Some Results on 4d Chern-Simons Theory: String Theory Realization, and Holography

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Scope of Presentation

A Review of 4d Chern-Simons theory

- 4d Chern-Simons theory has the action

$$S = \frac{1}{\hbar} \int_{Y \times \Sigma} C \wedge \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \Big), \qquad (1.1)$$

  where $\mathcal{A}$ is a complex-valued gauge field, Y is a 2-manifold, and Σ is a Riemann surface endowed with a holomorphic one-form $C = C(z)dz$.
- Topological along Y, but depends on the complex structure of Σ.
- It has a complex gauge group, denoted G.

- Initially derived from deformed, twisted $\mathcal{N} = 1$ SUSY gauge theory by Costello.[*]
- Subsequently studied in depth by Costello, Witten and Yamazaki.[†]
- Describes **integrable lattice models** of clasical statistical mechanics, such as the six-vertex and eight-vertex model.

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[*] K. Costello, *Supersymmetric gauge theory and the Yangian*, arXiv:1303.2632

[†] K. Costello, E. Witten, M. Yamazaki, *Gauge Theory and Integrability, I, II*, arXiv:1709.09993, 1802.01579

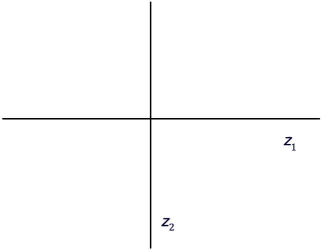
- Theory is unrenormalizable by power counting, as $\hbar$ has dimensions of inverse mass.
- But theory **can be quantized in perturbation theory** - all conceivable counterterms vanish via EOM.
- Moreover, BV quantization was used by Costello to show that the theory has a well-defined perturbation expansion.

- The action involves only the ratio $C/\hbar$ - naively, a zero of C corresponds to a point at which $\hbar \to \infty$.
- But the theory is only defined perturbatively, so C cannot have zeros, though it may have poles.
- This restricts Σ to one of the following possibilities:

$$\begin{aligned} \Sigma &= \mathbb{C}\,, & C &= dz\,, \quad \text{(rational)}\,, \\ \Sigma &= \mathbb{C}^{\times} = \mathbb{C}/\mathbb{Z}\,, & C &= \frac{dz}{z}\,, \quad \text{(trigonometric)}\,, \\ \Sigma &= E = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})\,, & C &= dz\,, \quad \text{(elliptic)}\,. \end{aligned} \tag{1.2}$$

- As shown, the three choices of Σ lead to rational, trigonometric and elliptic integrable lattice models.

- Costello, Witten and Yamazaki explicitly showed how to derive the **quasi-classical R-matrix** from correlation functions of crossed Wilson lines.
- E.g., the rational R-matrix for Wilson lines on $Y = \mathbb{R}^2$:



$$= I + \frac{\hbar\, c_{\rho,\rho'}}{z_1 - z_2} + \mathcal{O}(\hbar^2)$$

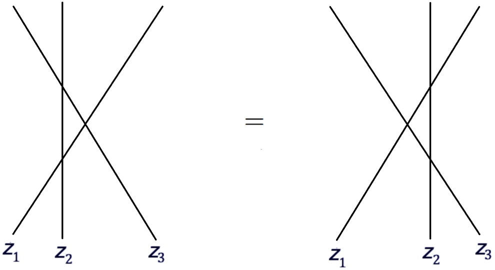
- Here the Wilson lines at z_1 and z_2 are respectively in representations ρ and ρ', with $c_{\rho,\rho'} = \sum_a T_{a,\rho} \otimes T_{a,\rho'}$.

- Such an R-matrix, denoted $R_{\rho\rho'}(z_1, z_2)$, is a solution of the **Yang-Baxter equation with spectral parameter**, i.e.,

$$R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3) = R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2)$$

- The YBE underlies the integrability of the integrable lattice models, as it leads to commuting transfer matrices.

- Can be realized in 4d CS theory due to the topological symmetry along Y.



- No singular behaviour arises in moving a Wilson line, as long as z_1, z_2 and z_3 are distinct.

- Outside of perturbation theory, 4d CS is not well-understood - path integral is **exponentially divergent**.
- **Question**: What is the nonperturbative definition of 4d CS theory?
- Suggestion[‡] - **Nonperturbative definition** comes from the **D4-NS5 system** of string theory, similar to how the D3-NS5 system realizes the nonperturbative 3d analytically-continued Chern-Simons theory.[§]
- Also, unlike 3d CS theory, much work on 4d CS theory has not involved canonical quantization, current algebras, and boundary theories.
- **Question**: Is there a boundary WZW theory for 4d CS theory?

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‡. E. Witten, *Integrable Lattice Models From Gauge Theory*, arXiv:1611.00592

§. E. Witten, *Fivebranes and Knots*, Quantum Topology **3** (1) (2012) 1–137

We shall attempt to answer these questions in today's talk. This talk is based on

- M. Ashwinkumar, K.-S. Png, M.-C. Tan, in progress
- M. Ashwinkumar, *Boundary Dynamics of 4d Chern-Simons Theory*, in progress

Summary of results : 4d CS from partial twist of D4-NS5 system

	$\tilde{V}$									
	Y		$\mathbb{R}$	Σ		$N\tilde{V}\subset T^*\tilde{V}$				
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
D4	×	×	×	×	×					
NS5	×	×		×	×	×	×			

- We begin with this brane configuration in type IIA string theory, where we have a stack of N D4-branes.
- Here, the D4-brane worldvolume is $Y \times \mathbb{R}_+ \times \Sigma$, with boundary conditions determined by an NS5-brane.

- Moreover, the worldvolume theory is partially twisted along $Y \times \mathbb{R}_+$.
- This twisting gives us 4 supercharges that are scalar along V. We take a linear combination of 2 of them, denoted $\mathcal{Q} = \kappa Q + \lambda Q'$ (for $\kappa, \lambda \in \mathbb{C}$), to define our theory.

- These 2 supercharges are distinguished since they lead to desirable $\mathcal{Q}$-invariant localization equations.
- In particular, for $\lambda = \overline{\kappa}$, they can be written as a **gradient flow equation**, i.e.,

$$\frac{dx^i}{dt} = -g^{i\bar{j}} \frac{\partial \overline{W}}{\partial x^{\bar{j}}} \tag{2.1}$$

for

$$W = \frac{ie^{i2\rho}}{g_5^2} \int_{Y \times \Sigma} dz \wedge \mathrm{Tr}\left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right). \tag{2.2}$$

- Such a gradient flow equation defines an integration cycle for the path integral over W that ensures its **convergence**.

- We can define a $\mathcal{Q}$-invariant action

$$\begin{aligned} S =& \{\mathcal{Q}, \widetilde{V}\} \\ &+ \frac{w-\bar{w}}{4}\frac{i\Psi}{2\pi}\int_{\partial M} dz_w \wedge \mathrm{Tr}\left(\mathcal{A}_w \wedge d\mathcal{A}_w + \frac{2}{3}\mathcal{A}_w \wedge \mathcal{A}_w \wedge \mathcal{A}_w\right), \end{aligned} \tag{2.3}$$

  that is $\mathcal{Q}$-exact up to a 4d Chern-Simons action.
- This action is equivalent to a 1d gauged A-model, with target space the space of all possible $\mathcal{A}_w$ fields, and the 4d Chern-Simons action as superpotential.

- This 1d A-model was shown by Witten[¶] to reduce exactly to a path integral over the boundary superpotential, with integration cycle, Γ, determined by localization equations.
- In this way, we end up with

$$\int_\Gamma D\mathcal{A}\,\exp\left(\frac{\Psi}{4\pi}\int_{\partial M}\; dz\wedge\mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)\right), \tag{2.4}$$

which for $\Psi=\frac{2i}{\hbar}$ is the (convergent) path integral for 4d Chern-Simons theory **for all** $\hbar$.

[¶]. A New Look at the Path Integral of Quantum Mechanics, arXiv:1009.6032

Summary of results : 4d CS with boundary and a 3d WZW model

- Consider 4d CS on $D \times \Sigma$, where D is a disk, with classical action

$$S = \frac{1}{\hbar} \int_{D \times \Sigma} dz \wedge \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \Big). \qquad (2.5)$$

  Here, $\mathcal{A} = \mathcal{A}_r dr + \mathcal{A}_\varphi d\varphi + \mathcal{A}_{\bar{z}} d\bar{z}$, where (r, φ) are polar coordinates on D and $(z, \bar{z})$ are complex coordinates on Σ.

