Resolution of singularities and the McKay correspondence

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Introduction

- What is the Equivalence $a = b$?

- $a \sim b$: a is equivalent to b

- **Equivalence relation**

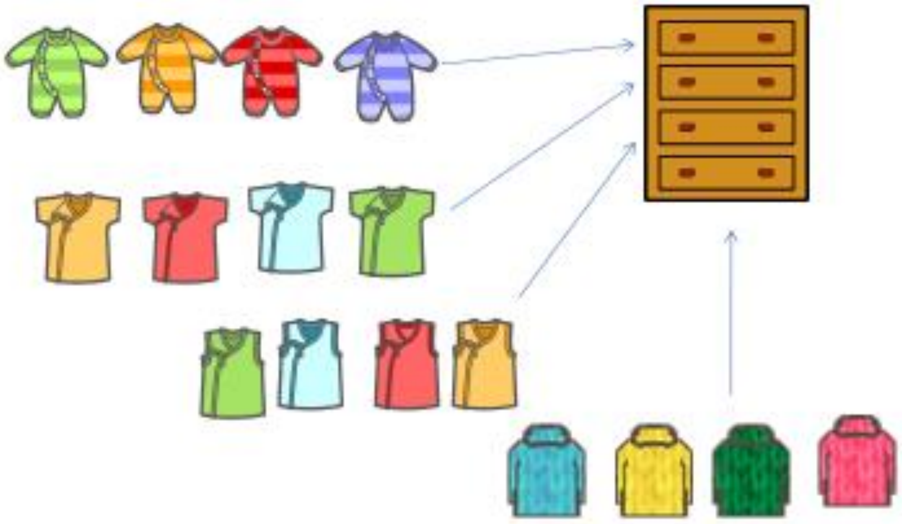
  For any objects a, b, and c:

- $a \sim a$ (reflexive property),
- if $a \sim b$ then $b \sim a$ (symmetric property), and
- if $a \sim b$ and $b \sim c$ then $a \sim c$ (transitive property).

Introduction

- By this equivalence relation, we can classify any set!

- Classification by shcape, colors , …, any property which satisfy the equivalence relation.

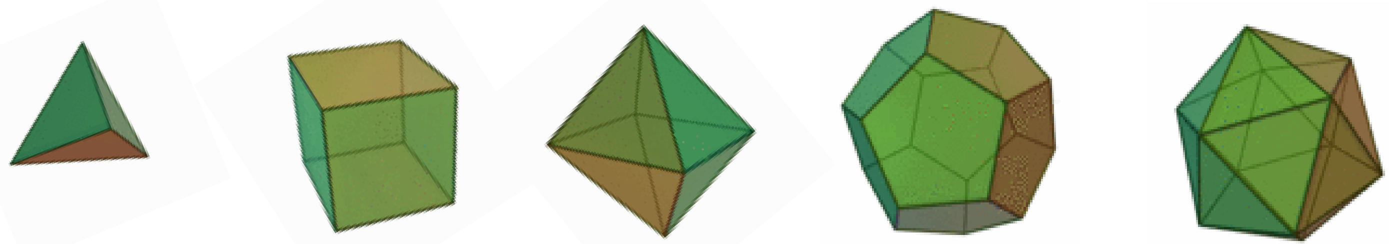


Geometry

- In geometry, we call the equivalence relation Invariant.

Euclid Geometry : length, angle are invariants.

- Regular convex polyhedrons



= Platonic solid (4 classical elements + universe)

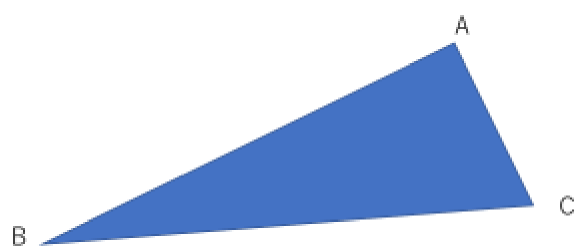
fire earth air universe water

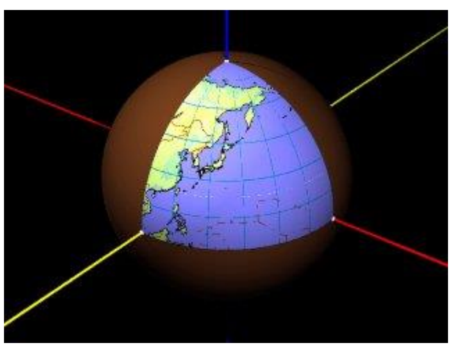
Geometry

Differential Geometry: curvature is an invariant,

Used in the theory of relativity (Einstein)

