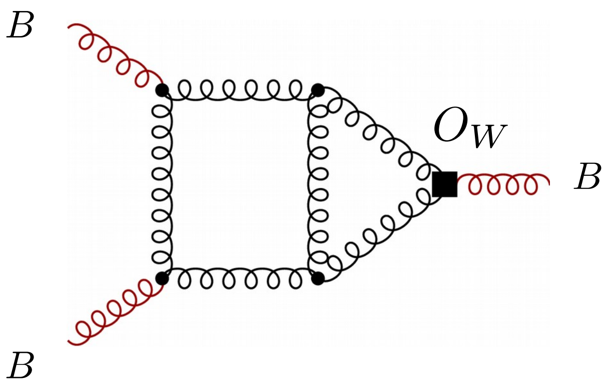

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- Need to compute at least 3-pt function with 3 derivatives
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Status:

- Have a result, now checking the 4-pt function.

Summary

- Presented the R* method - a powerful tool for extracting anomalous dimensions of arbitrary QFTs as well as decay rates in massless QCD
- Presented a new result for the five-loop beta function of Yang Mills Theory with fermions valid for arbitrary simple compact gauge group
- Briefly presented results for decay rates at N4LO; in particular a new result for the decay rate $H \rightarrow gg$
- Future Plans with R*:
  - Build fully automated tools for extracting anomalous dimensions for arbitrary (potentially non-renormalisable) QFTs, e.g. operators in the OPE of splitting functions, SM EFT, (Super-) Gravities, ..

Backup

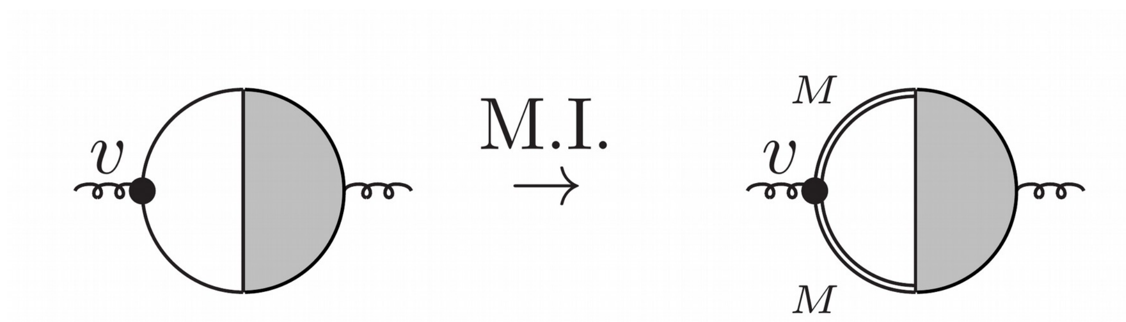
Global R^*

[Chetyrkin; Chetyrkin, Falcioini, FH, Vermaseren]

- Computationally more efficient than local approach
- Conceptually demanding and case specific (!)
- Used this method to compute all the Yang-Mills renormalisation constants for arbitrary gauge parameter dependence at 5 loops [arxiv:1709.08541]
- Method paper in progress

Global R^*

- Procedure
  - Insert a mass into propagators next to a particular vertex v:

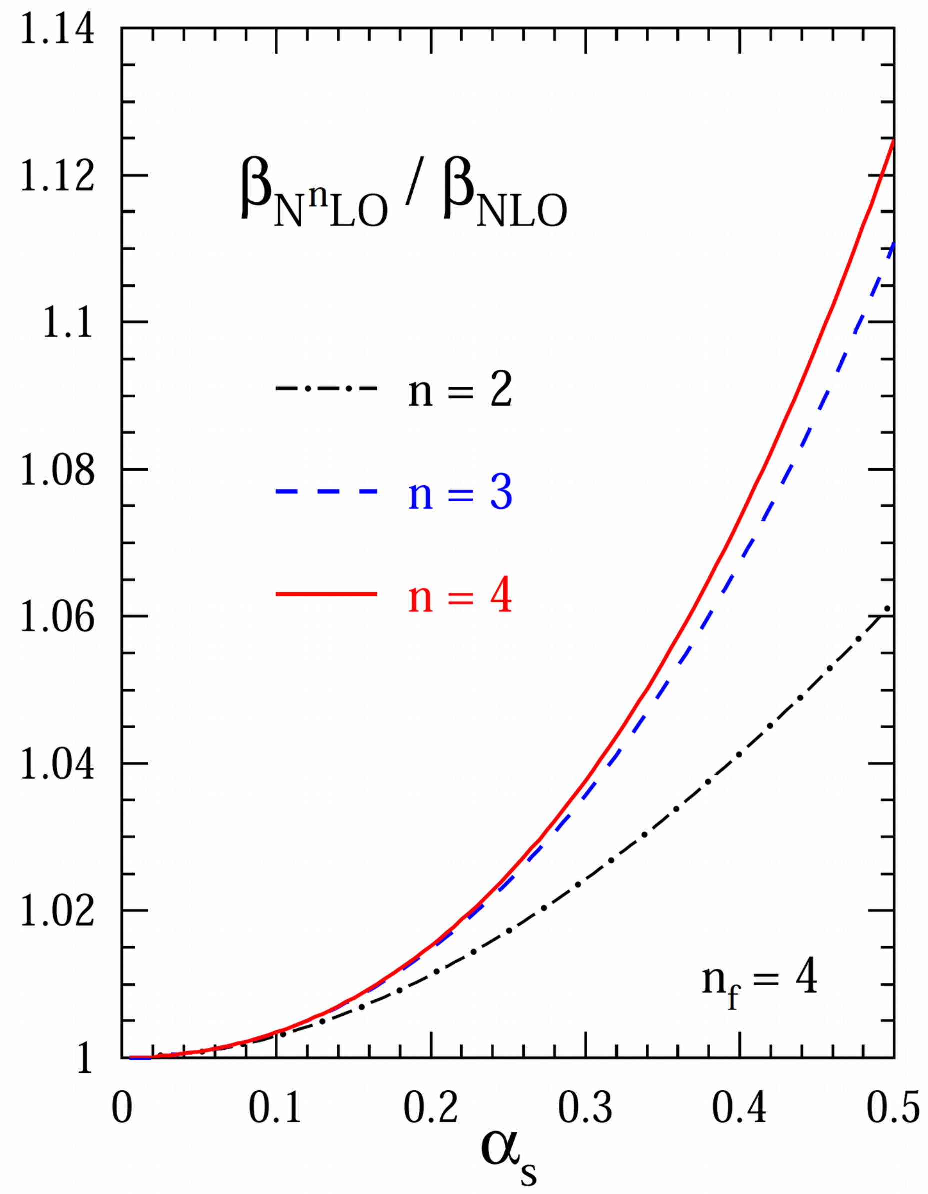


  - Derive the global UV counterterm
  - Expand around large M
  - Derive the global IR counterterm

$$Z(\Gamma(Q)) = \Gamma^{R/v}(M) + (\delta V^{R/v}(M) + \delta Z_V) * \delta Z^{-1}$$