- To ensure locality of EOM, and gauge invariance, we require the boundary condition $\mathcal{A}_{\bar{z}} = 0$.

- Using this boundary condition, we can show that 4d CS reduces to a boundary theory, i.e.,

$$\int Dg \ e^{-S(g)}, \tag{2.6}$$

where g is a map $g : \partial D \times \Sigma \to G$, and where

$$\begin{aligned} S(g) =& \frac{1}{\hbar} \int_{S^1 \times \Sigma} d\varphi \wedge dz \wedge d\bar{z} \ \mathrm{Tr}(\partial_\varphi g g^{-1} \partial_{\bar{z}} g g^{-1}) \\ &+ \frac{1}{3\hbar} \int_{D \times \Sigma} dz \wedge \mathrm{Tr}\left(dg g^{-1} \wedge dg g^{-1} \wedge dg g^{-1}\right). \end{aligned} \tag{2.7}$$

- This is a 3d analogue of the 2d chiral WZW model.

- The classical action is invariant under the $G \times G$ symmetry

$$g(\varphi, z, \bar{z}) \to \tilde{\Omega}(\varphi, z) g \Omega(z, \bar{z}), \tag{2.8}$$

  where Ω and $\tilde{\Omega}$ give rise to the conserved currents $J_\varphi = -\frac{2}{\hbar}\partial_\varphi g g^{-1}$ and $J_{\bar{z}} = -\frac{2}{\hbar} g^{-1}\partial_{\bar{z}} g$ respectively.
- We find a current algebra for J_φ by computing Poisson brackets and canonically quantizing:

$$\begin{aligned}\left[\mathrm{Tr} A J_\varphi(\varphi, z), \mathrm{Tr} B J_\varphi(\varphi', z')\right] =& i\delta(\varphi - \varphi')\delta(z - z')\mathrm{Tr}[A, B] J_\varphi(\varphi, z) \\ & - i\frac{2}{\hbar}\delta'(\varphi - \varphi')\delta(z - z')\mathrm{Tr} AB.\end{aligned}$$

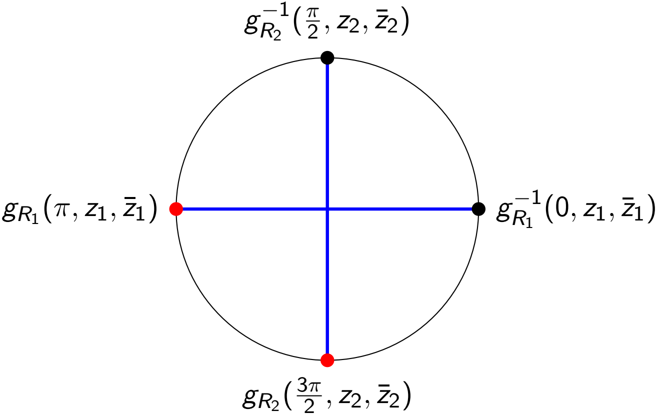
- This is an **"analytically-continued" toroidal Lie algebra**.

- A Wilson line in representation R can be described in terms of local operators of the boundary theory:

$$\mathcal{P}e^{\int_{t_i}^{t_f} \mathcal{A}} \to g_R^{-1}(t_f) g_R(t_i). \tag{2.9}$$

- Correlation functions of Wilson lines in 4d CS can therefore be **computed from the boundary theory.**
- For crossed, perpendicular Wilson lines, we have

$$\begin{aligned} &\langle \mathcal{P}e^{\int_{\pi, z_1, \bar{z}_1}^{0, z_1, \bar{z}_1} \mathcal{A}_{R_1}} \otimes \mathcal{P}e^{\int_{3\pi/2, z_2, \bar{z}_2}^{\pi/2, z_2, \bar{z}_2} \mathcal{A}_{R_2}} \rangle \\ =&\langle g_{R_1}^{-1}(0, z_1, \bar{z}_1) g_{R_1}(\pi, z_1, \bar{z}_1) \otimes g_{R_2}^{-1}(\pi/2, z_2, \bar{z}_2) g_{R_2}(3\pi/2, z_2, \bar{z}_2) \rangle. \end{aligned}$$



Perpendicular Wilson lines on D.

- We can compute the 4-pt. function via perturbation theory around $g = \mathbb{1}$:

$$g = e^{\phi_a T^a} = \mathbb{1} + \phi_a T^a + \dots$$

- Using the free-field propagator for ϕ_a, we arrive at

$$\begin{aligned}&\langle g_{R_1}^{-1}(0, z_1, \bar{z}_1) g_{R_1}(\pi, z_1, \bar{z}_1) \otimes g_{R_2}^{-1}(\pi/2, z_2, \bar{z}_2) g_{R_2}(3\pi/2, z_2, \bar{z}_2)\rangle \\ =&\mathbb{1} + \frac{\hbar}{z_1 - z_2} c_{R_1,R_2} + O(\hbar^2),\end{aligned}$$

which is **precisely** Costello, Witten and Yamazaki's result for the R-matrix to leading nontrivial order.

4d Chern-Simons theory from partial twist of D4-NS5 system

D4-brane worldvolume theory with NS5 boundary conditions

The low energy worldvolume theory of N coincident D4-branes on a flat manifold, $\mathcal{M}$, involves fields which transform as reps. of $SO_{\mathcal{M}}(5) \times SO_R(5)$:

$$\begin{aligned} A_M &: (\mathbf{5}, \mathbf{1}) \\ \phi_{\widehat{M}} &: (\mathbf{1}, \mathbf{5}) \\ \rho_{A\widehat{A}} &: (\mathbf{4}, \mathbf{4}) \end{aligned} \tag{3.1}$$

with the classical action of 5d $\mathcal{N} = 2$ SYM:

$$S = -\frac{1}{g_5^2}\int_{\mathcal{M}} d^5x \, \mathrm{Tr}\,\Big(\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}D_M\phi_{\widehat{M}}D^M\phi^{\widehat{M}} + \frac{1}{4}[\phi_{\widehat{M}},\phi_{\widehat{N}}][\phi^{\widehat{M}},\phi^{\widehat{N}}] \\ + i\rho^{A\widehat{A}}(\Gamma^M)_A{}^B D_M\rho_{B\widehat{A}} + \rho^{A\widehat{A}}(\Gamma^{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}}[\phi_{\widehat{M}},\rho_{A\widehat{B}}]\Big).$$

It is invariant under the SUSY transformations

$$\begin{aligned}
\delta A_M =& 2\zeta^{A\widehat{A}}(\Gamma_M)_A{}^B \rho_{B\widehat{A}} \\
\delta\phi^{\widehat{M}} =& -i2\zeta^{A\widehat{A}}(\Gamma^{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}} \rho_{A\widehat{B}} \\
\delta\rho_{A\widehat{A}} =& (\Gamma^M)_A{}^B D_M \phi^{\widehat{M}} (\Gamma_{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}} \zeta_{B\widehat{B}} - \frac{i}{2}(\Gamma_{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}} (\Gamma_{\widehat{N}})_{\widehat{B}\widehat{C}} [\phi^{\widehat{M}}, \phi^{\widehat{N}}] \zeta_A{}^{\widehat{C}} \\
& - \frac{i}{2} F^{MN} (\Gamma_{MN})_{AB} \zeta^B_{\widehat{A}}.
\end{aligned} \tag{3.2}$$

The stack of D4-branes shall be taken to end on an NS5-brane in the following type IIA brane configuration in flat Euclidean space

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
D4	×	×	×	×	×					
NS5	×	×		×	×	×	×			

where, e.g., an empty entry under '3' indicates that the brane is located at $x^3 = 0$. The scalar fields $\{\phi_{\widehat{1}}, \phi_{\widehat{2}}, \phi_{\widehat{3}}, \phi_{\widehat{4}}, \phi_{\widehat{5}}\}$ are understood to parametrize the $\{6, 7, 8, 9, 10\}$ directions, respectively.

The NS5-brane provides **boundary conditions** for the D4-brane worldvolume theory.

Partial twist

4d Chern-Simons theory on $Y \times \Sigma$ is topological-holomorphic:

- It has **diffeomorphism invariance** along the 2-manifold denoted Y.
- It has **holomorphic dependence** on the Riemann surface, Σ.

