Positive geometry in the diagonal limit of the conformal bootstrap

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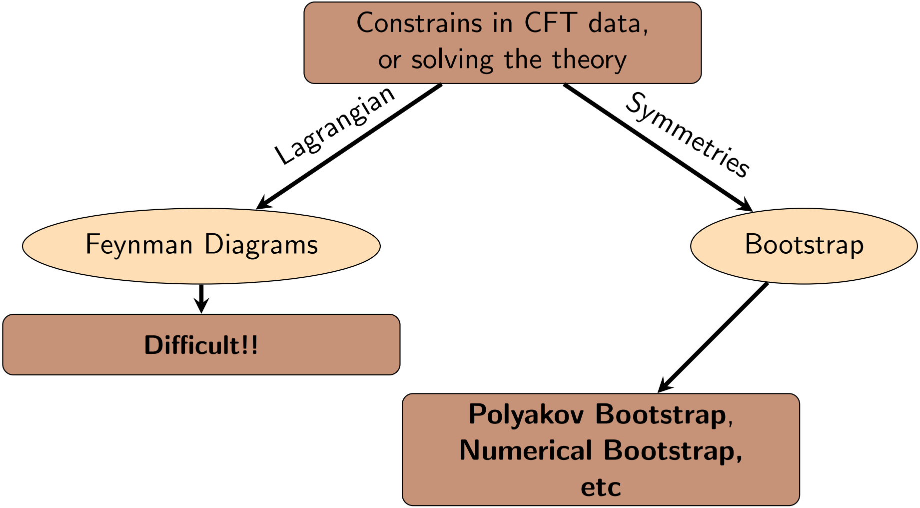
   - Kink from Positive Geometry


Constrains in CFT data,
or solving the theory
Lagrangian
Symmetries
Feynman Diagrams
Bootstrap
Difficult!!
Polyakov Bootstrap,
Numerical Bootstrap,
etc

Big Picture

arXiv:1812.07739v2
N.Arkani-Hamed, Y.T.Huang, S.H.Shao

Conformal bootstrap $\rightarrow$ geometry problem.

Unitarity $\rightarrow$ Taylor coefficients of four point function lie inside a polytope **U**

Crossing $\rightarrow$ Taylor coefficients of four point function lie on a plane **X**.

$\Big\}$ **The consistent solution of the conformal bootstrap entails finding of U $\cap$ X**

The polytope $\mathbf{U}$ is a cyclic polytope $\rightarrow$ face structure known

The conditions for intersection $\mathbf{U} \cap \mathbf{X}$ $\rightarrow$ New exact results of the spectrum

example
Analytic bounds on leading operators
Analytic bounds on sub-leading operators
Kink from the positive geometry.

Conformal Bootstrap

Transformations

$$x^\mu \to a^\mu + x^\mu$$
$$x^\mu \to \lambda x^\mu$$
$$x^\mu \to x^\mu \Lambda_\mu^\nu$$
$$x^\mu \to \frac{x^\mu - (x\cdot x) b^\mu}{1-2(b\cdot x)+(b\cdot b)(x\cdot x)}$$

Conformal transformations fixes

$$\langle\phi(x)\phi(y)\rangle = \frac{c}{|x-y|^{2\Delta}}\ , \quad \text{Normalize } c=1$$

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{|x_{12}|^{2\alpha_{123}}|x_{13}|^{2\alpha_{132}}|x_{23}|^{2\alpha_{33}}}, \quad \alpha_{ijk} = \frac{\Delta_i+\Delta_j-\Delta_k}{2}$$

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{\mathcal{A}(u,v)}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}$$

$$u = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\ \ v = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2} \tag{2.1}$$

u, v are cross ratios

Conformal Bootstrap

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{\mathcal{A}(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \;\; v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \;\; ; \;\; u = z\bar{z}, \;\; v = (1-z)(1-\bar{z}) \tag{2.2}$$

$$\mathcal{A}(u,v) = \sum_{\Delta,\ell} C_{\Delta,\ell}\, \mathcal{G}_{d,\Delta,\ell}(u,v) \tag{2.3}$$

Conformal blocks $\mathcal{G}_{\Delta,\ell}(u,v)$ are

1. Conformally invariant.
2. Consistent with factorization.
3. Consistent with OPE.

Conformal blocks are not crossing symmetric!

Conformal Bootstrap

Crossing Equation

$$\langle \phi(x_1)\,\phi(x_2)\phi(x_3)\,\phi(x_4)\rangle = \langle \phi(x_1)\phi(x_4)\,\phi(x_3)\phi(x_2)\rangle$$

$$\mathcal{A}(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} \mathcal{A}(v,u)$$

$$\sum_{\Delta,\ell} C_{\Delta,\ell} \underbrace{\left(\mathcal{G}_{d,\Delta,\ell}(u,v) - \left(\frac{u}{v}\right)^{\Delta_\phi} \mathcal{G}_{d,\Delta,\ell}(v,u)\right)}_{F^{\Delta_\phi}_{d,\Delta,\ell}(u,v)} = 0 \qquad (2.4)$$

Diagonal limit of Blocks

arXiv:1305.1321
M.Hogervorst, H.Osborn, S.Rychkov

$$u = z\bar{z}, \;\; v = (1-z)(1-\bar{z})$$

Diagonal limit $z \longrightarrow \bar{z}$

$$\mathcal{A}(z) = \sum_{\Delta,\ell} C_{\Delta,\ell}\mathcal{G}_{d,\Delta,\ell}(z)$$

$\mathcal{G}_{d,\Delta,\ell}(z)$ for ℓ is even

$$\mathcal{G}_{d,\Delta,\ell}(z) = \frac{\left(\frac{z^2}{1-z}\right)^{\Delta/2}(d-2)_\ell\left(\frac{\Delta+1}{2}\right)_{\frac{\ell}{2}}}{\left(\frac{d-2}{2}\right)_\ell\left(\frac{\Delta}{2}\right)_{\frac{\ell}{2}}\left(\frac{1}{2}(-d-\ell+\Delta+3)\right)_{\frac{\ell}{2}}}\sum_{r=0}^{\frac{\ell}{2}}\frac{\left(\frac{1}{2}\right)_r\binom{\frac{\ell}{2}}{r}\left(\frac{d-2+\ell}{2}\right)_r\left(\frac{2-d+\Delta-\ell}{2}\right)_{\frac{\ell}{2}-r}}{\left(\frac{1+\Delta}{2}\right)_r}$$
$$\times\;{}_3F_2\left(-\frac{d}{2}+\frac{\Delta}{2}+1, r+\frac{\Delta}{2}, \frac{\Delta}{2}; r+\frac{\Delta}{2}+\frac{1}{2}, -\frac{d}{2}+\Delta+1; \frac{z^2}{4(z-1)}\right).$$