Convergence study

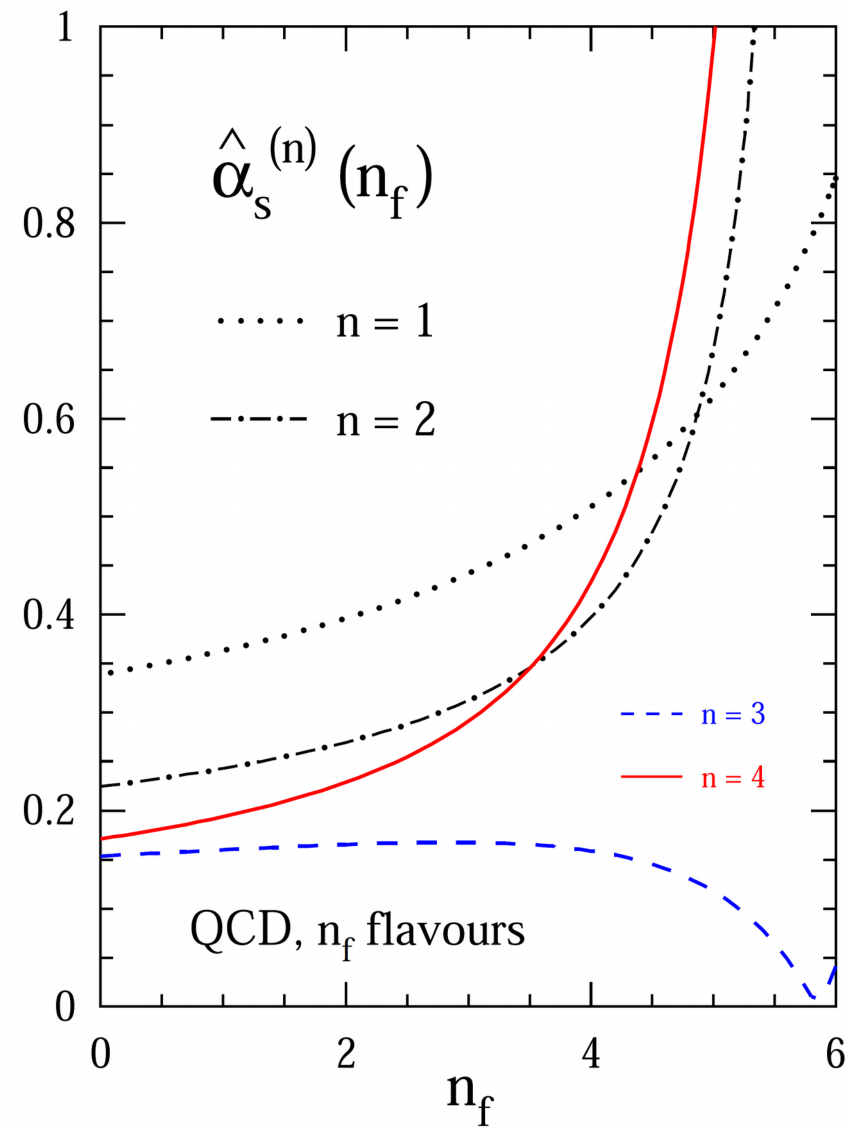
5-loop effects only visible for very high values of coupling, but even then perturbativity seems under great control.



n_f -convergence Study

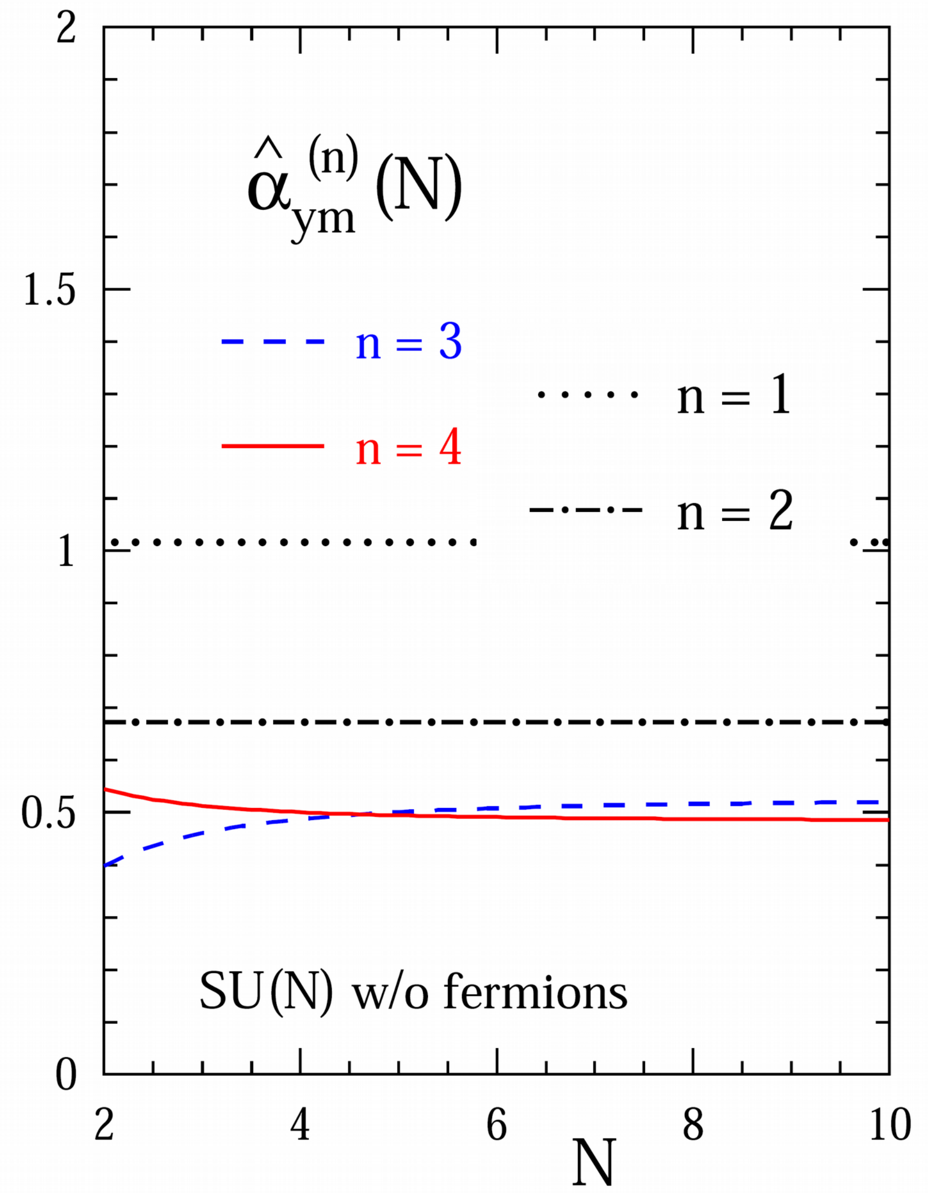
$$\widehat{\alpha}_{\mathrm{s}}^{(n)}(n_f) \;=\; 4\pi \left| \frac{\beta_{n-1}(n_f)}{4\,\beta_n(n_f)} \right|$$