To obtain it from the D4-NS5 system, we ought to perform a **partial twist** that leads to the above properties.

To this end, we shall take $\mathcal{M} = Y \times \mathbb{R}_+ \times \Sigma$, and we wish to twist the D4-brane worldvolume theory along $Y \times \mathbb{R}_+$.

This amounts to redefining the $SO_V(3)$ rotation group of $V = Y \times \mathbb{R}_+$ to be the diagonal subgroup

$$SO_V(3)' \subset SO_V(3) \times SO_R(3),$$

where $SO_R(3) \subset SO_R(5)$ rotates $\{\phi_{\widehat{1}}, \phi_{\widehat{2}}, \phi_{\widehat{3}}\}$.

Specifically, we are studying the following type IIA configuration:

	$\tilde{V}$									
	Y		$\mathbb{R}$	Σ		$N\tilde{V}\subset T^*\tilde{V}$				
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
D4	×	×	×	×	×					
NS5	×	×		×	×	×	×			

The twist arises in this configuration because $V \subset \tilde{V} = Y \times \mathbb{R}$, where $\tilde{V}$ is the zero section of the cotangent bundle $T^*\tilde{V}$, and 'coordinates' normal to $\tilde{V}$ in $T^*\tilde{V}$ must be components of one-forms, as we shall obtain via twisting.[‖]

[‖] M. Bershadsky, C. Vafa, V. Sadov, *D-branes and topological field theories*, *Nuclear Physics B* **463** (2-3) (1996) 420-434

Let us now implement the partial twist. Having performed the reductions $SO_{\mathcal{M}}(5) \to SO_V(3) \times SO_\Sigma(2)$ and $SO_R(5) \to SO_R(3) \times SO_R(2)$, we denote the relevant indices as

	$SO_V(3)$	$SO_R(3)$	$SO_\Sigma(2)$	$SO_R(2)$
Vector	$\alpha, \beta, \gamma, \ldots$	$\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \ldots$	$m, n, p, \ldots$	$\widehat{m}, \widehat{n}, \widehat{p}, \ldots$
Spinor	$\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \ldots$	$\widehat{\bar{\alpha}}, \widehat{\bar{\beta}}, \widehat{\bar{\gamma}}, \ldots$	$\bar{m}, \bar{n}, \bar{p}, \ldots$	$\widehat{\bar{m}}, \widehat{\bar{n}}, \widehat{\bar{p}}, \ldots$

Partial twisting amounts to setting the hatted $SO_R(3)$ indices to unhatted indices.

As a result, the scalar fields $\{\phi_{\widehat{1}}, \phi_{\widehat{2}}, \phi_{\widehat{3}}\}$ now transform as the components $\{\phi_1, \phi_2, \phi_3\}$ of a one-form on $Y \times \mathbb{R}_+$.

In addition, the spinor fields $\rho_{A\widehat{A}} = \rho_{\bar{\alpha}\bar{m}\widehat{\bar{\alpha}}\widehat{\bar{m}}}$ can be expanded after twisting as

$$\rho_{\bar{\alpha}\bar{m}\bar{\beta}\widehat{\bar{m}}} = \epsilon_{\bar{\alpha}\bar{\beta}}\eta_{\bar{m}\widehat{\bar{m}}} + (\sigma^{\alpha})_{\bar{\alpha}\bar{\beta}}\psi_{\alpha\bar{m}\widehat{\bar{m}}}, \tag{3.3}$$

where $\eta_{\bar{m}\widehat{\bar{m}}}$ and $\psi_{\alpha\bar{m}\widehat{\bar{m}}}$ transform as **1** and **3** under $SO_V(3)'$.

Here we have used the antisymmetric matrix $\epsilon_{\bar{\alpha}\bar{\beta}}$ and the symmetric matrix $(\sigma^{\alpha})_{\bar{\alpha}\bar{\gamma}} = (\sigma^{\alpha})_{\bar{\alpha}}{}^{\bar{\beta}}\epsilon_{\bar{\beta}\bar{\gamma}}$, where ϵ is the Levi-Civita symbol and σ^{α} are the Pauli matrices.

Likewise, we can expand the SUSY transformation parameters $\zeta_{A\widehat{A}} = \zeta_{\bar{\alpha}\bar{m}\widehat{\bar{\alpha}}\widehat{\bar{m}}}$ as

$$\zeta_{\bar{\alpha}\bar{m}\bar{\beta}\widehat{\bar{m}}} = \epsilon_{\bar{\alpha}\bar{\beta}}\zeta_{\bar{m}\widehat{\bar{m}}} + (\sigma^{\alpha})_{\bar{\alpha}\bar{\beta}}\zeta_{\alpha\bar{m}\widehat{\bar{m}}}. \tag{3.4}$$

Substituting these expansions into the SUSY transformations, we can obtain the partially twisted SUSY transformations.

However, we wish to pick a supercharge, $\mathcal{Q}$, **that is scalar along** V, w.r.t. which we shall eventually localize the theory.

We shall choose only ζ_{11} and ζ_{21} to be nonzero, and take a linear combination of the corresponding supercharges to be $\mathcal{Q}$.

This choice leads to localization equations that define an integration cycle for 4d Chern-Simons theory such that its **path integral is convergent**.

To see this, let $\zeta_{11} = \kappa$ and $\zeta_{21} = \lambda$, where $\kappa, \lambda \in \mathbb{C}$. The supercharge, $\mathcal{Q}$, generates the SUSY transformations

$$\delta A_\alpha = -2i\kappa\psi_{\alpha 22} + 2i\lambda\psi_{\alpha 12} \qquad \delta\eta_{11} = i\kappa\left(F_{45} + \left[\phi_{\hat{4}}, \phi_{\hat{5}}\right] + D_\beta\phi^\beta\right)$$

$$\delta\phi_\alpha = 2\kappa\psi_{\alpha 22} + 2\lambda\psi_{\alpha 12} \qquad \delta\eta_{12} = -i\lambda\left(D_4 - iD_5\right)\left(\phi_{\hat{4}} + i\phi_{\hat{5}}\right)$$

$$\delta A_4 = 2i\kappa\eta_{12} + 2i\lambda\eta_{22} \qquad \delta\eta_{21} = -i\lambda\left(F_{45} - \left[\phi_{\hat{4}}, \phi_{\hat{5}}\right] + D_\beta\phi^\beta\right)$$

$$\delta A_5 = -2\kappa\eta_{12} + 2\lambda\eta_{22} \qquad \delta\eta_{22} = -i\kappa\left(D_4 + iD_5\right)\left(\phi_{\hat{4}} + i\phi_{\hat{5}}\right)$$

$$\delta\phi_{\hat{4}} = 2\kappa\eta_{21} + 2\lambda\eta_{11} \qquad \delta\psi_{\alpha 12} = \kappa\left(\left[\phi_\alpha, \phi_{\hat{4}} + i\phi_{\hat{5}}\right] - iD_\alpha\left(\phi_{\hat{4}} + i\phi_{\hat{5}}\right)\right)$$

$$\delta\phi_{\hat{5}} = 2i\kappa\eta_{21} + 2i\lambda\eta_{11} \qquad \delta\psi_{\alpha 22} = \kappa\left(\left[\phi_\alpha, \phi_{\hat{4}} + i\phi_{\hat{5}}\right] + iD_\alpha\left(\phi_{\hat{4}} + i\phi_{\hat{5}}\right)\right)$$

$$\delta\psi_{\alpha 11} = \kappa\varepsilon_{\alpha\beta\gamma}\left(\frac{i}{2}F^{\beta\gamma} - \frac{i}{2}\left[\phi^\beta, \phi^\gamma\right] - D^\beta\phi^\gamma\right) + \lambda\left(F_{\alpha 4} - iF_{\alpha 5} + i\left(D_4 - iD_5\right)\phi_\alpha\right)$$

$$\delta\psi_{\alpha 21} = \kappa\left(-F_{\alpha 4} - iF_{\alpha 5} + i\left(D_4 + iD_5\right)\phi_\alpha\right) + \lambda\varepsilon_{\alpha\beta\gamma}\left(\frac{i}{2}F^{\beta\gamma} - \frac{i}{2}\left[\phi^\beta, \phi^\gamma\right] + D^\beta\phi^\gamma\right)$$

Let us consider the equations $\delta\psi_{\alpha 11} = 0$ and $\delta\psi_{\alpha 21} = 0$.