Bootstrap in diagonal limit

Crossing Equation

$$\mathcal{A}(z) = \left(\frac{z}{1-z}\right)^{2\Delta_\phi} \mathcal{A}(1-z)$$

$$1 + \sum_{\Delta,\ell} C_{\Delta,\ell}\mathcal{G}_{d,\Delta,\ell}(z) = \left(\frac{z}{1-z}\right)^{2\Delta_\phi} + \left(\frac{z}{1-z}\right)^{2\Delta_\phi} \sum_{\Delta,\ell} C_{\Delta,\ell}\mathcal{G}_{d,\Delta,\ell}(1-z)$$

$$\sum_{\Delta,\ell} C_{\Delta,\ell}\mathcal{F}_{d,\Delta,\ell}(z) = 1\,,$$

where

$$\mathcal{F}_{d,\Delta,\ell}(z) = \frac{(1-z)^{2\Delta_\phi}\,\mathcal{G}_{d,\Delta,\ell}(z) - z^{2\Delta_\phi}\,\mathcal{G}_{d,\Delta,\ell}(1-z)}{z^{2\Delta_\phi} - (1-z)^{2\Delta_\phi}}\,,$$

$$\sum_{\Delta,\ell} C_{\Delta,\ell}\mathcal{F}_{d,\Delta,\ell}(z) = 1\,,$$

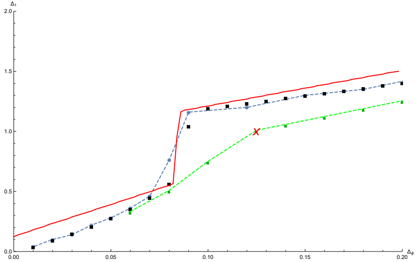
Taking derivatives of the equation around $z = 1/2$

$$\sum_{\Delta,\ell} C_{\Delta,\ell}\partial_z^{2m}\mathcal{F}_{d,\Delta,\ell}(z)|_{z=1/2} = 0\,,\ \ m > 0$$

Crossing condition can be satisfied?.

Obtain bounds on leading operator dimension Δ_1 .

The conditions says that Δ_1 should be below the curve.



$d = 2$

Positive Geometry in diagonal limit

Conformal bootstrap equations $\longrightarrow$ in the language of polytopes.

Taylor expansion around $z = \frac{1}{2}$ truncated upto $2N + 2$ terms.

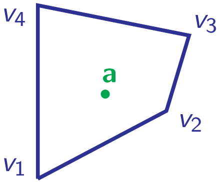
$y = z - \frac{1}{2}$

$$\mathcal{A}(z) = \mathcal{A}^0 + \mathcal{A}^1 y + \mathcal{A}^2 y^2 + \cdots + \mathcal{A}^{2N+1} y^{2N+1}$$
$$G_{d,\Delta,\ell}(z) = G^0_{d,\Delta,\ell} + G^1_{d,\Delta,\ell} y + G^2_{d,\Delta,\ell} y^2 + \cdots + G^{2N+1}_{d,\Delta,\ell} y^{2N+1}$$

Conformal bootstrap $\rightarrow$ $2N + 1$-dimensional geometry problem.

$$\mathcal{A}(z) \rightarrow \mathbf{A} = \begin{pmatrix} A^0 \\ A^1 \\ \vdots \\ A^{2N+1} \end{pmatrix} \; ; \; \mathcal{G}_{d,\Delta,\ell}(z) \rightarrow \mathbf{G}_{d,\Delta,\ell} = \begin{pmatrix} G^0_{d,\Delta,\ell} \\ G^1_{d,\Delta,\ell} \\ \vdots \\ G^{2N+1}_{d,\Delta,\ell} \end{pmatrix} \; : F^I \equiv \frac{1}{I!} \partial_z^I F\,(z)|_{z=1/2}\,, F = \mathcal{A} \text{ or } \mathcal{G}$$

$$\mathbf{A} = \sum_{\Delta,\ell} C_{\Delta,\ell} \mathbf{G}_{d,\Delta,\ell} \quad ; \quad C_{\Delta,\ell} > 0$$



Now lets take an example of vectors in 2d.

Lets say you know $\mathbf{a} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 \quad c_i > 0$

and $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4$ vectors form a convex tetragon $\rightarrow$ given to you

$\downarrow$

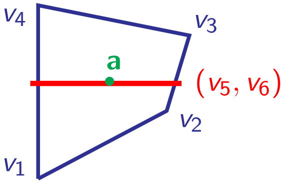
What can you comment about vector "**a**"?

$\downarrow$

Is "**a**" inside the tetragon or not?

$\downarrow$

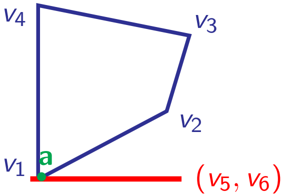
In this case you can say "**a**" is inside that tetragon if you know $\sum_i c_i = 1$



Now someone gives you further information
that "**a**" lies on line connecting two points, say the line is (v_5, v_6)

$\Downarrow$

Determining the intersection of the line with the tetragon you will be more sure about the region where **a** lies.



Given the line (v_5, v_6), you can get some idea where the tetragon will be.

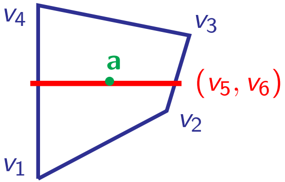
$\downarrow$

For example

You will be able to say that the smallest of $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4$ should be bounded,

$\downarrow$

otherwise it will not sometime intersect the tetragon.



For future reference,
$\sum_i c_i = 1$ defines the convex hull of the vectors v_i.
This four points actually form a polytope in 2d.
Convex polygon are cycilc polytope in 2d.

