Anomaly and the Modular Bootstrap

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based on 1904.04833
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Anomaly in CFT

- In Lagrangian field theory, the anomaly of a continuous symmetry is determined by the content of massless charged fermions.

  Anomaly $\leftrightarrow$ Massless charged fermions

- In CFT on R^d, all states are massless. Without a Lagrangian, the connection between anomaly and massless states is obscure.

The Quest

- What does anomaly entail in CFT?
- Is there a connection between anomaly and **light local operators**? Or equivalently, light states on $R^1 \times S^{d-1}$?

The Plan

- In 2d CFT, there is a powerful way to constrain the spectrum of local operators (states on $R^1 \times S^1$):
    Modular bootstrap.

- Let us consider the simplest setup:
    2d bosonic CFT with Z_2 flavor symmetry.

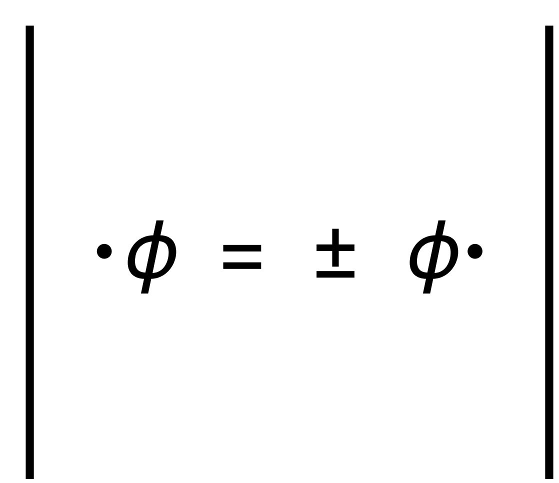
- We ask:
    How does the anomaly affect the universal bootstrap constraints on the spectrum?

Menu

- Introduce Z_2 anomaly using symmetry defects.
- Formulate modular bootstrap including Z_2 symmetry lines.
- Present bounds and discuss implications.
- Consider U(1) and relation to AdS_3 weak gravity conjecture.
- Bonus: Fusion category and the modular bootstrap.

Z2 symmetry lines

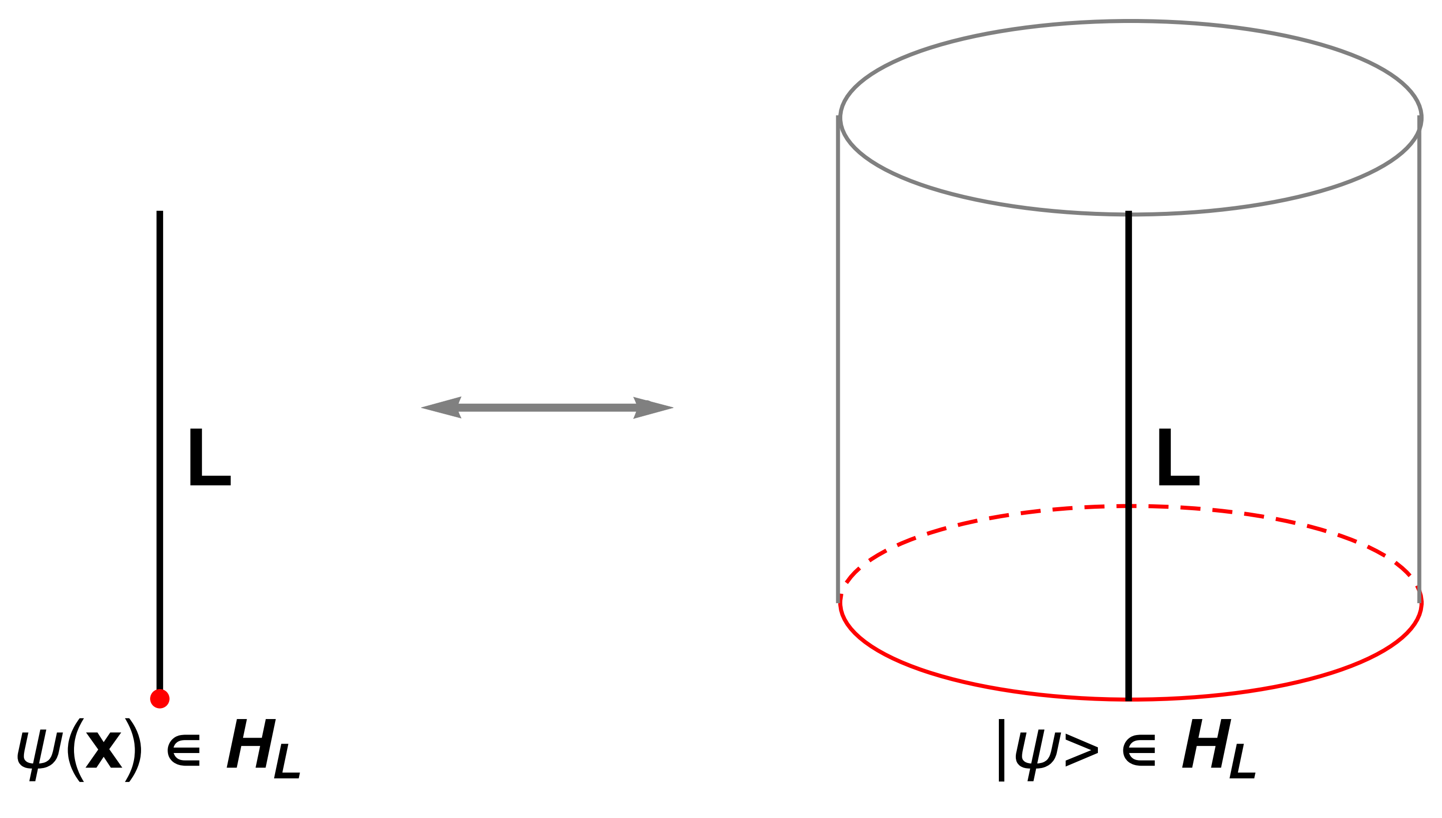
- 0-form symmetry → codimension 1 symmetry defects.
- In 2d, these are lines.
- Z_2 commutes with Virasoro^2 → the lines are topological.
- Operators are transformed by Z_2 when a line crosses.



Defect operators

- Lines can end on bosons, fermions or anyons.

- Via the cylinder map, these “defect operators” correspond to states on the cylinder quantized with “twisted” periodic boundary conditions.



Defect operators

- Defect operators are **point-like**, but **non-local**.
  c.f. Electron in QED must be attached to a Wilson line.

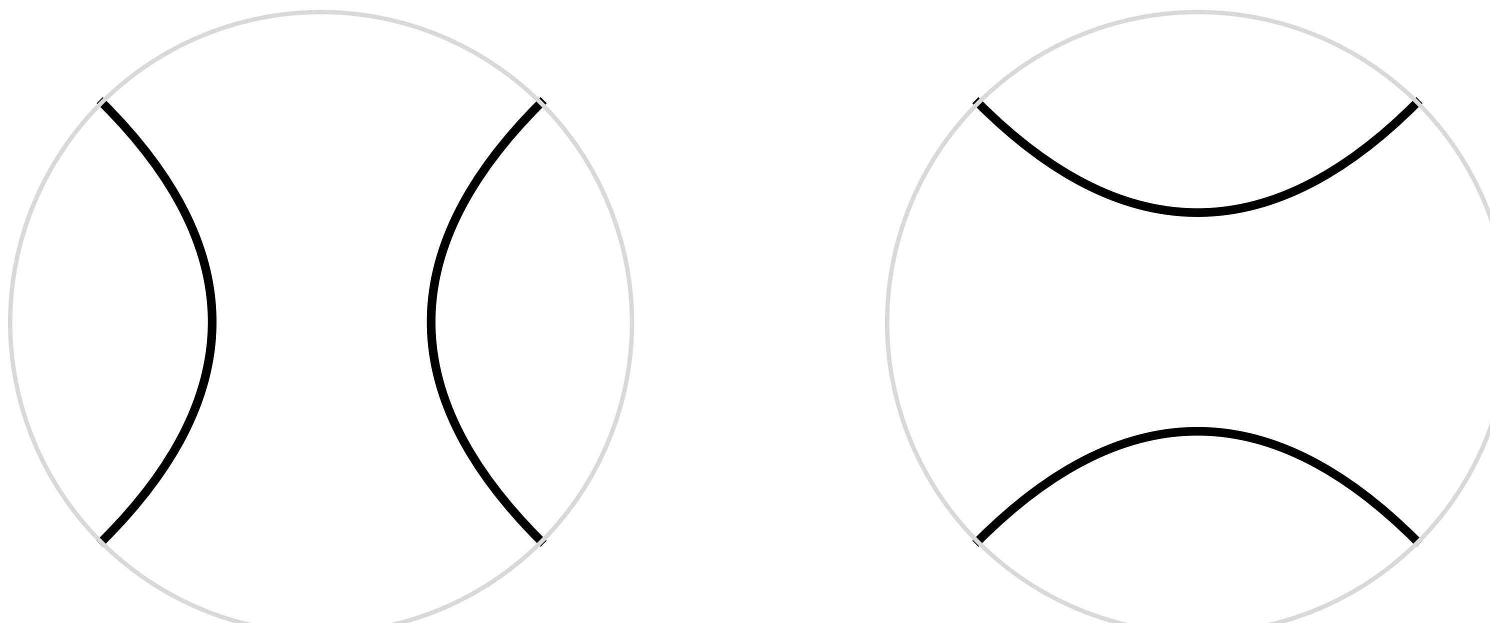
- Symmetry lines are topological → Spectrum of defect operators organize into Virasoro[2] families.