For $\lambda = \bar{\kappa}$, and $\kappa = |\kappa| e^{i\rho}$, these equations are equivalent, and are given by

$$\mathcal{F}_{\alpha\bar{z}} = -\frac{i}{4} e^{-i2\rho} \varepsilon_{\alpha\beta\gamma} \overline{\mathcal{F}}^{\beta\gamma}. \tag{3.5}$$

Here, we have defined the complex coordinates $z = x^4 + ix^5$ and $\bar{z} = x^4 - ix^5$, the complex gauge fields

$$\mathcal{A}_\alpha = A_\alpha + i\phi_\alpha, \quad \mathcal{A}_{\bar{z}} = \frac{1}{2}(A_4 + iA_5), \tag{3.6}$$

whereby we have the covariant derivatives $\mathcal{D}_\alpha = \partial_\alpha + [\mathcal{A}_\alpha, \cdot\ ]$ and $\mathcal{D}_{\bar{z}} = \partial_{\bar{z}} + [\mathcal{A}_{\bar{z}}, \cdot\ ]$, and the field strengths $\mathcal{F}_{\beta\gamma} = [\mathcal{D}_\beta, \mathcal{D}_\gamma]$, $\mathcal{F}_{\alpha\bar{z}} = [\mathcal{D}_\alpha, \mathcal{D}_{\bar{z}}]$.

The equation (3.5) is equivalent to

$$\mathcal{F}_{3\widetilde{\gamma}} = -ie^{-i2\rho}2\varepsilon_{\widetilde{\gamma}}{}^{\widetilde{\alpha}}\overline{\mathcal{F}}_{\widetilde{\alpha}z}, \quad \mathcal{F}_{3\bar{z}} \quad = -\frac{i}{4}e^{-i2\rho}\varepsilon^{\widetilde{\beta}\widetilde{\gamma}}\overline{\mathcal{F}}_{\widetilde{\beta}\widetilde{\gamma}}, \tag{3.7}$$

where $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma} = 1, 2$. They can be written in the gauge $A_3 = 0$ (with $x^3 = \tau$) as

$$\frac{dx^i}{d\tau} = -g^{i\bar{j}}\frac{\partial\overline{W}}{\partial x^{\bar{j}}} \tag{3.8}$$

for

$$W = \frac{ie^{i2\rho}}{g_5^2}\int_{Y\times\Sigma} dz \wedge \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) \tag{3.9}$$

and the field-space metric

$$g = -\frac{1}{2g_5^2}\int_{Y\times\Sigma} d^2z d^2x \ \mathrm{Tr}(\delta\mathcal{A}^{\widetilde{\alpha}} \otimes \overline{\mathcal{A}}_{\widetilde{\alpha}} + \delta\overline{\mathcal{A}}^{\widetilde{\alpha}} \otimes \mathcal{A}_{\widetilde{\alpha}} + 4\delta A_{\bar{z}} \otimes \delta A_z + 4\delta A_z \otimes \delta A_{\bar{z}}), \tag{3.10}$$

i.e., **gradient flow equations!**

We perform the following convenient redefinitions:

$$\sigma = \frac{1}{\sqrt{2}}\left(\phi_{\widehat{5}} - i\phi_{\widehat{4}}\right), \quad \bar{\sigma} = \frac{1}{\sqrt{2}}\left(\phi_{\widehat{5}} + i\phi_{\widehat{4}}\right), \tag{3.11}$$

$$\begin{aligned}
\chi_\alpha &= \frac{(1-i)}{2^{5/4}}\psi_{\alpha 11} + \frac{(-1-i)}{2^{5/4}}\psi_{\alpha 21}, & \widetilde{\chi}_\alpha &= \frac{(-1-i)}{2^{5/4}}\psi_{\alpha 11} + \frac{(1-i)}{2^{5/4}}\psi_{\alpha 21} \\
\eta &= \frac{(1+i)}{2^{1/4}}\eta_{11} + \frac{(1-i)}{2^{1/4}}\eta_{21}, & \widetilde{\eta} &= \frac{(-1+i)}{2^{1/4}}\eta_{11} + \frac{(-1-i)}{2^{1/4}}\eta_{21} \\
\psi_\alpha &= \frac{(1+i)}{2^{3/4}}\psi_{\alpha 12} + \frac{(-1+i)}{2^{3/4}}\psi_{\alpha 22}, & \widetilde{\psi}_\alpha &= \frac{(-1+i)}{2^{3/4}}\psi_{\alpha 12} + \frac{(1+i)}{2^{3/4}}\psi_{\alpha 22} \\
\Upsilon &= \frac{(1-i)}{2^{3/4}}\eta_{12} + \frac{(1+i)}{2^{3/4}}\eta_{22}, & \widetilde{\Upsilon} &= \frac{(-1-i)}{2^{3/4}}\eta_{12} + \frac{(-1+i)}{2^{3/4}}\eta_{22},
\end{aligned} \tag{3.12}$$

$$u = \frac{1}{2^{1/4}}\left[(1+i)\kappa + (1-i)\lambda\right], \quad v = \frac{1}{2^{1/4}}\left[(-1+i)\kappa + (-1-i)\lambda\right]. \tag{3.13}$$

The supersymmetry transformations are then (upon rescaling δ)

$$\begin{aligned}
\delta_t A_\alpha &= i\psi_\alpha + it\widetilde{\psi}_\alpha & \delta_t \eta &= t\left(F_{45} + D_\alpha \phi^\alpha\right) + [\bar{\sigma}, \sigma] \\
\delta_t \phi_\alpha &= it\psi_\alpha - i\widetilde{\psi}_\alpha & \delta_t \widetilde{\eta} &= -\left(F_{45} + D_\alpha \phi^\alpha\right) + t\,[\bar{\sigma}, \sigma] \\
\delta_t A_4 &= i\Upsilon + it\widetilde{\Upsilon} & \delta_t \psi_\alpha &= D_\alpha \sigma + t\,[\phi_\alpha, \sigma] \\
\delta_t A_5 &= it\Upsilon - i\widetilde{\Upsilon} & \delta_t \widetilde{\psi}_\alpha &= tD_\alpha \sigma - [\phi_\alpha, \sigma] \\
\delta_t \sigma &= 0 & \delta_t \Upsilon &= D_4 \sigma + tD_5 \sigma \\
\delta_t \bar{\sigma} &= i\eta + it\widetilde{\eta} & \delta_t \widetilde{\Upsilon} &= tD_4 \sigma - D_5 \sigma
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\delta_t \chi_\alpha &= \frac{1}{2}\left[F_{\alpha 4} + D_5 \phi_\alpha + \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\left(F^{\beta\gamma} - \left[\phi^\beta, \phi^\gamma\right]\right)\right] + \frac{1}{2}t\left[F_{\alpha 5} - D_4 \phi_\alpha + \varepsilon_{\alpha\beta\gamma} D^\beta \phi^\gamma\right] \\
\delta_t \widetilde{\chi}_\alpha &= \frac{1}{2}t\left[F_{\alpha 4} + D_5 \phi_\alpha - \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\left(F^{\beta\gamma} - \left[\phi^\beta, \phi^\gamma\right]\right)\right] - \frac{1}{2}\left[F_{\alpha 5} - D_4 \phi_\alpha - \varepsilon_{\alpha\beta\gamma} D^\beta \phi^\gamma\right]
\end{aligned}$$

so we now have $\mathcal{Q} = \mathcal{Q}_L + t\mathcal{Q}_R$, $t = v/u$. Henceforth, we write $\delta\chi_\alpha = \mathcal{V}_\alpha(t)$ and $\delta\widetilde{\chi}_\alpha = t\widetilde{\mathcal{V}}_\alpha(t)$.

The transformations now take a form very similar to those of GL-twisted $\mathcal{N} = 4$ SYM, as considered by Kapustin and Witten.

In fact, taking $\Sigma = \mathbb{C}^{\times}$, whereby the x^5 direction is S^1, we can dimensionally reduce along the latter to obtain precisely the transformations of Kapustin and Witten via $A_5 \to \phi_4$, $\chi_\alpha \to \chi^{+}_{\alpha 4}$, $\widetilde{\chi}_\alpha \to \chi^{-}_{\alpha 4}$, $\psi_4 \to \Upsilon$, $\widetilde{\psi}_4 \to \widetilde{\Upsilon}$.