We will play same game with

$$\mathbf{A} = \sum_{\Delta,\ell} C_{\Delta,\ell} \mathbf{G}_{d,\Delta,\ell} \quad ; \quad C_{\Delta,\ell} > 0$$

We can consider the expansion of $\mathbf{A} \to t\,\mathbf{A}$
The cone spanned by $\mathbf{G}_{d,\Delta,\ell} \to \alpha_{\Delta,\ell}\mathbf{G}_{d,\Delta,\ell}$, $\alpha_{\Delta,\ell} > 0$
i.e

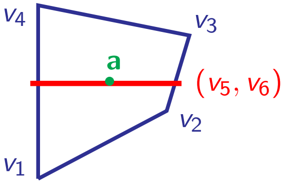
$$\mathbf{A} = t\begin{pmatrix}1\\ \vec{\mathcal{A}}\end{pmatrix} \text{ in terms of } \mathbf{G}_{d,\Delta,\ell} = \alpha_{\Delta,\ell}\begin{pmatrix}1\\ \vec{G}_{d,\Delta,\ell}\end{pmatrix}\,, \quad .$$

Gives $\sum \alpha_{\Delta,\ell} C_{\Delta,\ell} \equiv \sum C'_{\Delta,\ell} = t.$
So we have

$$\vec{\mathcal{A}} = \sum_{\Delta,\ell} \lambda_{\Delta,\ell}\vec{G}_{d,\Delta,\ell}\,, \quad \lambda_{\Delta,\ell} = \frac{C'_{\Delta,\ell}}{\sum C'_{\Delta,\ell}}$$

$\sum_{\Delta,\ell}\lambda_{\Delta,\ell} = 1 \to$ convex hull of $\vec{G}_{d,\Delta,\ell} \to$ a polytope in $\mathbb{R}^{2N+1}$.

$\mathbf{A} = \sum_{\Delta,\ell} C_{\Delta,\ell}\mathbf{G}_{d,\Delta,\ell}$; $C_{\Delta,\ell} > 0 \to$ projective polytope in $\mathbb{P}^{2N+1}$.



We will play same game with

$$\mathbf{A} = \sum_{\Delta,\ell} C_{\Delta,\ell} \mathbf{G}_{d,\Delta,\ell} \quad ; \quad C_{\Delta,\ell} > 0$$

So our next task will be to show that
We get a cyclic polytopes from the vectors $\mathbf{G}_{d,\Delta,\ell}$ (from Unitarity)

Also to show that $\mathbf{A}$ lies on a plane (from Crossing) that intersects the polytope.

Cyclic Polytopes

arXiv:1812.07739v2
N.Arkani-Hamed, Y.T.Huang, S.H.Shao

Cyclic polytope which vertices have an ordering $v_1, \dots v_n$ such that

$$\langle \mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \dots, \mathbf{v}_{i_D} \rangle\,,\ \text{have same sign}\ \forall\ i_1 < i_2 < \cdots < i_D\,.$$

Faces of cyclic polytope are known

Positivity Criteria

arXiv:1812.07739v2
N.Arkani-Hamed, Y.T.Huang, S.H.Shao

From CFT spectrum, the block vectors can be ordered simply in terms of increasing Δ.

Ordered set of vectors $(i_i, i_2, \cdots i_{D+1}) \quad \Delta_{i_1} < \Delta_{i_2} < \cdots < \Delta_{i_{D+1}}$.

Conditions for a cyclic polytope translates into

$$\langle i_1, i_2, \cdots, i_{D+1} \rangle \equiv \epsilon_{I_1 I_2 \cdots I_{D+1}} G^{I_1}_{d,\Delta_{i_1},\ell} \cdots G^{I_{D+1}}_{d,\Delta_{i_{D+1}},\ell}, \quad \text{same sign},$$

"i" is short hand for $\rightarrow \mathbf{G}_{d,\Delta_i,\ell} = \begin{pmatrix} G^0_{d,\Delta_i,\ell} \\ G^1_{d,\Delta_i,\ell} \\ \vdots \\ G^{2N+1}_{d,\Delta_i,\ell} \end{pmatrix}$

the positivity of a $D-$dimensional unitary polytope.

We give a single shot verification for the positivity.

Define,

$$F_{m,n} = \frac{1}{m!}\ \partial_{\Delta}{}^{n}\ \partial_{z}{}^{m} G_{d,\Delta,\ell}(z)|_{z=1/2}$$

Then construct $\mathbf{K}_{2N+1}$, $(2N+1)\times(2N+1)$ matrix,

$$\mathbf{K}_{2N+1}(d,\Delta,\ell) = \begin{pmatrix} F_{0,0} & F_{1,0} & .. & .. & F_{2N+1,0} \\ F_{0,1} & F_{1,1} & .. & .. & .. \\ .. & .. & F_{i,j} & .. & .. \\ .. & .. & .. & .. & .. \\ F_{0,2N+1} & .. & .. & .. & F_{2N+1,2N+1} \end{pmatrix}$$

Condition for positivity

$$g_i = \frac{|\mathbf{K}_i(d,\Delta,\ell)|\,|\mathbf{K}_{i-2}(d,\Delta,\ell)|}{|\mathbf{K}_{i-1}(d,\Delta,\ell)|^2} > 0\,,$$

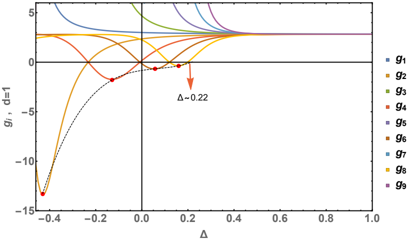
$\Delta \gg d, \ell$ limit of diagonal block

Leading order Block,

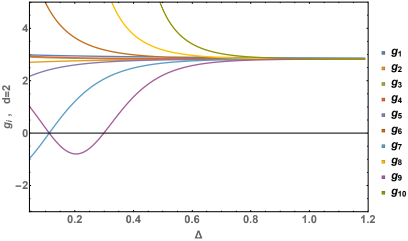
$$\mathcal{G}^{approx}_{d,\Delta,\ell}(z) = \frac{\sqrt{\pi}2^{-\frac{3d}{2}+2\Delta+3}\left(\sqrt{1-Z}+1\right)^{d/2}\left(\frac{Z}{Z+2\sqrt{1-Z}-2}\right)^{-\frac{\Delta}{2}}\Gamma(d+\ell-2)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{d}{2}+\ell-1\right)}\left(1+O\left(\frac{1}{\Delta}\right)\right).$$