This represents the value for α_s for which nth order is a quarter of the previous order.



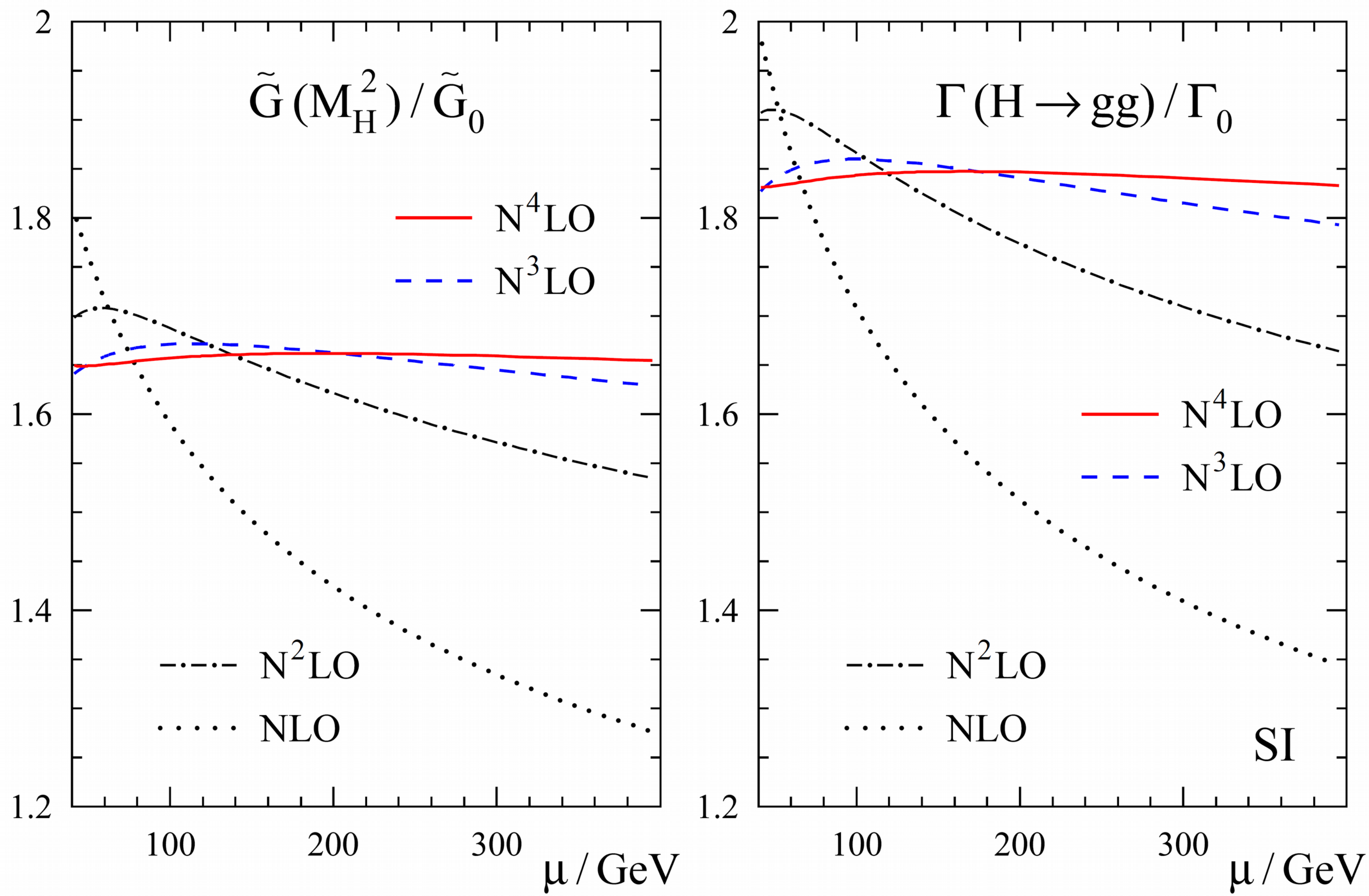
N-convergence study

$$\hat{\alpha}^{(n)}_{\mathrm{YM}}(N) \;=\; 