$$Z_{\mathcal{L}}(\tau,\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}_{\mathcal{L}}} e^{2\pi i(\tau L_0 - \bar{\tau}\bar{L}_0 - \frac{c+\bar{c}}{12})} = \sum_{h,\bar{h}} n_{h,\bar{h}} \chi_h(\tau)\chi_{\bar{h}}(\bar{\tau}), \quad n_{h,\bar{h}} \in \mathbb{Z}_{\geq 0}$$

- Non-locality:
  Spins $h - \bar{h}$ no longer need to be integer.
  We will see that the fractional part obeys a **spin selection rule** determined by the anomaly.

Crossing

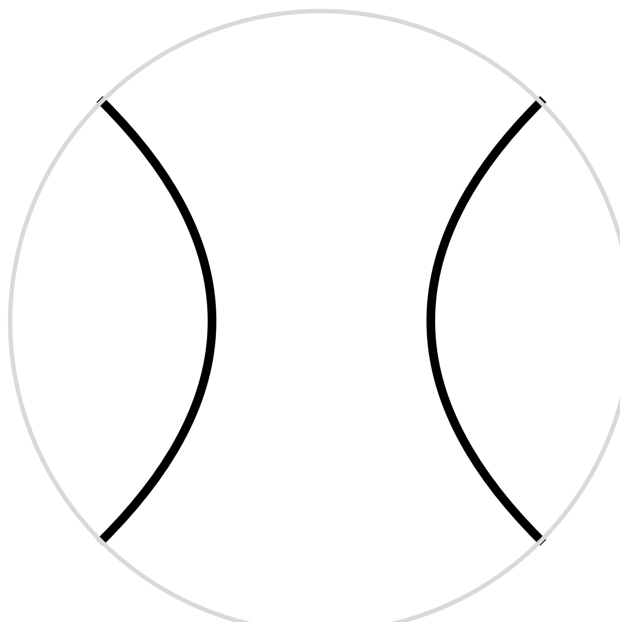
- Consider these two local configurations of lines:



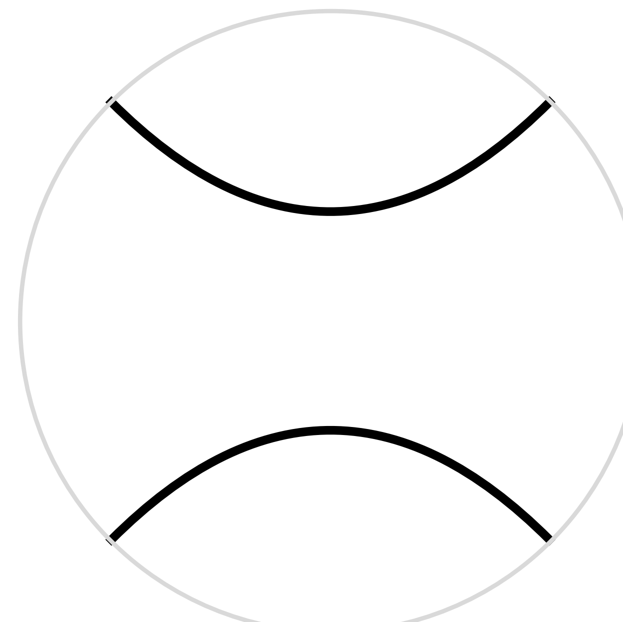
- On each side, the state on the circle is the vacuum.
- So they must be proportional to each other.

Crossing

- Crossing relation:



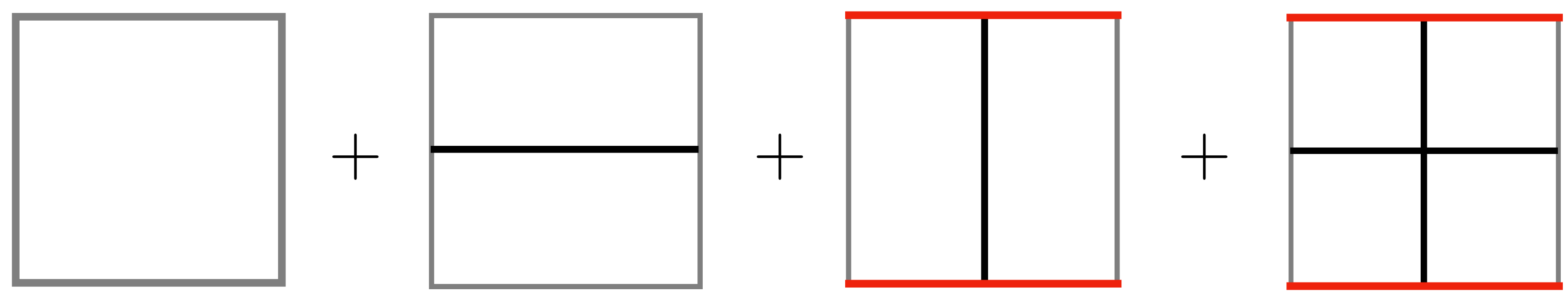
$$= \alpha$$

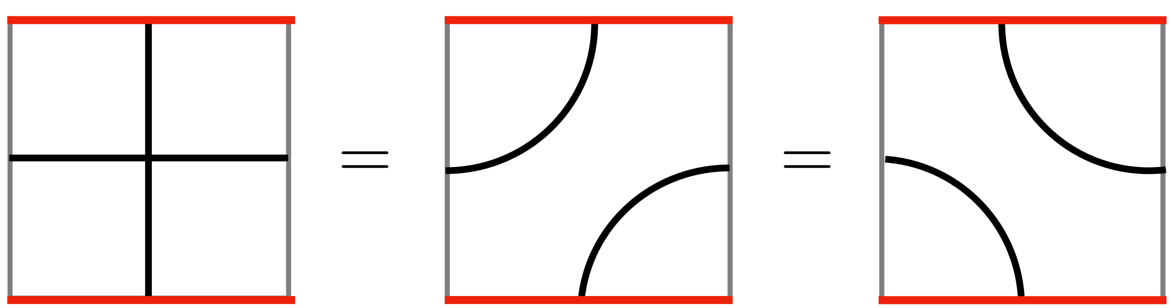


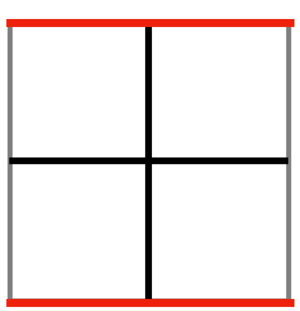
$$\alpha = \pm 1$$

Obstruction to orbifold

- Can we gauge the Z_2? The “would-be” partition function:

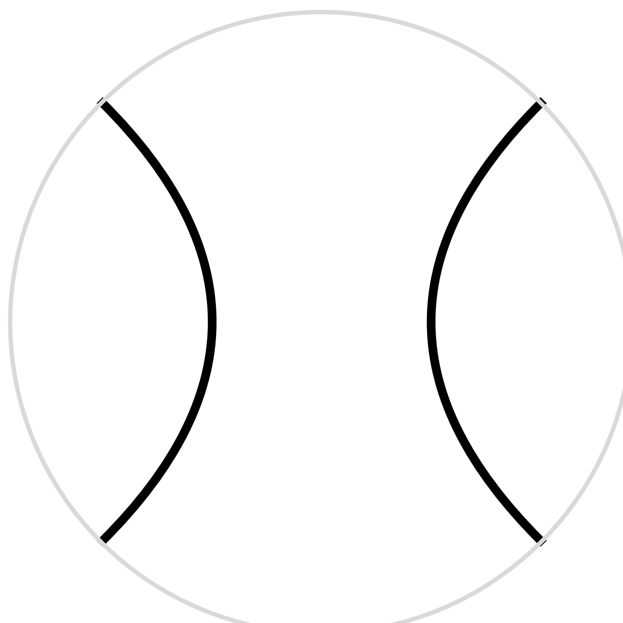
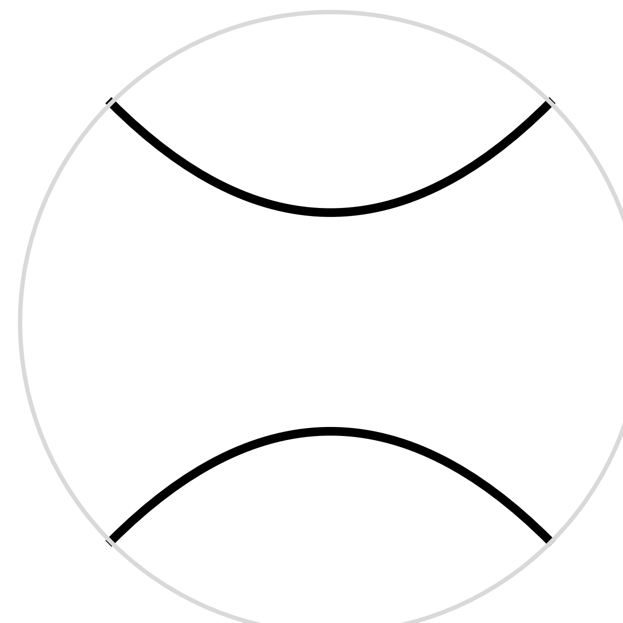
$Z_{\text{orbifold}} = \frac{1}{2}\Big($  $\Big)$

-  if $\alpha = 1$.

-  ill-defined if $\alpha = -1$ → **Cannot gauge.**

Anomaly

- Crossing relation:

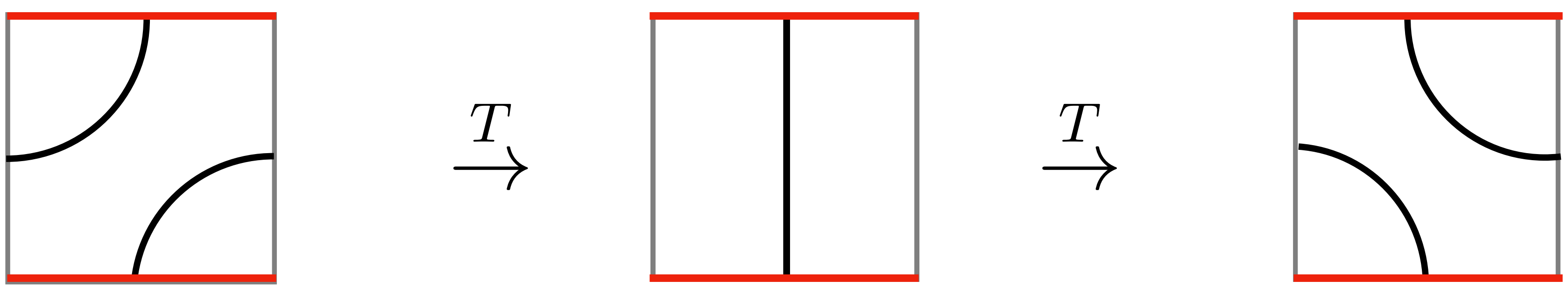
 $= \alpha$ 

- Non-anomalous: $\alpha = 1$

- Anomalous: $\alpha = -1$

Spin selection rule

- The spins of defect operators are constrained by considering the T transformation:

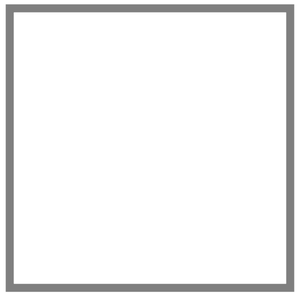


- Acting on each state, $T^2 = e^{4\pi i s} = \alpha$.

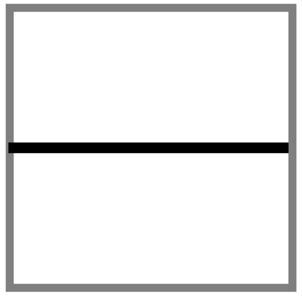
- The spins of defect operators in H_L must obey

$$s \in \begin{cases} \mathbb{Z}/2 & \alpha = 1 \\ \mathbb{Z}/2 + 1/4 & \alpha = -1 \end{cases}$$

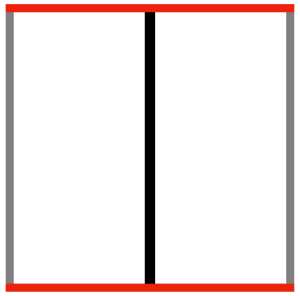
Partition functions

- 

$$Z(\tau,\bar{\tau}) = \sum_{h,\bar{h}} (n^+_{h,\bar{h}} + n^-_{h,\bar{h}})\chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$$