Computing g_i analytically

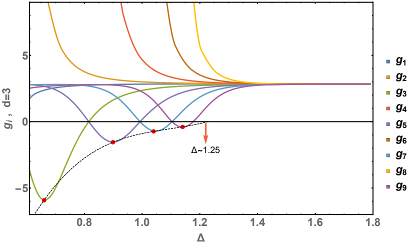
$$g_i \approx 2\sqrt{2}: \quad \forall\ i, \quad \Delta \gg d, \ell\,.$$



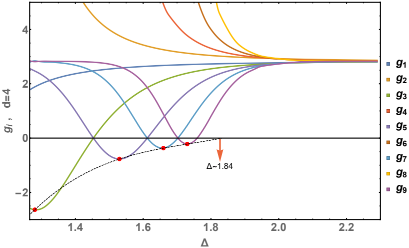
(a) g_i *vs* Δ for $d = 1$



(b) g_i *vs* Δ for $d = 2$



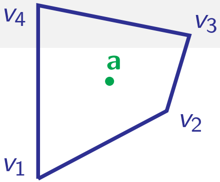
(c) g_i *vs* Δ for $d = 3$



(d) g_i *vs* Δ for $d = 4$

Figure: g_i *vs* Δ for scalar blocks for various d, plot range is $\Delta > \frac{d-2}{2}$

Unitary



So far we have learnt that
Unitarity demands

$$\mathbf{A} = \sum_{\Delta,\ell} C_{\Delta,\ell} \mathbf{G}_{d,\Delta,\ell} \quad ; \quad C_{\Delta,\ell} > 0$$

$\mathbf{A}$ lies inside the polytope spanned by block vectors $\mathbf{G}_{d,\Delta,\ell}$

Now we turn to Crossing Symmetry

$$\mathcal{A}(z) = \left(\frac{z}{1-z}\right)^{2\Delta_\phi} \mathcal{A}(1-z)\,.$$

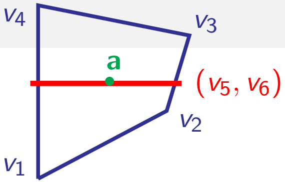
Taylor Expand around $z = 1/2$,
This equation relates odd $\mathcal{A}^{2n+1}$ in terms of the even $\mathcal{A}^{2n}$

$$\mathcal{A}(z) \to \boldsymbol{\mathcal{A}} = \begin{pmatrix} \mathcal{A}^0 \\ \mathcal{A}^1 \\ \vdots \\ \mathcal{A}^{2N+1} \end{pmatrix}$$

This in turn defines a hyperplane $\mathbf{X}[\Delta_\phi]$ which is a $(2N+2) \times (N+1)$ matrix in $\mathbb{P}^{2N+1}$.

Crossing Symmetry demands
$\mathbf{A}$ lies on the hyperplane $\mathbf{X}[\Delta_\phi]$

Crossing



$$\mathcal{A}(z) \to \boldsymbol{\mathcal{A}} = \begin{pmatrix} \mathcal{A}^0 \\ \mathcal{A}^1 \\ \vdots \\ \mathcal{A}^{2N+1} \end{pmatrix}$$

Crossing Symmetry demands

A lies on the hyperplane $\mathbf{X}[\Delta_\phi]$

For example $N = 2$

$$\mathbf{A} = \begin{pmatrix} \mathcal{A}^0 \\ 4\Delta_\phi \mathcal{A}^0 \\ \mathcal{A}^2 \\ \frac{16}{3}\left(\Delta_\phi - 4\Delta_\phi^3\right)\mathcal{A}^0 + 4\Delta_\phi \mathcal{A}^2 \\ \frac{64}{15}\Delta_\phi\left(32\Delta_\phi^4 - 20\Delta_\phi^2 + 3\right)\mathcal{A}^0 - \frac{16}{3}\Delta_\phi\left(4\Delta_\phi^2 - 1\right)\mathcal{A}^2 + 4\Delta_\phi \mathcal{A}^4 \\ \vdots \end{pmatrix} \in \mathbb{P}^{2N+1}$$

and

$$\mathbf{X}[\Delta_\phi] = \begin{array}{ccc} \mathcal{A}^0 & \mathcal{A}^2 & \mathcal{A}^4 \end{array}$$

$$\mathbf{X}[\Delta_\phi] = \begin{pmatrix} 1 & 0 & 0 \\ 4\Delta_\phi & 0 & 0 \\ 0 & 1 & 0 \\ \frac{16}{3}\left(\Delta_\phi - 4\Delta_\phi^3\right) & 0 & 0 \\ 0 & 0 & 1 \\ \frac{64}{15}\Delta_\phi\left(32\Delta_\phi^4 - 20\Delta_\phi^2 + 3\right) & \frac{16}{3}\left(\Delta_\phi - 4\Delta_\phi^3\right) & 4\Delta_\phi \end{pmatrix}$$

Implementing Bootstrap

arXiv:1812.07739v2
N.Arkani-Hamed, Y.T.Huang, S.H.Shao

Now we have both ingredients for bootstrap in the projective picture.

Unitarity demands

$\mathbf{A}$ lies inside the polytope spanned by block vectors $\mathbf{G}_{d,\Delta,\ell}$

Crossing Symmetry demands

$\mathbf{A}$ lies on the hyperplane $\mathbf{X}[\Delta_\phi]$

i.e. the consistent solution of bootstrap entails the region

$\mathbf{U}[\Delta] \cap \mathbf{X}[\Delta_\phi]$.

arXiv:1812.07739v2
N.Arkani-Hamed, Y.T.Huang, S.H.Shao

The question now is given $\mathbf{U}[\Delta]$,
what are the conditions determining the intersection with $\mathbf{X}[\Delta_\phi]$.

The Story is
$k-$plane intersects with a $D-$dimensional polytope with a $D-k$ face at a point given by,

$$\mathbf{v}_1\langle\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4,\ldots,\mathbf{v}_{D-k},\mathbf{X}\rangle-\mathbf{v}_2\langle\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_4,\ldots,\mathbf{v}_{D-k},\mathbf{X}\rangle+\ldots.$$

For a point inside the polytope and satisfying the intersection property above,

$$\langle\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4,\ldots,\mathbf{v}_{D-k},\mathbf{X}\rangle\,,-\langle\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_4,\ldots,\mathbf{v}_{d-k},\mathbf{X}\rangle\,,\langle\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_4,\ldots,\mathbf{v}_{D-k},\mathbf{X}\rangle$$

must have the same sign.