4\pi\, N \left| \frac{\beta_{n-1}(N)}{4\,\beta_n(N)} \right|$$



With this parameterisation the series converges for

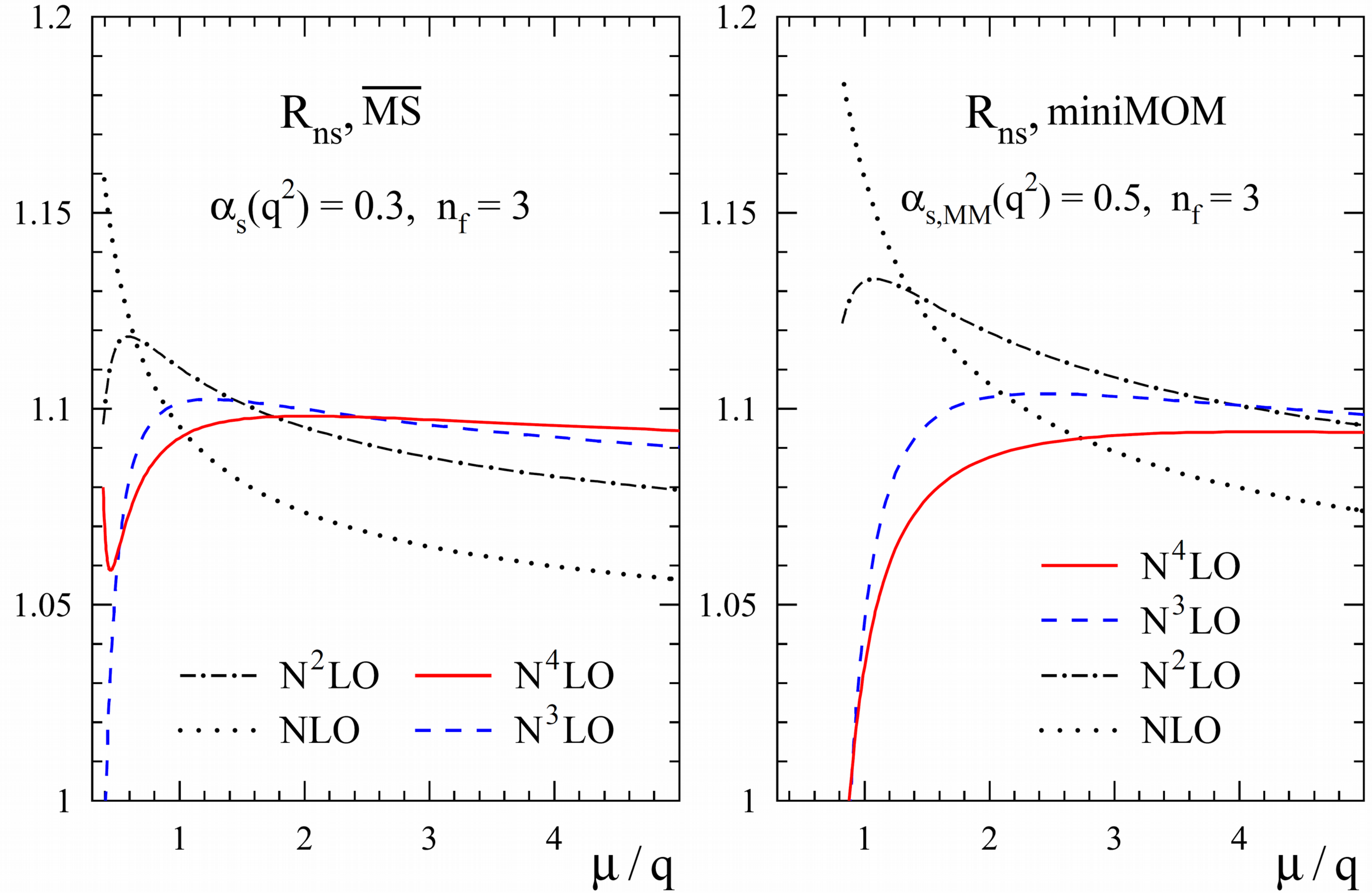
$$\hat{\alpha}^{(n)}_{\mathrm{YM}}(N) < 4$$