To construct an action suitable for localization, we require that it is $\mathcal{Q}$-exact up to some metric-independent term.

To this end we require that the rescaled supersymmetry variation

$$\delta_t = \delta_L + t\delta_R \tag{3.15}$$

is nilpotent up to gauge transformations. This is achieved by introducing auxiliary fields $(H_\alpha, \widetilde{H}_\alpha, P)$ that modify the SUSY variations to

$$\begin{aligned}
\delta_t \chi_\alpha &= H_\alpha & \delta\bar{\sigma} &= i\eta + it\widetilde{\eta} \\
\delta_t \widetilde{\chi}_\alpha &= \widetilde{H}_\alpha & \delta\eta &= tP + [\bar{\sigma}, \sigma] \\
\delta_t H_\alpha &= -i\left(1+t^2\right)[\sigma, \chi_\alpha] & \delta\widetilde{\eta} &= -P + t\,[\bar{\sigma}, \sigma] \\
\delta_t \widetilde{H}_\alpha &= -i\left(1+t^2\right)[\sigma, \widetilde{\chi}_\alpha] & \delta P &= -it[\sigma, \eta] + i\left[\sigma, \widetilde{\eta}\right]
\end{aligned} \tag{3.16}$$

We shall require that our action gives the original transformations on-shell.

As a result, for any field Φ, we have the SUSY algebra

$$\delta_t^2 \Phi = -i(1+t^2)\mathcal{L}_\sigma(\Phi), \tag{3.17}$$

where $\mathcal{L}_\sigma(\Phi)$ is the change in Φ due to a gauge transformation generated by σ, to first order.

We shall define the $\mathcal{Q}$-exact part of our action to be $\delta_t \widetilde{V}$, where $\widetilde{V} = \widetilde{V}_1 + \widetilde{V}_2$.

Here,

$$\widetilde{V}_1 = \frac{2}{{g_5}^2}\int_{\mathcal{M}} d^5x \left(\frac{4}{1+t^2}\right)\mathrm{Tr}\left(\chi_\alpha\left(\frac{1}{2}H^\alpha - \mathcal{V}^\alpha\right) + \widetilde{\chi}\left(\frac{1}{2}\widetilde{H}^\alpha - t\widetilde{\mathcal{V}}^\alpha\right)\right),$$

while

$$\widetilde{V}_2 = -\frac{1}{2t}(\delta_L - t\delta_R)\widetilde{V}_2'$$

with

$$\widetilde{V}_2' = \frac{2}{{g_5}^2}\int_{\mathcal{M}} d^5x\ \mathrm{Tr}\left(-\frac{1}{2}\eta\widetilde{\eta} - i\bar{\sigma}\left(F_{45} + D_\alpha\phi^\alpha\right)\right).$$

The $\mathcal{Q}$-exact action, upon integrating out auxiliary fields, takes the form (suppressing fermions)

$$\begin{aligned} S_1 =& \frac{1}{{g_5}^2}\int_{\mathcal{M}} d^5x \ \mathrm{Tr}\bigg(\frac{-4}{1+t^2}\left(\mathcal{V}^\alpha\mathcal{V}_\alpha + t^2\widetilde{\mathcal{V}}^\alpha\widetilde{\mathcal{V}}_\alpha\right) - (F_{45}+D_\alpha\phi^\alpha)^2 \\ & - 2D_m\bar{\sigma}D^m\sigma + [\bar{\sigma},\sigma]^2 - 2[\phi_\alpha,\sigma][\phi^\alpha,\bar{\sigma}] + 2\partial_\alpha(\bar{\sigma}D^\alpha\sigma) + \dots\bigg). \end{aligned}$$

The first line is just

$$\begin{aligned} -\frac{1}{{g_5}^2}\int_{\mathcal{M}} d^5x \ \mathrm{Tr}\bigg(& F_{\alpha m}F^{\alpha m} + F_{45}F^{45} + \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} + D_m\phi_\alpha D^m\phi^\alpha + D_\alpha\phi_\beta D^\alpha\phi^\beta \\ & + \frac{1}{2}[\phi_\alpha,\phi_\beta][\phi^\alpha,\phi^\beta] + \partial_\alpha\left(\phi^\alpha D_\beta\phi^\beta\right) - \partial_\gamma\left(\phi_\delta D^\delta\phi^\gamma\right) + 2\partial_\alpha(F_{45}\phi^\alpha)\bigg) + S_t \end{aligned}$$

Apart from the t-dependent term S_t and total derivative terms, we have the standard terms of 5d $\mathcal{N} = 2$ SYM (partially twisted).

S_t takes the form

$$S_t = \frac{1}{{g_5}^2} \int_{\mathcal{M}} d^5x \ \varepsilon^{\alpha\beta\gamma} \mathrm{Tr} \left(2 \left(\frac{t - t^{-1}}{t + t^{-1}} \right) \left(\frac{1}{2} F_{\alpha 4} F_{\beta\gamma} + \frac{1}{2} \partial_\alpha \left(\phi_\beta D_4 \phi_\gamma \right) + \partial_\alpha \left(F_{\beta 5} \phi_\gamma \right) \right) \right.$$
$$\left. - \left(\frac{4}{t + t^{-1}} \right) \left(\frac{1}{2} F_{\alpha 5} F_{\beta\gamma} + \frac{1}{2} \partial_\alpha \left(\phi_\beta D_5 \phi_\gamma \right) + \partial_\alpha \left(F_{\beta 4} \phi_\gamma \right) \right) \right) . \tag{3.18}$$

We choose to cancel this term by adding $-S_t$ to the action.

Boundary conditions/action

We may obtain the explicit NS5 boundary data at the origin of $\mathbb{R}_+$ ($x^3 = 0$) by lifting them from GL-twisted 4d $\mathcal{N} = 4$ SYM. Firstly, we obtain the Dirichlet boundary conditions

$$\phi_3 = 0|_{\partial\mathcal{M}}, \quad \sigma = 0|_{\partial\mathcal{M}}, \quad \overline{\sigma} = 0|_{\partial\mathcal{M}}, \tag{3.19}$$

whereby the total derivative terms in the $\mathcal{Q}$-exact action are just zero.

The fields $\{\phi_1, \phi_2\}$ and $\{A_1, A_2, A_4\}$ obey generalized Neumann boundary conditions, which imply a Dirichlet boundary condition on A_3.

These conditions are implied by including the boundary action

$$S_{\partial\mathcal{M}} = \frac{1}{{g_5}^2}\int_{\partial M} d^4x \,\mathrm{Tr}\left(\left(t+t^{-1}\right)\left(\frac{1}{2}\varepsilon^{\tilde{\alpha}\tilde{\beta}} D_5\phi_{\tilde{\alpha}}\phi_{\tilde{\beta}}\right) \quad + \left(\frac{t+t^{-1}}{t-t^{-1}}\right)\varepsilon^{ijk}\left(A_i\partial_j A_k + \frac{2}{3}A_iA_jA_k\right)\right),$$

where $\tilde{\alpha}, \tilde{\beta} = 1, 2$ and $i, j, k = 1, 2, 4$.

In addition, the boundary conditions on the fermionic fields are projection conditions.