A further simplification occurs
when one of the vertex vectors is
the identity operator $\mathbf{v}_0 = (1, 0, 0 \ldots, 0)$
or the infinity operator $\mathbf{v}_\infty = (0, 0 \ldots, 0, 1)$
since this reduces the dimensionality of the problem.

$N = 1$: Bounds on scalar operator

The two-dimensional facets consists of the following two sets

$$(0, i, i+1), \quad (i, i+1, \infty)$$

where "i" is $\mathbf{G}_{d,\Delta_i,0}$, 0 is the identity operator $\mathbf{G}_{d,\Delta_0,0} = (1, 0, \cdots, 0)$ and ∞ is $\mathbf{G}_{d,\Delta_\infty,0} = (0, 0, \cdots, 1)$.

The subscripts i and i+1 label two operators $\Delta_i < \Delta_{i+1}$ with nothing in between

The crossing plane $\mathbf{X}$ is one-dimensional, which is a 4×2 matrix

$$\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 4\Delta_\phi & 0 \\ 0 & 1 \\ \frac{16}{3}\left(\Delta_\phi - 4\Delta_\phi^3\right) & 4\Delta_\phi \end{pmatrix}$$

arXiv:1812.07739v2
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Using the sign rule of determinant
The crossing plane $\mathbf{X}$ intersects with the face $(0, i, i+1)$ if and only if

$$\langle \mathbf{X}, i, i+1 \rangle, \quad -\langle \mathbf{X}, 0, i+1 \rangle, \quad \langle \mathbf{X}, 0, i \rangle, \qquad \text{have same sign}$$

Similarly the crossing plane $\mathbf{X}$ intersects with the face $(i, i+1, \infty)$ if and only if

$$\langle \mathbf{X}, i, i+1 \rangle, \quad -\langle \mathbf{X}, \infty, i+1 \rangle, \quad \langle \mathbf{X}, \infty, i \rangle \qquad \text{have same sign}$$

The crossing plane intersects with the polytope iff either one of the two conditions is satisfied.

Of course generically if one condition is satisfied the other will not be;

To extract useful constraints from these conditions,
it is often useful to derive
necessary
(but not necessarily sufficient)
conditions by projecting the geometry to lower dimensions.

For example

We take $\mathbf{X}$ to be intersect $\mathbf{U}[\{\Delta_i\}]$ on both kinds of faces by forcing

$$\langle \mathbf{X}, i, i+1 \rangle = 0$$

Also we take projection through identity $\langle 0, \mathbf{X}, \Delta \rangle = 0$.

The crossing plane intersects with the block curve at two points. These two points are the solution to the equation $\langle 0, \mathbf{X}, \Delta \rangle = 0, \quad \Delta_+$ and Δ_-

There must exist at least an operator with dimension Δ satisfying

$$\Delta_- < \Delta < \Delta_+$$

.

The solutions Δ_+, Δ_- for large Δ_ϕ

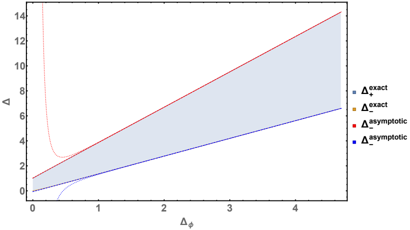
$$\Delta_+ = 2\sqrt{2}\Delta_\phi + \frac{\left(2\sqrt{2}-3\right)d+6}{4\sqrt{2}} + \frac{12-d(d+6)}{128\sqrt{2}\Delta_\phi} - \frac{3(d(d(d+2)-44)+88)}{2048\sqrt{2}\Delta_\phi^2}$$
$$+ \frac{d(-d(d+6)(37d-282)-7392)+15216}{131072\sqrt{2}\Delta_\phi^3} + O\left(\frac{1}{\Delta_\phi^4}\right)$$

$$\Delta_- = \sqrt{2}\Delta_\phi + \frac{1}{8}\left(4-3\sqrt{2}\right)d - \frac{(d-6)d+12}{64\sqrt{2}\Delta_\phi} - \frac{3\left((d-4)^2d-32\right)}{512\sqrt{2}\Delta_\phi^2}$$
$$+ \frac{d(d((372-37d)d-1188)+480)-1680}{16384\sqrt{2}\Delta_\phi^3} + O\left(\frac{1}{\Delta_\phi^4}\right)$$

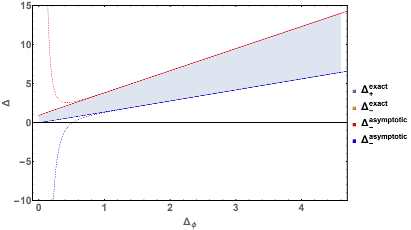
We can also expand around $\Delta_\Phi = \Delta_\phi + a$
We can choose a whatever we want.

$$\Delta_+ = 2\sqrt{2}\Delta_\Phi + \frac{1}{8}\left(-16\sqrt{2}a + \left(4 - 3\sqrt{2}\right)d + 6\sqrt{2}\right) + \frac{12 - d(d+6)}{128\sqrt{2}\Delta_\Phi}$$
$$+ \frac{-16a(d(d+6) - 12) - 3(d(d(d+2) - 44) + 88)}{2048\sqrt{2}\Delta_\Phi^2}$$
$$+ \frac{-1024a^2(d(d+6) - 12) - 384a(d(d(d+2) - 44) + 88) + d(-d(d+6)(37d - 282) - 7392) + 15216}{131072\sqrt{2}\Delta_\Phi^3}$$
$$+ O\left(\frac{1}{\Delta_\Phi^4}\right)$$