$\tilde{G}(M_H^2)/\tilde{G}_0$
$\Gamma(H \to gg)/\Gamma_0$
N^{4}LO
N^{3}LO
N^{2}LO
NLO
SI
2
1.8
1.6
1.4
1.2
100
200
300
μ / GeV

H→gg n_f dependence

$$
\begin{aligned}
K_{\mathrm{OS}}(n_f=1) &= 1 + 7.188498\,\alpha_s + 32.61874\,\alpha_s^2 + 112.031\,\alpha_s^3 + 300.278\,\alpha_s^4 + \ldots \\
K_{\mathrm{OS}}(n_f=3) &= 1 + 6.445775\,\alpha_s + 23.69992\,\alpha_s^2 + 56.1329\,\alpha_s^3 + 64.5259\,\alpha_s^4 + \ldots \\
K_{\mathrm{OS}}(n_f=5) &= 1 + 5.703052\,\alpha_s + 15.51204\,\alpha_s^2 + 12.6660\,\alpha_s^3 - 69.3287\,\alpha_s^4 + \ldots \\
K_{\mathrm{OS}}(n_f=7) &= 1 + 4.960329\,\alpha_s + 8.055116\,\alpha_s^2 - 19.2021\,\alpha_s^3 - 120.458\,\alpha_s^4 + \ldots \\
K_{\mathrm{OS}}(n_f=9) &= 1 + 4.217606\,\alpha_s + 1.329135\,\alpha_s^2 - 40.3039\,\alpha_s^3 - 107.042\,\alpha_s^4 + \ldots
\end{aligned}
$$



$R_{ns}, \overline{MS}$
$\alpha_s(q^2) = 0.3, \ n_f = 3$
R_{ns}, miniMOM
$\alpha_{s,MM}(q^2) = 0.5, \ n_f = 3$
N^{2}LO
NLO
N^{4}LO
N^{3}LO
1.2
1.15
1.1
1.05
1
1
2
3
4
μ / q

Analytic continuation π^2-terms

$$\begin{aligned} G(q^2) &= 1 + 3.952348\,\alpha_s + (10.629125 - \underline{3.673611})\,\alpha_s^2 \\ &\quad + (28.57606 - \underline{35.42782})\,\alpha_s^3 + (89.55798 - \underline{164.81711})\,\alpha_s^4 \end{aligned}$$

- Underlined contributions stem from analytic continuation and can be predicted from lower orders.
- Cancellations are observed between the genuine and analytic continuation terms – but the precise cancellation pattern changes at different loop orders.

$$\begin{aligned} g_4 &= 1267.05129 - \underline{1048.43622} - (394.681626 - \underline{281.704409})\,n_f \\ &\quad + (37.9589880 - \underline{25.1937144})\,n_f^2 - (1.28868582 - \underline{0.89082162})\,n_f^3 \\ &\quad + (0.01284135 - \underline{0.01026045})\,n_f^4 \ . \end{aligned}$$