Finally, the 4d boundary conditions were shown to imply that $\delta(A_i + w\phi_i) = 0$ for $w = \frac{t-t^{-1}}{2}$. The lift of this to 5d gives

$$\delta(A_{\tilde{\alpha}} + w\phi_{\tilde{\alpha}}) = 0$$

and

$$\delta(A_4 + wA_5) = 0.$$

Localization to 4d Chern-Simons theory

Our total action now takes the form

$$S = \delta_t \widetilde{V} - S_t + S_{\partial \mathcal{M}}. \tag{3.20}$$

In fact,

$$-S_t + S_{\partial\mathcal{M}} = \frac{w-\bar{w}}{4}\frac{i\Psi}{2\pi}\int_{\partial M} dz_w \wedge \mathrm{Tr}\left(\mathcal{A}_w \wedge d\mathcal{A}_w + \frac{2}{3}\mathcal{A}_w \wedge \mathcal{A}_w \wedge \mathcal{A}_w\right)$$

where

$$\Psi = \frac{4\pi i}{{g_5}^2}\left(\frac{t-t^{-1}}{t+t^{-1}} - \frac{t+t^{-1}}{t-t^{-1}}\right).$$

Here, we have defined the complex coordinates z_w, $\overline{z}_w$ with corresponding derivatives

$$\begin{aligned} \partial_{z_w} &= \frac{1}{2}(\partial_4 + \overline{w}\partial_5) \\ \partial_{\overline{z}_w} &= \frac{1}{2}(\partial_4 + w\partial_5), \end{aligned} \tag{3.21}$$

and the complexified gauge fields

$$\mathcal{A}_{w\tilde{\alpha}} = A_{\tilde{\alpha}} + w\phi_{\tilde{\alpha}} \tag{3.22}$$

(for $\tilde{\alpha} = 1, 2$) and

$$\mathcal{A}_{w\bar{z}_w} = \frac{1}{2}\left(A_4 + wA_5\right) \tag{3.23}$$

that are $\mathcal{Q}$-invariant along the boundary. Hence, the non-$\mathcal{Q}$-exact 4d CS term **is $\mathcal{Q}$-invariant**, and we have a $\mathcal{Q}$-invariant **5d topological-holomorphic theory.**

In what follows we shall consider $t \neq \pm i$, as this implies that the theory is **completely independent of** t.

Now, we localize by adding the $\mathcal{Q}$-exact term

$$\begin{aligned} &-\frac{1}{\epsilon}\{\mathcal{Q}, \int_{\mathcal{M}} \mathrm{Tr}\,(\chi_\alpha \mathcal{V}^\alpha + \widetilde{\chi}'_\alpha \widetilde{\mathcal{V}}^\alpha + \eta' \mathcal{V}_0)\} \\ =&-\frac{1}{\epsilon}\int_{\mathcal{M}} \mathrm{Tr}\,(\mathcal{V}_\alpha \mathcal{V}^\alpha + \widetilde{\mathcal{V}}_\alpha \widetilde{\mathcal{V}}^\alpha + \mathcal{V}_0 \mathcal{V}_0 + \ldots), \end{aligned} \tag{3.24}$$

where $\mathcal{V}_0 = F_{45} + D_\alpha \phi^\alpha$, and $\{\mathcal{Q}, \chi_\alpha\} = \mathcal{V}_\alpha(t)$, $\{\mathcal{Q}, \widetilde{\chi}'_\alpha\} = \widetilde{\mathcal{V}}_\alpha(t)$, and $\{\mathcal{Q}, \widetilde{\eta}'_\alpha\} = \widetilde{\mathcal{V}}_0$.

Then, for $t \in \mathbb{R}$, we have the localization configurations

$$\begin{aligned} \mathcal{V}_\alpha(t) &= 0 \\ \widetilde{\mathcal{V}}_\alpha(t) &= 0 \\ \mathcal{V}_0 &= 0. \end{aligned} \tag{3.25}$$

In fact for $t \in \mathbb{R}$, we retrieve the gradient flow equations from $\mathcal{V}_\alpha(t) = 0$ and $\widetilde{\mathcal{V}}_\alpha(t) = 0$.

This choice of t allowed - for any finite, fixed Ψ there is always a convenient choice of $t \in \mathbb{R}$, and we have freedom to choose t.

The remaining localization equations (for σ) are trivial.

The 5d partially twisted theory can be interpreted as a **1d gauged A-model**, with target space $\mathfrak{A}$, the space of all $\mathcal{A}_w$ fields, and gauge group H, the space of maps from $Y \times \Sigma$ to $U(N)$.

For example, with the metric

$$g=-\frac{1}{2g_5^2}\int_{Y\times\Sigma}d^2zd^2x\ \mathrm{Tr}(\delta\mathcal{A}^{\widetilde{\alpha}}\otimes\overline{\mathcal{A}}_{\widetilde{\alpha}}+\delta\overline{\mathcal{A}}^{\widetilde{\alpha}}\otimes\mathcal{A}_{\widetilde{\alpha}}+4\delta A_{\bar{z}}\otimes\delta A_z+4\delta A_z\otimes\delta A_{\bar{z}}),$$

moment map

$$\mu=-\frac{1}{g_5^2}(D_{\widetilde{\alpha}}\phi^{\widetilde{\alpha}}+F_{45}),$$

and superpotential

$$W=-\frac{e^{i\alpha}}{g_5^2}\int_{Y\times\Sigma}dz\wedge\mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right),$$

the standard terms $\frac{1}{4}|dW|^2+|\mu|^2$ are equal to

$$-\frac{1}{g_5^2}\mathrm{Tr}\left(\frac{1}{2}F^{\tilde{\alpha}\tilde{\beta}}F_{\tilde{\alpha}\tilde{\beta}}+D^{\tilde{\alpha}}\phi^{\tilde{\beta}}D_{\tilde{\alpha}}\phi_{\tilde{\beta}}+\frac{1}{2}[\phi^{\tilde{\alpha}},\phi^{\tilde{\beta}}][\phi_{\tilde{\alpha}},\phi_{\tilde{\beta}}]+4F^{\tilde{\alpha}}{}_{\bar{z}}F_{\tilde{\alpha}z}+4D_z\phi_{\tilde{\alpha}}D_{\bar{z}}\phi^{\tilde{\alpha}}-4F_{z\bar{z}}F_{z\bar{z}}\right).$$

Such a 1d model localizes to its boundary superpotential.**

Hence, our 5d theory is equivalent to

$$\int_\Gamma D\mathcal{A}\, \exp\left(\frac{\Psi}{4\pi}\int_{\partial M} dz \wedge \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)\right).$$

Here, we have assumed that there is no fermion number anomaly, and used the path integral's independence of w to set $w = i$.

We also require boundary conditions $\mathcal{A} \in \mathrm{Crit}\ W$ and $\mu = 0$ at infinity on $\mathbb{R}_+$.

**. E. Witten, A New Look at the Path Integral of Quantum Mechanics, arXiv:1009.6032

For $\frac{1}{\hbar} = \frac{-i\Psi}{2}$, this is the path integral for 4d Chern-Simons theory, **defined beyond perturbation theory** with integration cycle Γ.

To obtain lattice, we use **F-strings** ending on D4-brane boundary to realize Wilson lines.

4d Chern-Simons theory with boundary and a 3d WZW model

The 3d Chiral WZW Model

4d Chern-Simons theory defined on $D \times \Sigma$, where D is a disk, is

$$S = \frac{1}{\hbar} \int_{D \times \Sigma} dz \wedge \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \Big), \tag{4.1}$$

where $\mathcal{A}$ is the partial connection $\mathcal{A} = \mathcal{A}_r dr + \mathcal{A}_\varphi d\varphi + \mathcal{A}_{\bar{z}} d\bar{z}$. Varying S gives

$$\delta S = \frac{1}{\hbar} \int_{D \times \Sigma} dz \wedge \mathrm{Tr}\Big(\delta\mathcal{A} \wedge \mathcal{F} + d(\delta\mathcal{A} \wedge \mathcal{A}) \Big). \tag{4.2}$$

To have EOM free from boundary corrections, we impose $\mathcal{A}_{\bar{z}} = 0|_{\partial D}$.

Observe that

$$S = -\frac{1}{\hbar}\int_{D\times\Sigma} z\mathrm{Tr}\Big(F\wedge F\Big) + \frac{1}{\hbar}\int_{\partial D\times\Sigma} z\mathrm{Tr}\Big(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big), \tag{4.3}$$

where $\mathcal{A}$ has been extended to a *full* connection over $D \times \Sigma$, i.e., $\mathcal{A} = \mathcal{A}_r dr + \mathcal{A}_\varphi d\varphi + \mathcal{A}_z dz + \mathcal{A}_{\bar{z}} d\bar{z}$.

The boundary term on the RHS of (4.3) vanishes using $\mathcal{A}_{\bar{z}} = 0|_{\partial D}$ as well as $\mathcal{A}_z = 0|_{\partial D}$.

The remaining term is gauge invariant under large gauge transformations

$$\mathcal{A} \to U\mathcal{A}U^{-1} - dUU^{-1}. \tag{4.4}$$

However, we ought to restrict U such that the boundary conditions $\mathcal{A}_{\bar{z}} = \mathcal{A}_z = 0|_{\partial D}$ are preserved. We shall achieve this by insisting that U tends to the identity element of G at the boundary.