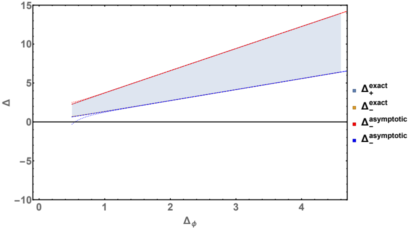
$$\Delta_- = \sqrt{2}\Delta_\Phi + \left(\frac{1}{8}\left(4 - 3\sqrt{2}\right)d - \sqrt{2}a\right) - \frac{(d-6)d + 12}{64\sqrt{2}\Delta_\Phi} + \frac{-8a((d-6)d + 12) - 3\left((d-4)^2 d - 32\right)}{512\sqrt{2}\Delta_\Phi^2}$$
$$+ \frac{-256a^2((d-6)d + 12) - 192a\left((d-4)^2 d - 32\right) + d(d((372 - 37d)d - 1188) + 480) - 1680}{16384\sqrt{2}\Delta_\Phi^3}$$
$$+ O\left(\frac{1}{\Delta_\Phi^4}\right)$$



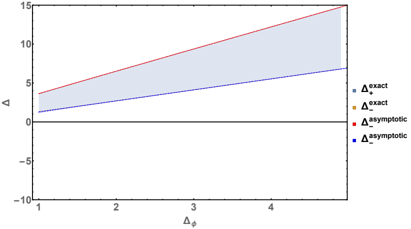
(a) Δ_+, Δ_- vs Δ_ϕ for $d = 1$



(b) Δ_+, Δ_- vs Δ_ϕ for $d = 2$



(c) Δ_+, Δ_- vs Δ_ϕ for $d = 3$



(d) Δ_+, Δ_- vs Δ_ϕ for $d = 4$

Figure: Solid lines represent Δ_+, Δ_- using exact block, dashed lines represent $a = 1$ and dotted are $a = 0$.

Interpretation of Δ_+ and Δ_- from numerical bootstrap

Crossing Symmetry

$$1+\sum_{\Delta,\ell} C_{\Delta,\ell}\mathcal{G}_{d,\Delta,\ell}(z)=\left(\frac{z}{1-z}\right)^{2\Delta_\phi}+\left(\frac{z}{1-z}\right)^{2\Delta_\phi}\sum_{\Delta,\ell} C_{\Delta,\ell}\mathcal{G}_{d,\Delta,\ell}(1-z)$$

Rearrange the equation a bit

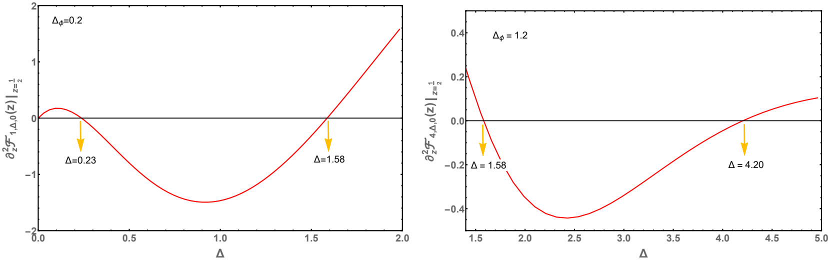
$$\sum_{\Delta,\ell} C_{\Delta,\ell}\mathcal{F}_{d,\Delta,\ell}(z)=1\,,$$

where

$$\mathcal{F}_{d,\Delta,\ell}(z)=\frac{(1-z)^{2\Delta_\phi}\mathcal{G}_{d,\Delta,\ell}(z)-z^{2\Delta_\phi}\mathcal{G}_{d,\Delta,\ell}(1-z)}{z^{2\Delta_\phi}-(1-z)^{2\Delta_\phi}}\,,$$

We can write

$$\sum_{\Delta,\ell} C_{\Delta,\ell}\partial_z^2\mathcal{F}_{d,\Delta,\ell}(z)|_{z=1/2}=0\,,$$



(a) $\partial_z^2 \mathcal{F}_{d,\Delta,\ell}(z)|_{z=1/2}$ vs Δ for $d=1$ (b) $\partial_z^2 \mathcal{F}_{d,\Delta,\ell}(z)|_{z=1/2}$ vs Δ for $d=4$

Figure

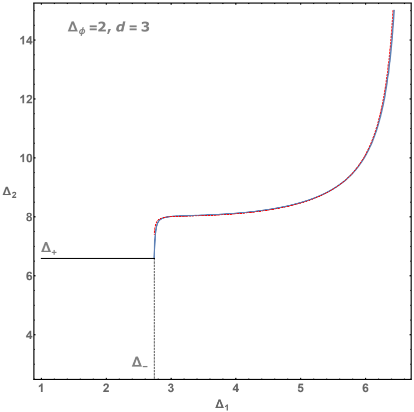
It is clear that $\partial_z^2 \mathcal{F}_{d,\Delta,0}(z)|_{z=1/2}$ changes its sign at indicated values. At least there should be one operator below Δ_+ (the larger value) in order to satisfy equation $\sum_{\Delta,\ell} C_{\Delta,\ell}\partial_z^2 \mathcal{F}_{d,\Delta,\ell}(z)|_{z=1/2} = 0\,,$

Constraints on the first two operator $\Delta_1,\ \Delta_2$.

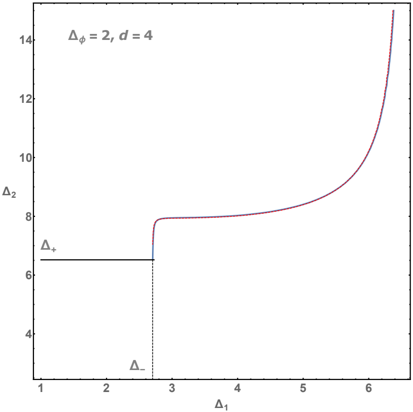
For $\Delta_1 < \Delta_-$ we should have $\Delta_2 < \Delta_+$,
since there should atleast one operator between (Δ_-, Δ_+)
as we observed in $N = 1$ case.

And also $\Delta_1 > \Delta_+$ not allowed if Δ_1 is the leading operator.

Necessary conditions from $N = 2$ is
For $\Delta_- < \Delta_1 < \Delta_+$, Δ_2 must be below the curve $\langle \mathbf{X}, 0, \Delta_1, \Delta_2 \rangle = 0$
otherwise some of the sign rule of determinant will not satisfied.



(a)



(b)

Figure: Black solid line represents Δ_+ and black dashed Δ_-. The region below the curve is allowed. Before the black dashed line, $\Delta_1 < \Delta_-$ and hence Δ_2 must be smaller than Δ_+. After the dashed line $\Delta_- < \Delta_1$, Δ_2 must be below the curve $\langle \mathbf{X}, 0, \Delta_1, \Delta_2 \rangle = 0$. Finally $\Delta_1 > \Delta_+$ is ruled out.