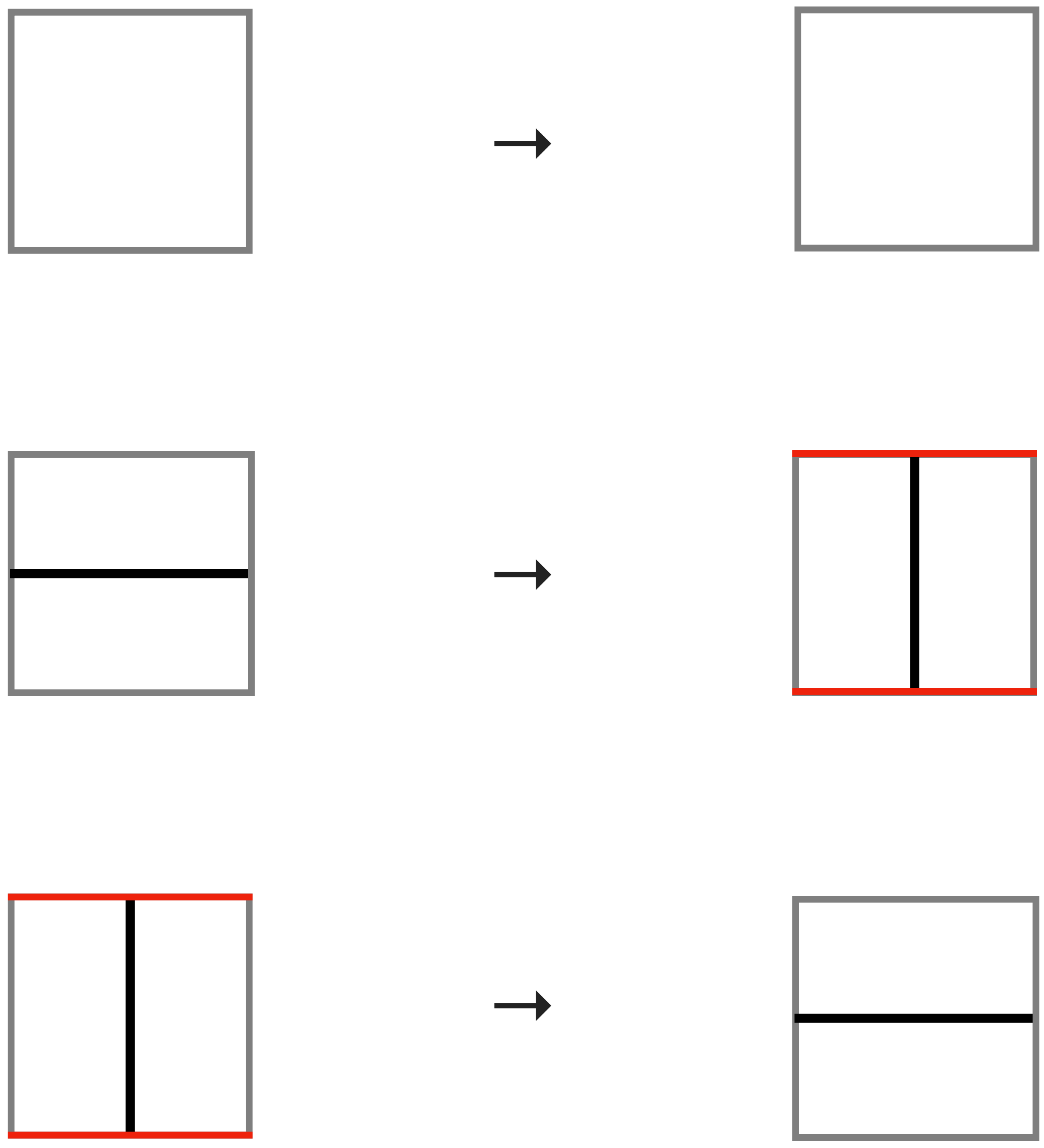
- 

$$Z^{\mathcal{L}}(\tau,\bar{\tau}) = \sum_{h,\bar{h}} (n^+_{h,\bar{h}} - n^-_{h,\bar{h}})\chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$$

- 

$$Z_{\mathcal{L}}(\tau,\bar{\tau}) = \sum_{h,\bar{h}} n^{\mathcal{L}}_{h,\bar{h}}\chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$$

Modular S transform



Positive basis

- To derive constraints by bootstrap, we need to work with objects that have positive expansions.
- Consider combinations of partition functions that count states without sign:

$$Z^{+}(\tau,\bar{\tau}) = \tfrac{1}{2}[Z(\tau,\bar{\tau}) + Z^{\mathcal{L}}(\tau,\bar{\tau})] = \textstyle\sum_{h,\bar{h}} n^{+}_{h,\bar{h}}\chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$$

$$Z^{-}(\tau,\bar{\tau}) = \tfrac{1}{2}[Z(\tau,\bar{\tau}) - Z^{\mathcal{L}}(\tau,\bar{\tau})] = \textstyle\sum_{h,\bar{h}} n^{-}_{h,\bar{h}}\chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$$

$$Z_{\mathcal{L}}(\tau,\bar{\tau}) = \textstyle\sum_{h,\bar{h}} n^{\mathcal{L}}_{h,\bar{h}}\chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$$

Modular crossing

- The modular crossing equation written in the positive basis is

$$\begin{pmatrix} Z^+(-1/\tau, -1/\bar{\tau}) \\ Z^-(-1/\tau, -1/\bar{\tau}) \\ Z_{\mathcal{L}}(-1/\tau, -1/\bar{\tau}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} Z^+(\tau, \bar{\tau}) \\ Z^-(\tau, \bar{\tau}) \\ Z_{\mathcal{L}}(\tau, \bar{\tau}) \end{pmatrix}$$

- **Anomaly** determines the **spin content** of H_L.

Modular bootstrap

- We can expand this equation in characters

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \sum_{\mathcal{H}^+} n^+_{h,\bar{h}} \left[\begin{pmatrix}1\\0\\0\end{pmatrix} \chi_h(-\tfrac{1}{\tau})\chi_{\bar{h}}(-\tfrac{1}{\bar{\tau}}) - \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\1\end{pmatrix} \chi_h(\tau)\chi_{\bar{h}}(\bar{\tau}) \right]$$

$$+ \sum_{\mathcal{H}^-} n^-_{h,\bar{h}} \left[\begin{pmatrix}0\\1\\0\end{pmatrix} \chi_h(-\tfrac{1}{\tau})\chi_{\bar{h}}(-\tfrac{1}{\bar{\tau}}) - \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\-1\end{pmatrix} \chi_h(\tau)\chi_{\bar{h}}(\bar{\tau}) \right]$$

$$+ \sum_{\mathcal{H}_\mathcal{L}} n^\mathcal{L}_{h,\bar{h}} \left[\begin{pmatrix}0\\0\\1\end{pmatrix} \chi_h(-\tfrac{1}{\tau})\chi_{\bar{h}}(-\tfrac{1}{\bar{\tau}}) - \begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\0\end{pmatrix} \chi_h(\tau)\chi_{\bar{h}}(\bar{\tau}) \right]$$

Modular bootstrap

- Schematically,

$$\vec{0} = \sum_{\mathcal{H}^+} n^+_{h,\bar{h}} \vec{\mathcal{X}}^+_{h,\bar{h}}(\tau,\bar{\tau}) + \sum_{\mathcal{H}^-} n^-_{h,\bar{h}} \vec{\mathcal{X}}^-_{h,\bar{h}}(\tau,\bar{\tau}) + \sum_{\mathcal{H}_{\mathcal{L}}} n^{\mathcal{L}}_{h,\bar{h}} \vec{\mathcal{X}}^{\mathcal{L}}_{h,\bar{h}}(\tau,\bar{\tau})$$

- To proceed, make some assumption about the spectrum. *e.g.* Lightest odd primary has scaling dimension Δ.

- Try to disprove the assumption:
  → Act by a vector-valued linear functional $\mathbb{L}$.
  → If we manage to make $\mathbb{L}[\mathcal{X}^*]$ non-negative for all allowed $h, \bar{h}$, then we have a contradiction.
  → Profit!

Modular bootstrap

- We make use of
  Virasoro^2 symmetry
  Positivity of degeneracies
  Modular covariance
  Spin selection rule for H_L ← Anomaly dependent!

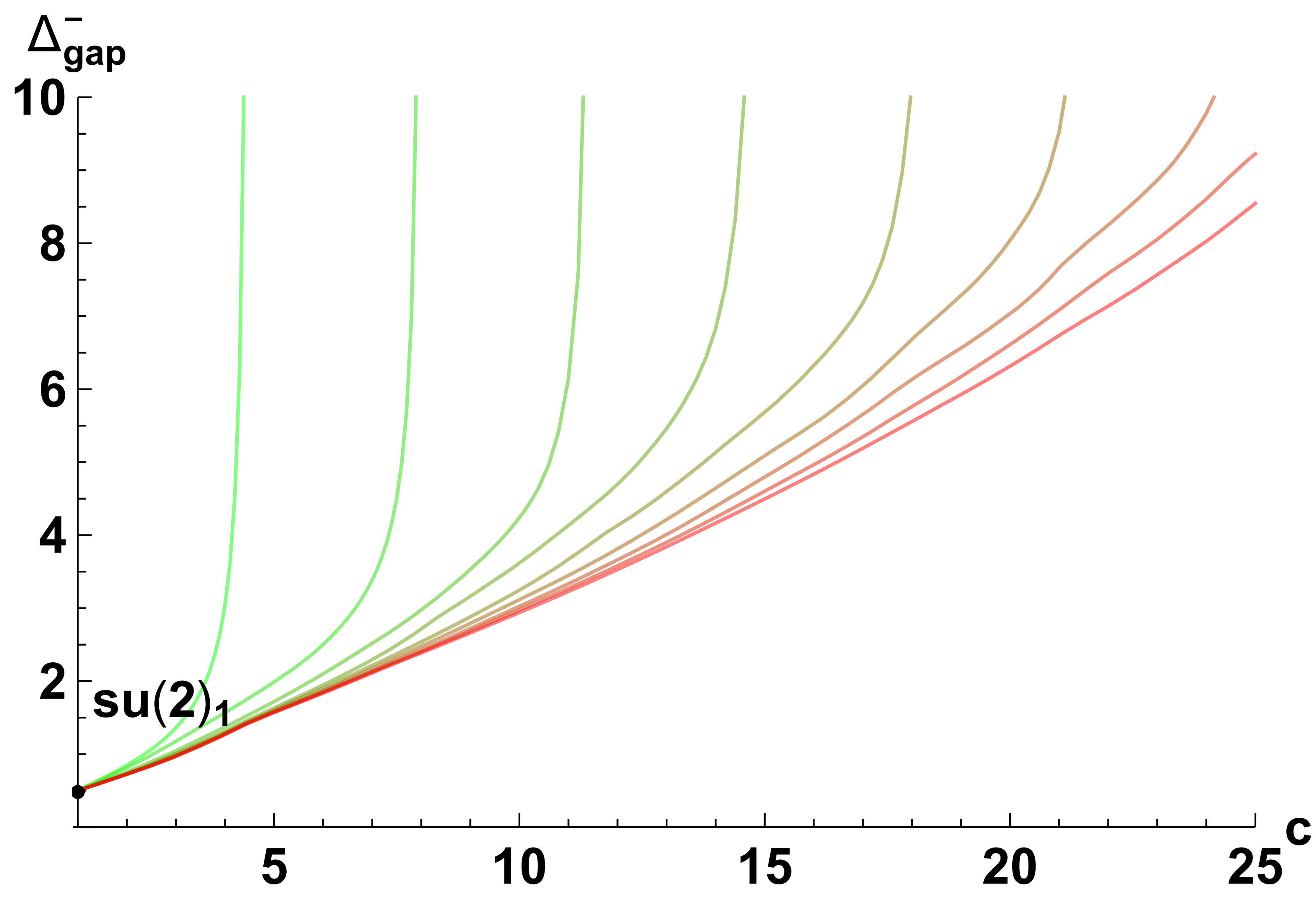
- …to produce constraints on the spectrum:
  Bound on the gap in each sector
  Bound on the scalar gap