Using $\mathcal{A}_{\bar{z}} = 0|_{\partial D}$, we find

$$S = \frac{1}{\hbar}\int dz \wedge dr \wedge d\varphi \wedge d\bar{z} \; \mathrm{Tr}\Big(2\mathcal{A}_{\bar{z}}\mathcal{F}_{r\varphi} - \mathcal{A}_r\partial_{\bar{z}}\mathcal{A}_\varphi + \mathcal{A}_\varphi\partial_{\bar{z}}\mathcal{A}_r\Big). \tag{4.5}$$

Varying $A_{\bar{z}}$ gives $\mathcal{F}_{r\varphi} = 0$. Solved by

$$\mathcal{A}_r = -\partial_r g g^{-1}, \;\; \mathcal{A}_\varphi = -\partial_\varphi g g^{-1}, \tag{4.6}$$

where $g : D \times \Sigma \to G$.

Then, substituting into S, we find

$$\begin{aligned} S(g) =& \frac{1}{\hbar} \int_{S^1 \times \Sigma} d\varphi \wedge dz \wedge d\bar{z} \ \mathrm{Tr}(\partial_\varphi g g^{-1} \partial_{\bar{z}} g g^{-1}) \\ &+ \frac{1}{3\hbar} \int_{D \times \Sigma} dz \wedge \mathrm{Tr} \left(dgg^{-1} \wedge dgg^{-1} \wedge dgg^{-1} \right). \end{aligned} \tag{4.7}$$

Also, no Jacobian appears when transforming the measure, i.e.,

$$\frac{1}{\mathrm{vol}\ G} \int D\mathcal{A}_r D\mathcal{A}_\varphi \ \delta(\mathcal{F}_{r\varphi}) = \frac{1}{\mathrm{vol}\ G} \int Dg. \tag{4.8}$$

Now, $\mathcal{A} \to U\mathcal{A}U^{-1} - dUU^{-1}$ amounts to $g \to Ug$, so we may change the value of g in the interior without changing its boundary value.

Hence, the action only depends on the value of g on the boundary, so we can divide out $\mathrm{vol}\ G$ to obtain

$$\int Dg\ e^{-S(g)}, \tag{4.9}$$

where g is now a map $g : \partial D \times \Sigma \to G$. This is a **3d "chiral" WZW model.**

This model has a local $G \times G$ symmetry under

$$g(\varphi, z, \bar{z}) \to \tilde{\Omega}(\varphi, z) g \Omega(z, \bar{z}). \tag{4.10}$$

$\tilde{\Omega}$ and Ω correspond, respectively, to the conserved currents $J_\varphi = -\frac{2}{\hbar}\partial_\varphi g g^{-1}$ and $J_{\bar{z}} = -\frac{2}{\hbar} g^{-1} \partial_{\bar{z}} g$, that obey $\partial_\varphi J_{\bar{z}} = 0$ and $\partial_{\bar{z}} J_\varphi = 0$.

We can use J_φ to derive a current algebra.

Current Algebra via Canonical Quantization

To compute Poisson brackets of J_φ, we shall first take $\bar{z}$ to be the time direction.

In general, for an action first order in time with variables ϕ^i,

$$I = \int dt \mathscr{A}(\phi) \frac{d\phi^i}{dt}, \tag{4.11}$$

we have

$$\delta I = \int dt \ \omega_{ij} \delta\phi^i \frac{d\phi^j}{dt}, \tag{4.12}$$

where $\omega_{ij} = \frac{\partial}{\partial\phi^i}\mathscr{A}_j - \frac{\partial}{\partial\phi^j}\mathscr{A}_i$ is the symplectic structure on the classical phase space.

The Poisson bracket of any two functions X and Y on the phase space is then defined by

$$[X, Y]_{PB} = \omega^{ij} \frac{\partial X}{\partial \phi^i} \frac{\partial Y}{\partial \phi^j}, \tag{4.13}$$

where $\omega^{jk}\omega_{kl} = \delta^j_l$.

Since

$$\delta S = -\frac{2}{\hbar}\int d\varphi \wedge dz \wedge d\bar{z}\ \mathrm{Tr}\left(g^{-1}\delta g \partial_{\varphi}(g^{-1}\partial_{\bar{z}} g)\right), \tag{4.14}$$

we have

$$\omega = 1_{\mathfrak{g}} \otimes \frac{(-2)}{\hbar}\frac{\partial}{\partial\varphi} \otimes 1_{z},$$

where $1_{\mathfrak{g}}$ acts on the Lie algebra index, $\frac{(-2)}{\hbar}\frac{\partial}{\partial\varphi}$ acts on the φ coordinate, and 1_z acts on the z coordinate.

Its inverse is

$$\delta^{ab}\frac{(-\hbar)}{2}\theta(\varphi-\varphi')\delta(z-z'). \tag{4.15}$$

Let $X = \mathrm{Tr}\ A\frac{\partial g}{\partial \varphi}g^{-1}(\varphi, z)$ and $Y = \mathrm{Tr}\ B\frac{\partial g}{\partial \varphi'}g^{-1}(\varphi', z')$, where $A, B \in \mathfrak{g}$.

We compute the Poisson brackets $[X, Y]_{PB}$, and canonically quantize such that $[X, Y]_{PB} \to -i[X, Y]$. In this manner, we arrive at the **current algebra**

$$\begin{aligned}\left[\mathrm{Tr} A J_\varphi(\varphi, z), \mathrm{Tr} B J_\varphi(\varphi', z')\right] =& i\delta(\varphi - \varphi')\delta(z - z')\mathrm{Tr}[A, B]J_\varphi(\varphi, z)\\ &- i\frac{2}{\hbar}\delta'(\varphi - \varphi')\delta(z - z')\mathrm{Tr} AB.\end{aligned}$$

Expanding currents in Fourier modes along $S^1 = \partial D$,

$$J_\varphi(\varphi, z) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} J_\varphi^n(z) e^{in\varphi}, \tag{4.16}$$

gives

$$\begin{aligned}\left[\mathrm{Tr} A J_\varphi^n(z), \mathrm{Tr} B J_\varphi^m(z')\right] =& i\mathrm{Tr}[A, B] J_\varphi^{n+m}(z)\delta(z - z') \\ & - (2\pi i)\frac{2}{\hbar}(in\delta_{m+n,0})\delta(z - z')\mathrm{Tr} AB,\end{aligned}$$

a $\mathfrak{g}$ Kac-Moody algebra with **holomorphic** generators. But note that there is no quantization condition on $\hbar$.

Now let $z = \epsilon t + i\theta$, and compactify the θ direction to be valued in $[0, 2\pi]$, and take $\epsilon \to 0$. Expanding as

$$J^n_\varphi(\theta) = \frac{1}{2\pi} \sum_{\tilde{n}=-\infty}^{\infty} J^{n,\tilde{n}}_\varphi e^{i\tilde{n}\theta}, \tag{4.17}$$

we find

$$\begin{aligned}\left[\mathrm{Tr} A J^{n,\tilde{n}}_\varphi, \mathrm{Tr} B J^{m,\tilde{m}}_\varphi\right] =& i\mathrm{Tr}[A, B] J^{n+m,\tilde{n}+\tilde{m}}_\varphi \\ & - (2\pi i)^2 \frac{2}{\hbar} n \delta_{m+n,0} \delta_{\tilde{m}+\tilde{n},0} \mathrm{Tr} AB.\end{aligned} \tag{4.18}$$

This is a two-toroidal Lie algebra. So our original algebra is an is an **"analytically-continued" toroidal Lie algebra**.

R-matrix from Local Boundary Operators

Consider Wilson lines along D ending on ∂D. These can be expressed in terms of **local boundary operators** since $\mathcal{A}|_D$ is pure gauge.