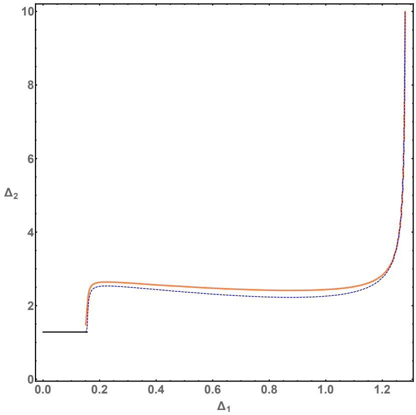


Figure: How the curve $\langle \mathbf{X}, 0, \Delta_1, \Delta_2 \rangle = 0$ changes if we put the second operator with spin? In figure we have taken 2D ising model $\Delta_\phi = \frac{1}{8}$ and used the spin $\ell = 2$ block for the operator Δ_2 *i.e* we used $\mathbf{G}_{2,\Delta_2,2}$ instead of $\mathbf{G}_{2,\Delta_2,0}$. One can see the feature $\Delta = 2$ is allowed for $\ell = 2$. Orange line is using spin-2 block for Δ_2 and blue dashed line is using scalar block for Δ_2.

Kink from Positive Geometry

$$d = 2$$

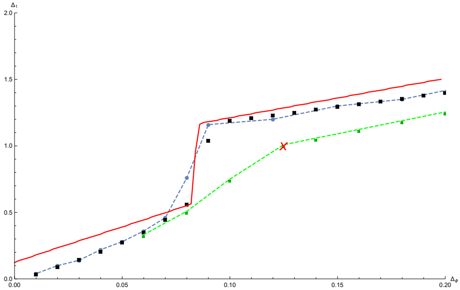


Figure: Δ_1 *vs* Δ_ϕ

Kink from Positive Geometry

We consider 10 scalar operators Δ_i
where Δ_0 is the identity operator and Δ_9 is the infinity vector.

Δ_1 is the leading operator and $\Delta_i, i \geq 2$ are chosen randomly to be above Δ_1 (but ordered).

The intersection conditions are now checked.

For $N = 1$ we find essentially the same results as from Δ_+

For $N = 2$, we find that there is a kink type feature in the plot as in the figure,

Kink from Positive Geometry

$$d = 2$$

Δ_1 can't be above the red line

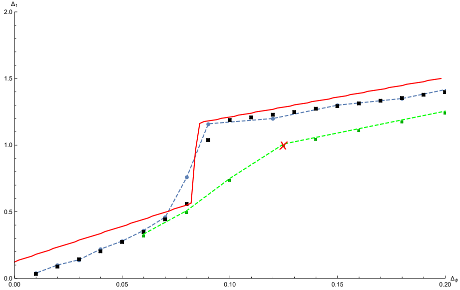


Figure: Δ_1 *vs* Δ_ϕ

Thank You

Polytopes

arXiv:1812.07739v2
N.Arkani-Hamed, Y.T.Huang, S.H.Shao

Computations

Given a polytopes in $\mathbb{P}^D$ built out of $\{\mathbf{v}_i\}$
$(\mathbf{v}_{i_1},\ \mathbf{v}_{i_2},\ \ldots\ \mathbf{v}_{i_D}) \rightarrow$ facets
we need,

$\langle \mathbf{v}_i, \mathbf{v}_{i_1}, \ldots \mathbf{v}_{i_D} \rangle$ have the same sign $\forall\ i$.

$\mathbf{v}_i$ *or* i we simply refer to $\mathbf{G}_{d,\Delta,\ell}$

An example: 2d polygons.

Three points $v_1,\ v_2,\ v_3$ in 2d plane projectively associated with three-vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$.

If $v_1,\ v_2,\ v_3$ collinear $\rightarrow\ \langle \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\rangle = 0$,
i.e.

$$det \begin{pmatrix} v_1^{(x)} & v_1^{(y)} & v_1^{(z)} \\ v_2^{(x)} & v_2^{(y)} & v_2^{(z)} \\ v_3^{(x)} & v_3^{(y)} & v_3^{(z)} \end{pmatrix} = 0\,.$$

If v_3 is not on the line $(v_1 v_2) \rightarrow\ \langle \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\rangle > 0$ *or* $\langle \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\rangle < 0$,

If $v_4,\ v_3$ is on same side of $(v_1 v_2) \rightarrow\ \langle \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\rangle\,,\ \langle \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_4\rangle$ same sign

This generalize to 2d convex n-gon
formed by the vectors $\mathbf{v}_1, \ \mathbf{v}_2, \ \ldots \mathbf{v}_n$.
$(v_{i_1} v_{i_2})$ is a edge if

$\langle \mathbf{v}_i, \ \mathbf{v}_{i_1}, \ \mathbf{v}_{i_2} \rangle$ have same sign ; $\forall \ i$.

In general D-dimension will be

$\langle \mathbf{v}_i, \ \mathbf{v}_{i_1}, \ \mathbf{v}_{i_2} \ldots \mathbf{v}_{i_d} \rangle$ have same sign ; $\forall \ i$.

Backup Intersection explain

To see this

We again go back to 2d and ask what is the intersection the two lines spanned by the point pairs (ab) and (cd).

Point of intersection is $\langle \mathbf{c}, \mathbf{d}, \mathbf{b} \rangle \, \mathbf{a} - \langle \mathbf{c}, \mathbf{d}, \mathbf{a} \rangle \, \mathbf{b}$.

To prove that this point is indeed collinear with (ab) and (cd) one have to use $\langle \mathbf{v}_1, \; \mathbf{v}_2, \; \mathbf{v}_3 \rangle = 0$.