Bound on Gap

	Non-anomalous	Anomalous
Even	✓	✓
Odd		✓
Twisted		

Odd gap

- Bound on odd gap only exists when Z_2 is anomalous.



Odd gap

- When Z_2 is non-anomalous, the lightest charged operator can be arbitrarily heavy.

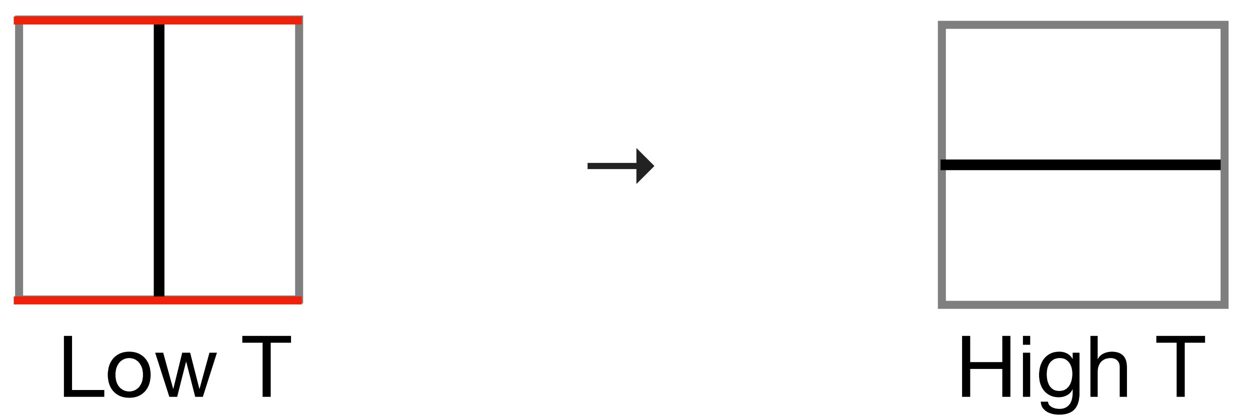
  → Non-anomalous symmetry can be hard to detect.

- In contrast, when Z_2 is anomalous, there must be “light” enough charged operators.

  → **Anomalous symmetry is difficult to “hide”.**

A Cardy-like argument

- Consider the modular S transform in the Cardy limit:



- RHS has Cardy growth determined by the effective vacuum energy $E_0 = c/12 - \Delta_0$.

- **Anomalous** spin selection rule implies that the lightest defect operator has $\Delta_0 \geq 1/4$.

A Cardy-like argument

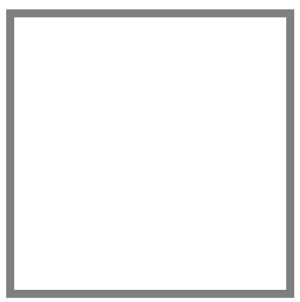
- **Anomalous:**

Cardy growth of $n^+ + n^-$ > Cardy growth of $n^+ - n^-$.

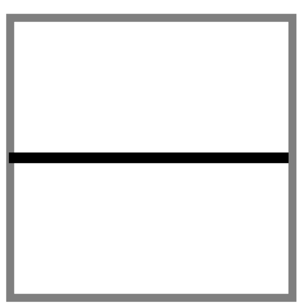
- Therefore, n^- has nontrivial Cardy growth.

- Lightest odd operator < Onset of Cardy regime.

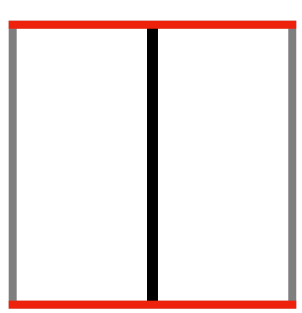
Partition functions

- 

$$Z(\tau,\bar{\tau}) = \sum_{h,\bar{h}} (n^{+}_{h,\bar{h}} + n^{-}_{h,\bar{h}})\chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$$

- 

$$Z^{\mathcal{L}}(\tau,\bar{\tau}) = \sum_{h,\bar{h}} (n^{+}_{h,\bar{h}} - n^{-}_{h,\bar{h}})\chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$$

- 

$$Z_{\mathcal{L}}(\tau,\bar{\tau}) = \sum_{h,\bar{h}} n^{\mathcal{L}}_{h,\bar{h}}\chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$$

Even gap

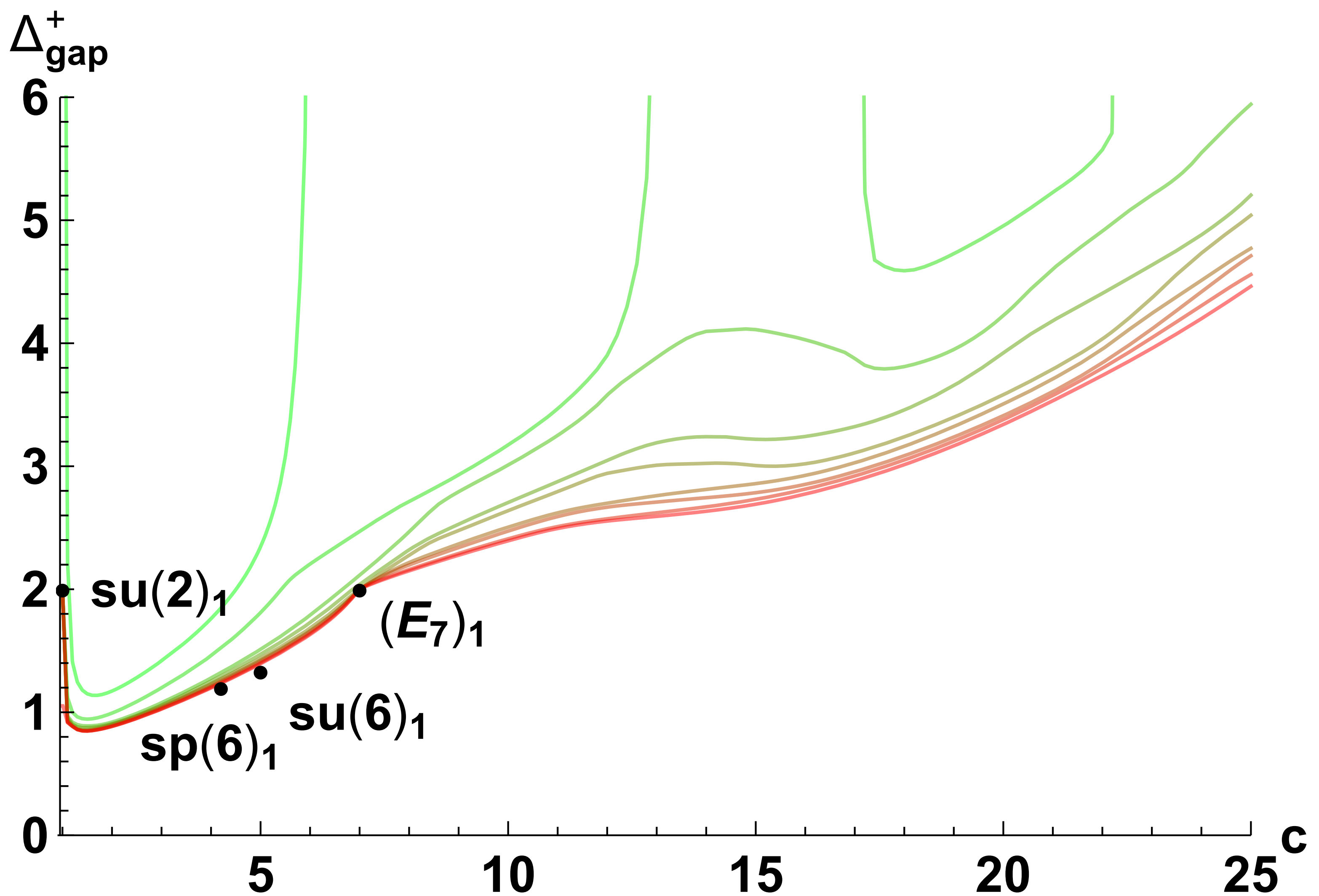
- Bounds on the even gap depend on the anomaly, but in a less drastic way.