E.g., for a Wilson line in representation R,

$$\mathcal{P}e^{\int_{t_i}^{t_f}\mathcal{A}} = g_R^{-1}(t_f)\mathcal{P}e^{\int_{t_i}^{t_f}\mathcal{A}'}g_R(t_i) \qquad (4.19)$$

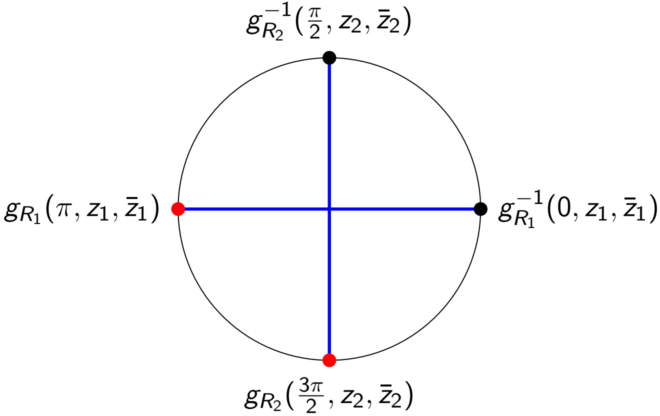
where $\mathcal{A} = g\mathcal{A}'g^{-1} - dgg^{-1}$. Setting $\mathcal{A}' = 0$, we find that

$$\mathcal{P}e^{\int_{t_i}^{t_f}(-dgg^{-1})} = g_R^{-1}(t_f)g_R(t_i). \qquad (4.20)$$

We can thus compute correlation functions of Wilson lines via correlators of such boundary operators.

Let us try to retrieve the R-matrix, using

$$\begin{aligned}&\langle \mathcal{P}e^{\int_{\pi,z_1,\bar{z}_1}^{0,z_1,\bar{z}_1}\mathcal{A}_{R_1}}\otimes\mathcal{P}e^{\int_{3\pi/2,z_2,\bar{z}_2}^{\pi/2,z_2,\bar{z}_2}\mathcal{A}_{R_2}}\rangle\\ =&\langle g_{R_1}^{-1}(0,z_1,\bar{z}_1)g_{R_1}(\pi,z_1,\bar{z}_1)\otimes g_{R_2}^{-1}(\pi/2,z_2,\bar{z}_2)g_{R_2}(3\pi/2,z_2,\bar{z}_2)\rangle.\end{aligned}$$



Perpendicular Wilson lines on D.

Bulk R-matrix computation (to order $\hbar$) used perturbation theory around $\mathcal{A} = 0$ and free field propagators.

So we consider perturbation theory around $g = \mathbb{1}$:

$$g = e^{\phi_a T^a} = \mathbb{1} + \phi_a T^a + \dots$$

whereby the 3d WZW kinetic term is

$$\begin{aligned} &\frac{1}{\hbar}\int_{S^1\times\Sigma} d\varphi\wedge dz\wedge d\bar{z}\,\mathrm{Tr}\big(\partial_\varphi g g^{-1}\partial_{\bar{z}} g g^{-1}\big) \\ =&-\frac{1}{\hbar}\int_{S^1\times\Sigma} d\varphi\wedge dz\wedge d\bar{z}\quad \phi^a\partial_\varphi\partial_{\bar{z}}\phi_a + \dots . \end{aligned} \tag{4.21}$$

We construct the generating functional

$$\begin{aligned} Z_0[J] &= \frac{\int D\phi e^{-\frac{1}{\hbar}\int_{S^1\times\Sigma} d\varphi\wedge dz\wedge d\bar{z}(-\phi^a\partial_\varphi\partial_{\bar{z}}\phi_a+\hbar J_a\phi^a)}}{\int D\phi e^{-\frac{1}{\hbar}\int_{S^1\times\Sigma} d\varphi\wedge dz\wedge d\bar{z}(-\phi^a\partial_\varphi\partial_{\bar{z}}\phi_a)}} \\ &= \exp\left(-\frac{\hbar}{4}\int d^3x\int d^3y J_a(x)\Delta^{ab}(x-y)J_b(y)\right), \end{aligned} \tag{4.22}$$

where $x=(\varphi,z,\bar{z})$, $y=(\varphi',z',\bar{z}')$, and Δ^{ab} is the propagator which obeys

$$\partial_\varphi\partial_{\bar{z}}\Delta^{ab}(x)=\delta^{ab}\delta(x). \tag{4.23}$$

It is given explicitly by

$$\Delta^{ab}(x) = \delta^{ab}\frac{1}{2\pi i}\frac{1}{z}\widetilde{\Delta}_{\varphi}. \tag{4.24}$$

where,

$$\widetilde{\Delta}_{\varphi} = \frac{1}{2\pi}\left(\sum_{k=1}^{\infty}\frac{e^{ik\varphi}}{ik} + \varphi + \sum_{k=-\infty}^{-1}\frac{e^{ik\varphi}}{ik}\right), \tag{4.25}$$

defined with a branch cut. The two point function for ϕ is

$$\langle \phi^a(x)\phi^b(y)\rangle = -\frac{\hbar}{2}\Delta^{ab}(x-y). \tag{4.26}$$

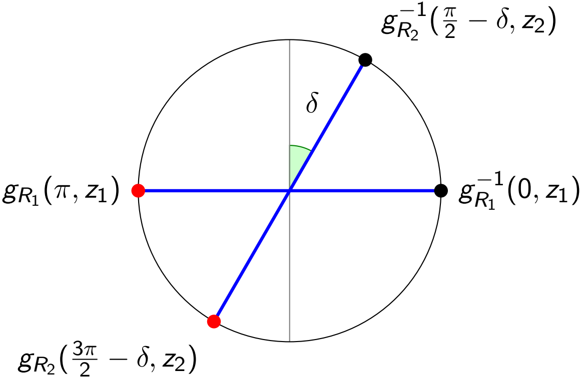
Now

$$\begin{aligned}&\langle g_{R_1}^{-1}(0,z_1,\bar{z}_1)g_{R_1}(\pi,z_1,\bar{z}_1)\otimes g_{R_2}^{-1}(\pi/2,z_2,\bar{z}_2)g_{R_2}(3\pi/2,z_2,\bar{z}_2)\rangle\\ =&\mathbb{1}+\langle\phi_a(0,z_1)\phi_c(\pi/2,z_2)\rangle T^a_{R_1}\otimes T^c_{R_2}-\langle\phi_a(\pi,z_1)\phi_c(\pi/2,z_2)\rangle T^a_{R_1}\otimes T^c_{R_2}\\ -&\langle\phi_a(2\pi,z_1)\phi_c(3\pi/2,z_2)\rangle T^a_{R_1}\otimes T^c_{R_2}+\langle\phi_a(\pi,z_1)\phi_c(3\pi/2,z_2)\rangle T^a_{R_1}\otimes T^c_{R_2}+\dots\end{aligned}$$

Finally, using the 2 pt. function for ϕ we have

$$\begin{aligned}&\langle g_{R_1}^{-1}(0,z_1,\bar{z}_1)g_{R_1}(\pi,z_1,\bar{z}_1)\otimes g_{R_2}^{-1}(\pi/2,z_2,\bar{z}_2)g_{R_2}(3\pi/2,z_2,\bar{z}_2)\rangle\\ =&\mathbb{1}+\frac{1}{2\pi i}\frac{\hbar}{z_1-z_2}T^a_{R_1}\otimes T_{aR_2}+\dots\end{aligned}$$

If we use the conventions of CWY, we find precise agreement with their computation.



Non-perpendicular Wilson lines on D.

Here, the four-point function is

$$\langle g_{R_1}^{-1}(0,z_1)g_{R_1}(\pi,z_1)\otimes g_{R_2}^{-1}(\pi/2-\delta,z_2)g_{R_2}(3\pi/2-\delta,z_2)\rangle$$
$$=\mathbb{1}+\frac{1}{2\pi i}\frac{\hbar}{z_1-z_2}\frac{1}{2}\left(1+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(\frac{k\pi}{2})\cos(k\delta)}{k}\right)T^a_{R_1}\otimes T_{aR_2}.$$

The sum is δ-independent and equal to $\pi/4$, so once again we have agreement with CWY.

Conclusion and Future Directions

- We have made use of string theory to derive an integration cycle that allows us to define 4d CS theory nonperturbatively.
- We have also found a new 3d WZW model dual to 4d CS theory, governed by a novel toroidal Lie algebra. This WZW model could be used to learn more about 4d CS.

- Future work involves including D2-branes in the D4-NS5 system to realize surface defects in the 4d CS theory, which then allows us to study integrable field theories.
- D5-NS5, D6-NS5 systems can be studied to realize higher dim. Chern-Simons theories, e.g., 5d CS and affine Yangian, etc.
- For the 3d WZW model, future work involves computing R-matrix to higher order in $\hbar$, framing anomaly, OPEs, etc.

Thank you for your attention!