A 2-plane $\mathbf{v}_1, \; \mathbf{v}_2, \; \mathbf{v}_3$ intersects a line $\mathbf{v}_a, \; \mathbf{v}_b$ in $\mathbb{P}^3$ at the point

$$\mathbf{v}_a \left\langle \mathbf{v}_b, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \right\rangle - \mathbf{v}_b \left\langle \mathbf{v}_a, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \right\rangle \; .$$

Generalization of it a k-plan $\mathbf{X}$ intersects with a $D-k$-face of the polytope of D-dimension at a point, which is given as

$$\mathbf{v}_1 \left\langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \cdots, \mathbf{v}_{D-k}, \mathbf{X} \right\rangle - \mathbf{v}_2 \left\langle \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \cdots, \mathbf{v}_{D-k}, \mathbf{X} \right\rangle + \mathbf{v}_3 \left\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4, \cdots, \mathbf{v}_{D-k}, \mathbf{X} \right\rangle + \cdots ,$$

This point id interior of the polytope iff

$$\left\langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \cdots, \mathbf{v}_{D-k}, \mathbf{X} \right\rangle , \;\; -\left\langle \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \cdots, \mathbf{v}_{D-k}, \mathbf{X} \right\rangle ,$$

$$\left\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4, \cdots, \mathbf{v}_{D-k}, \mathbf{X} \right\rangle , \cdots , \quad \text{have same sign}$$

Backup Positivity Criteria

N.Arkani Hamed, Yu-tin Huang, ,
K.Sen,A.Sinha,A.Zahed

$$\langle i_1, i_2, \cdots, i_{D+1} \rangle \equiv \epsilon_{I_1 I_2 \cdots I_{D+1}} G^{I_1}_{d,\Delta_{i_1},\ell} \cdots G^{I_{D+1}}_{d,\Delta_{i_{D+1}},\ell}\,, \quad \text{same sign}\,,$$

The function

$$f_{D+1} = c_1 G^0_{d,\Delta,\ell} + c_2 G^1_{d,\Delta,\ell} + c_3 G^3_{d,\Delta,\ell} + \cdots + G^D_{d,\Delta,\ell} = 0$$

can't have a solutions.
So what are the constrains that block should have ?

Normalize the block vector by

$$G^0_{d,\Delta,\ell} = 1$$

By induction.

For D=1, $f_2 = c_1 + c_2 G^1_{d,\Delta,\ell} = 0$ can not have a solutions.

$$\Rightarrow \quad g_1 = \left(G^1_{d,\Delta,\ell}\right)' > 0$$

$$\text{For } D = 2$$
$$f_3 = c_1 + c_2 G^1_{d,\Delta,\ell} + c_3 G^2_{d,\Delta,\ell} = 0 \;\; \text{can't have solutions,}$$
$$\Downarrow$$
$$c_2\left(G^1_{d,\Delta,\ell}\right)' + c_3\left(G^2_{d,\Delta,\ell}\right)' = \left(G^1_{d,\Delta,\ell}\right)'\left(c_2 + c_3\frac{\left(G^2_{d,\Delta,\ell}\right)'}{\left(G^1_{d,\Delta,\ell}\right)'}\right) = 0$$
can't have solution.

$$\Rightarrow \;\; g_2 = \left(\frac{\left(G^2_{d,\Delta,\ell}\right)'}{\left(G^1_{d,\Delta,\ell}\right)'}\right)' > 0 \;\; \textit{if} \;\; g_1 = \left(G^1_{d,\Delta,\ell}\right)' > 0.$$

Similarly for D=3, $\quad g_3 = \left(\dfrac{\left(\dfrac{\left(G^3_{d,\Delta,\ell}\right)'}{\left(G^1_{d,\Delta,\ell}\right)'}\right)'}{\left(\dfrac{\left(G^2_{d,\Delta,\ell}\right)'}{\left(G^1_{d,\Delta,\ell}\right)'}\right)'}\right)' > 0$

and so on.

We give a single shot verification for the positivity.
No need for induction method

We define,

$$F_{m,n} = \frac{1}{m!}\ \partial_\Delta{}^n\ \partial_z{}^m G_{d,\Delta,\ell}(z)|_{z=1/2} \tag{12.1}$$

Then construct $\mathbf{K}_{2N+1}$, $(2N+1)\times(2N+1)$ matrix,

$$\mathbf{K}_{2N+1}(d,\Delta,\ell) = \begin{pmatrix} F_{0,0} & F_{1,0} & .. & .. & F_{2N+1,0} \\ F_{0,1} & F_{1,1} & .. & .. & .. \\ .. & .. & F_{i,j} & .. & .. \\ .. & .. & .. & .. & .. \\ F_{0,2N+1} & .. & .. & .. & F_{2N+1,2N+1} \end{pmatrix}$$

Condition for positivity discussed above,
can be written in a more generic format ,

$$g_i = \frac{|\mathbf{K}_i(d,\Delta,\ell)|\,|\mathbf{K}_{i-2}(d,\Delta,\ell)|}{|\mathbf{K}_{i-1}(d,\Delta,\ell)|^2} > 0\,, \tag{12.2}$$

This is results is equivalent to previous induction method results.

Positivity criterion in $\Delta \gg d, \ell$ limit

Block Vectors

$$\frac{\partial_z^m \mathcal{G}_{d,\Delta,\ell}(z)}{m!}\bigg|_{z=\frac{1}{2}} = \frac{\sqrt{\pi}\left(2+\frac{3}{\sqrt{2}}\right)^{d/2}\left(12\sqrt{2}+17\right)^{-\frac{\Delta}{2}}\Delta^m 2^{-2d+2\Delta+\frac{3m}{2}+3}\Gamma(d+\ell-2)}{\Gamma\left(\frac{d-1}{2}\right)(2)_{m-1}\Gamma\left(\frac{d}{2}+\ell-1\right)}\left[1+O\left(\frac{1}{\Delta}\right)\right],$$

$F_{m,n}$ matrix

$$F_{m,n} = \frac{\sqrt{\pi}\left(2+\frac{3}{\sqrt{2}}\right)^{d/2}\left(12-8\sqrt{2}\right)^{\Delta}2^{-2d+\frac{3m}{2}+3}\Delta^m(-\Delta)^{-n}\Gamma(d+\ell-2)}{\Gamma\left(\frac{d-1}{2}\right)(2)_{m-1}\Gamma\left(\frac{d}{2}+\ell-1\right)}$$
$$U\left(-n, m-n+1, -\Delta\log\left(12-8\sqrt{2}\right)\right)\left(1+O(\frac{1}{\Delta})\right)$$

computing g_i analytically

$$g_i \approx 2\sqrt{2}: \quad \forall\, i, \quad \Delta \gg d, \ell\,.$$