- Saturated or almost saturated by several WZW models.

- We can analytically write down a cubic-derivative functional to derive a bound valid at large central charge.
  → Same slope as Simeon's bound.

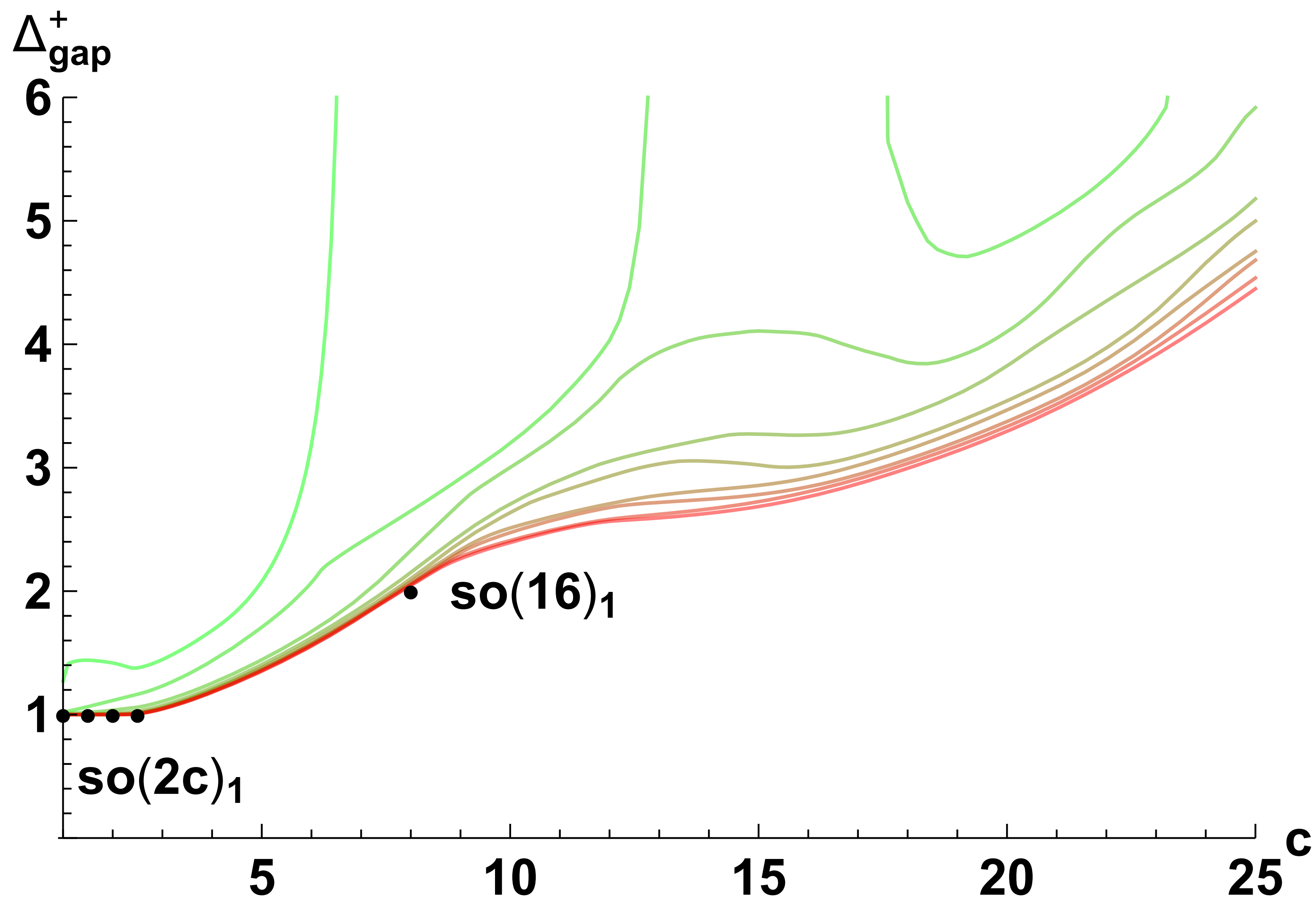
Even gap

- **Anomalous**



Even gap

- **Non-anomalous**



Even scalar gap and RG

- Given a CFT with a symmetry, a natural question is whether it is reachable by symmetry-preserving RG flows.

- More precisely, can we reach it without fine-tuning?

- This requires the absence of symmetry-preserving relevant scalar primaries.

- Otherwise, close to the fixed point, the relevant deformation is generically turned on, and drives the flow further towards the IR.

Even scalar gap and RG

- For small values of the central charge, bootstrap shows that there necessarily exists a symmetry-preserving relevant scalar primary.

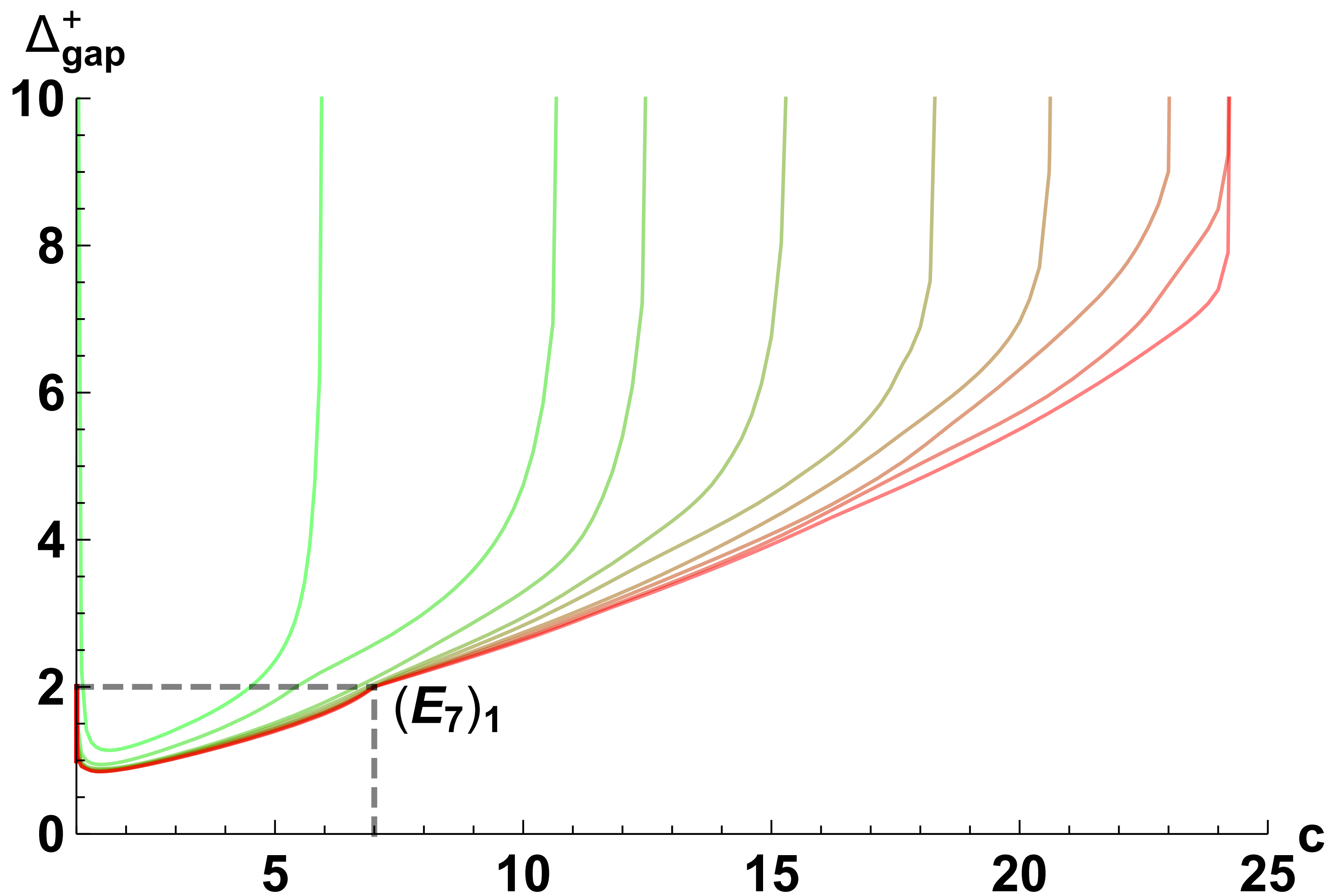
- No-Go:

  CFT with Z_2 symmetry cannot be obtained by RG flow that only preserves Z_2, and without fine-tuning, if the central charge is in the following range:

  Anomalous: $1 < c < 7$
  Non-anomalous: $1 < c < 7.81$

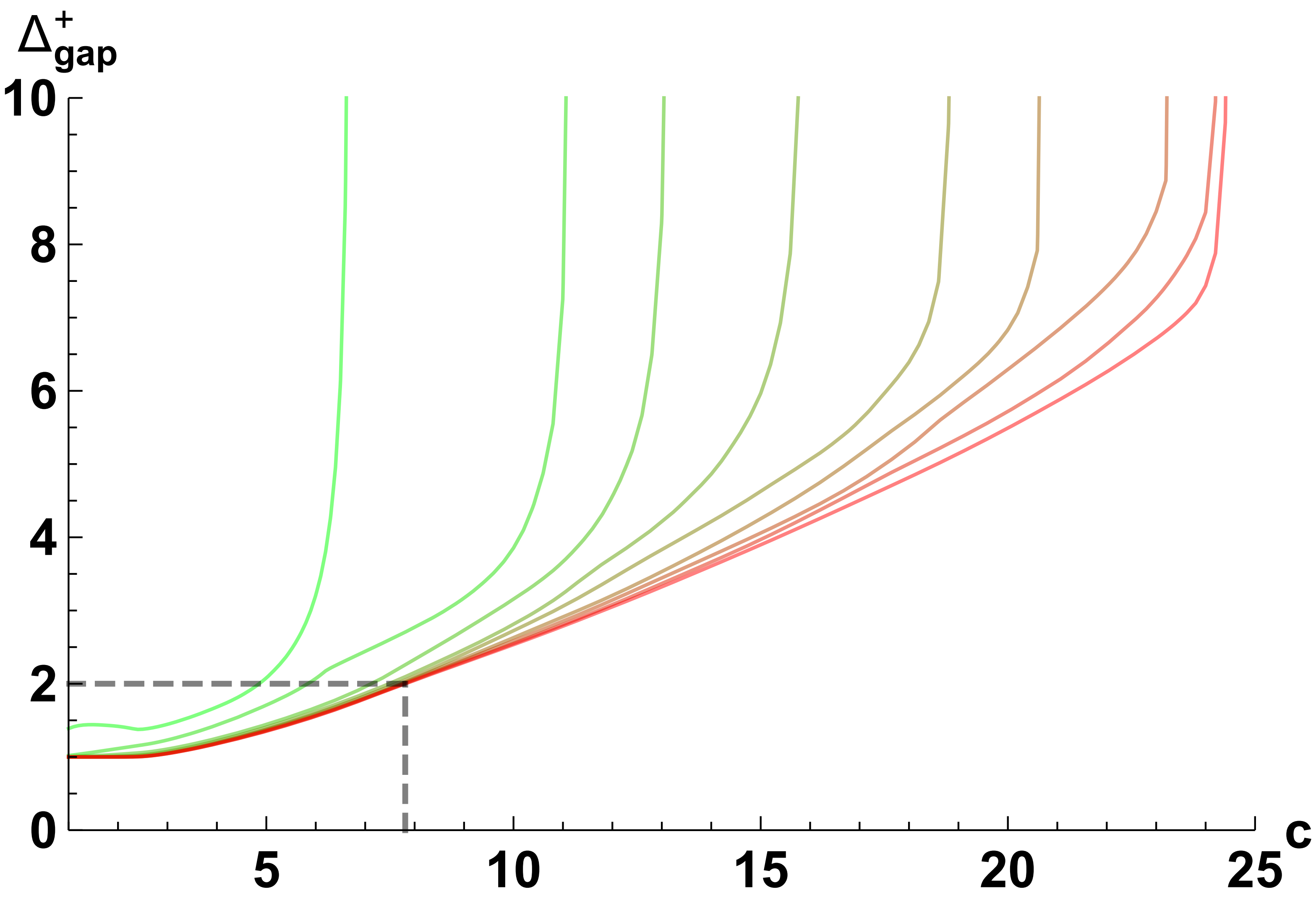
Even scalar gap

- **Anomalous**



Even scalar gap

- **Non-anomalous**



A moment of Z_2en

A moment of Z_2en

- Usually, the anomaly of a discrete symmetry constrains the **gapped** phase of QFT:

  Either there is a TQFT to match the anomaly…

  Or the symmetry is spontaneously broken.

- Here, we use Z_2 anomaly to constrain local operators in a **gapless** phase.

A moment of Z_2en

- Universal bound on the odd gap exists for anomalous Z_2

    …does not exist for non-anomalous Z_2.

- Universal bound on the even gap exists and differ for anomalous and non-anomalous Z_2

    …saturated by various WZW models.

- Universal bound on the even scalar gap

    …led to a No-Go result about RG flows.

A moment of Z_2en

- Our results have direct implications on any symmetry group G that contains Z_2 as a subgroup.

- Anomaly of G → Anomaly of subgroup.

- Universal bound on odd operators under anomalous Z_2 → Universal bound on charged operators under G, if the Z_2 subgroup is anomalous.

- We will see an explicit example momentarily.

A moment of Z_2en

- Consider G = U(1).
- Wait…a bound on the lightest U(1) charged operator? Smells like the weak gravity conjecture in AdS_3.
- Why is there only a bound when anomalous?

U(1) and WGC

- Modular bootstrap with **holomorphic** “U(1)” flavor symmetry has been studied by
  Benjamin-Dyer-Fitzpatrick-Kachru [1603.09745]
  Montero-Shiu-Soler [1606.08438].

- They obtained bounds on the lightest charged operator.

- They claimed to prove the AdS_3 version of the weak gravity conjecture.

- …The interpretation of their results requires a closer examination.

U(1) and WGC

- Generically, a holomorphic J generates not U(1) but R.
- More precisely, the symmetry group generated by J and $\bar{J}$ together may have topology T^2, but the compact directions are generated by combinations of J and $\bar{J}$.
- Holomorphic J is always anomalous.
- The existence of a bound on the lightest charged operator has more to do with **anomaly**.

U(1) and WGC

- One should consider general non-holomorphic U(1), especially if making connection to WGC.
- The anomaly of U(1) is characterized by integer $(k - \bar{k})/2$.
- The anomaly of the Z_2 subgroup is $\alpha = (-1)^{(k-\bar{k})/2}$.
- Our Z_2 odd bound → AdS_3 WGC for U(1), odd $(k - \bar{k})/2$.
  c.f. Montero-Shiu-Soler for holomorphic U(1), $\bar{k} = 0$.
- We believe that a bootstrap bound on charged operators can be derived for general anomalous U(1), $k - \bar{k} \neq 0$.

U(1) and WGC

- But there is no bound for non-anomalous U(1), $k - \bar{k} = 0$.
- Counter-example at $c=1$:
  Free compact boson with radius R.
- Momentum U(1) and winding U(1), both non-anomalous.
- Charged operators become arbitrarily heavy for arbitrarily small or large radius R.
- Tensor with CFTs → Counter-examples for other c values.

Outlook

- Generalize to other discrete groups.
  → Connection to condensed matter physics.

- Generalize to non-symmetry lines.
  → Fusion categories (sneak preview next).

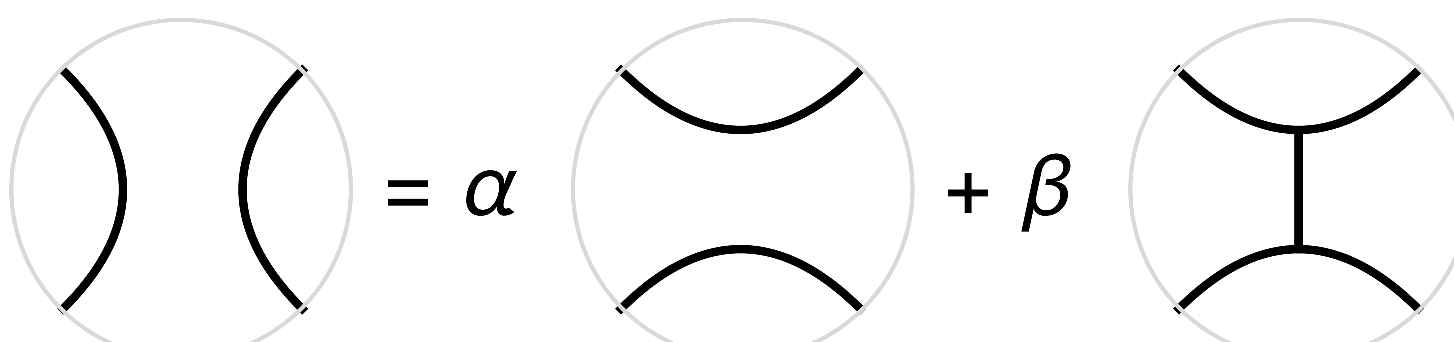
- Relation between anomaly and bounds on charged operators in higher dimensions?
  → Cannot work for discrete symmetry, since any unitary bosonic anomaly can be carried by TQFT with identity being the unique local operator.
  → What about continuous symmetries?

Fusion categories

- Topological lines can form a fusion algebra that is not a group, but a ring.

- This generalizes:
  Symmetry group → Grothendieck ring
  Anomaly → Fusion category

- A fusion category includes information about the crossing relations of lines.



Fibonacci

- Fusion algebra (Grothendieck ring):
  $W^2 = I + W$.

- Operators have charges

$$\langle W \rangle = \frac{1 \pm \sqrt{5}}{2} \, .$$

- There is one fusion category compatible with unitarity.
  → Realized in tricritical Ising, etc.

- There is one that is necessarily non-unitary.
  → Realized in Lee-Yang CFT.

Fibonacci

- Similar to Z_2, the two fusion categories for Fibonacci have different spin selection rules.

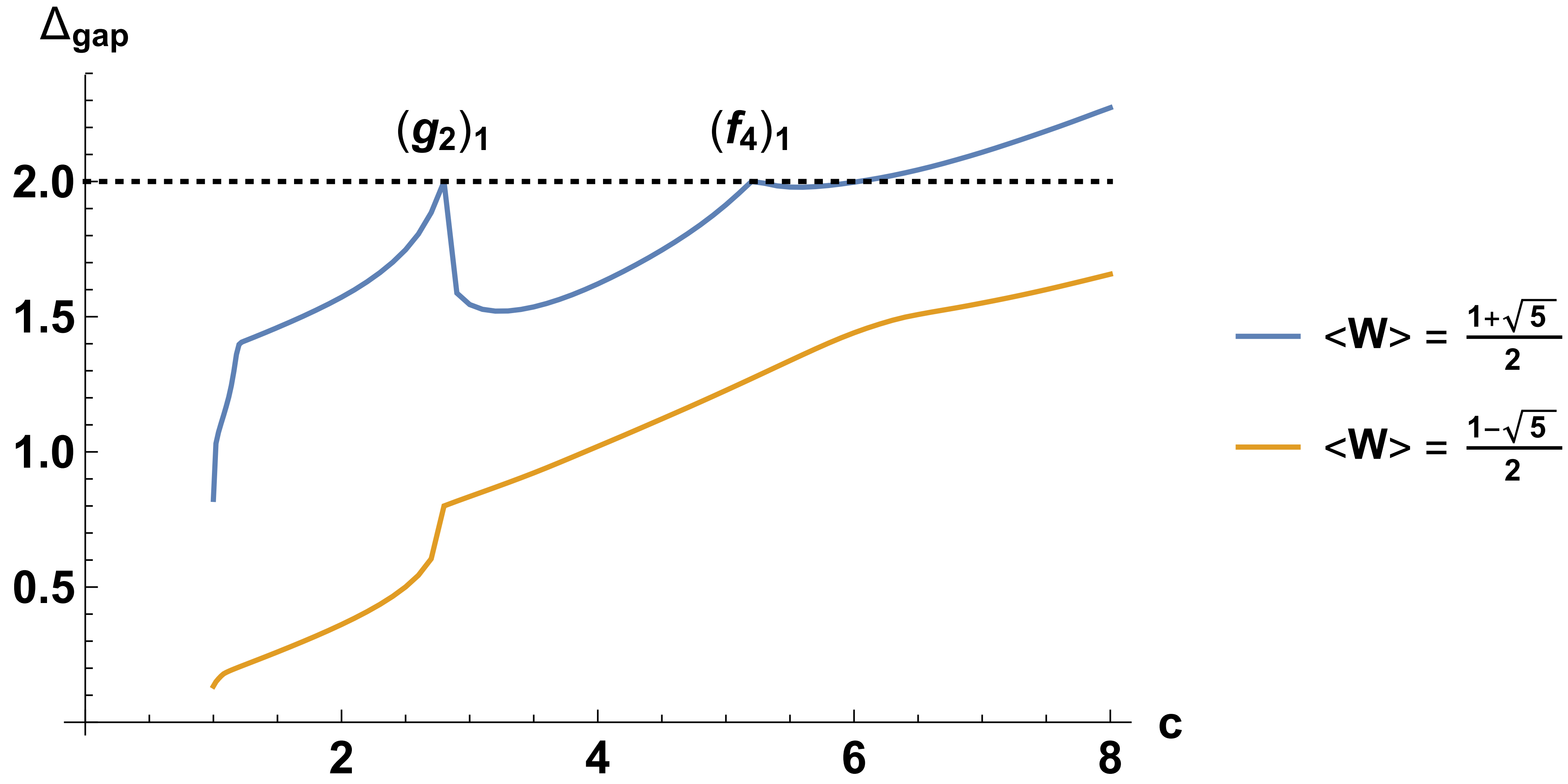
- Fusion category compatible with CFT unitarity:

$$s \in \mathbb{Z} \pm \left\{0, \frac{2}{5}\right\}$$

- Fusion category incompatible with CFT unitarity:

$$s \in \mathbb{Z} \pm \left\{0, \frac{1}{5}\right\}$$

Fibonacci (unitary)



Thank you!

ありがとうございました！