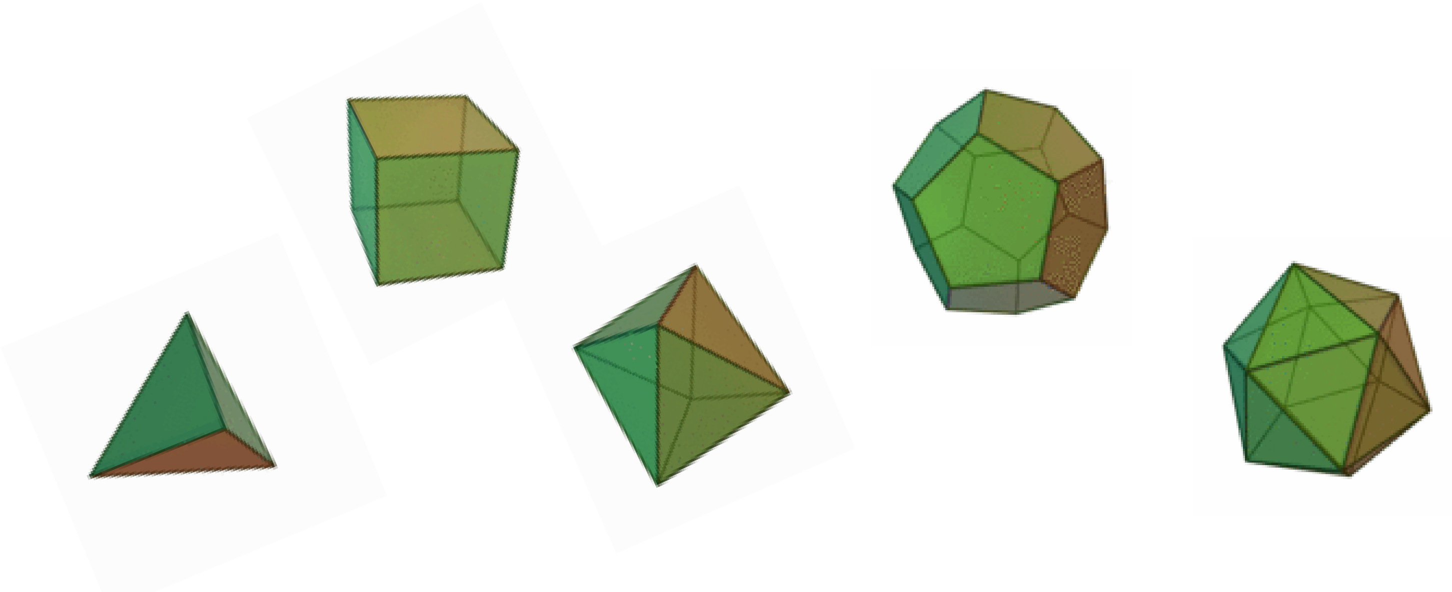
A+B+C=?





Geometry

Topology genus g and Euler number n are Invariants. ($g=n-2$)

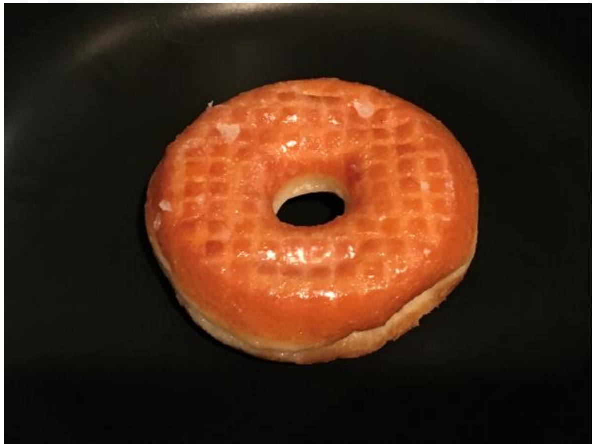


Euler number n = #{vertex}-#{edge}+#{face} $=2$

genus=#{holes} $g=2-2=0$

Geometry

- **Topology**
- genus (=#{holes}) 1



g=2

g=3

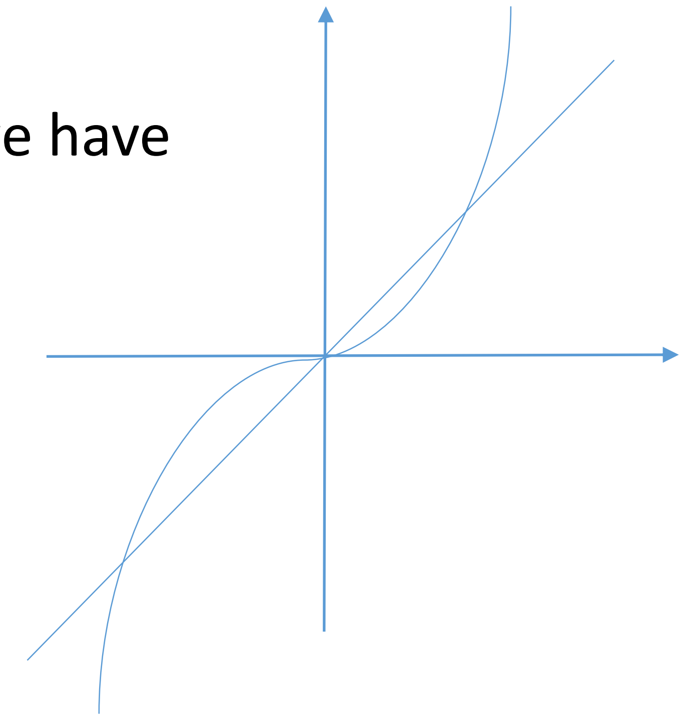
Geometry

・**Algebraic Geometry**

geometry defined by polynomial equations $f(x,y)=0,\ g(x,y)=0.$

Two varieties are equivalent when we have Rational map between them.

(Birational equivalence)



Coodinate (Descartes)

Singularity

- Singularity is special point, something different from other part.

- Big bang?
- Black hole?

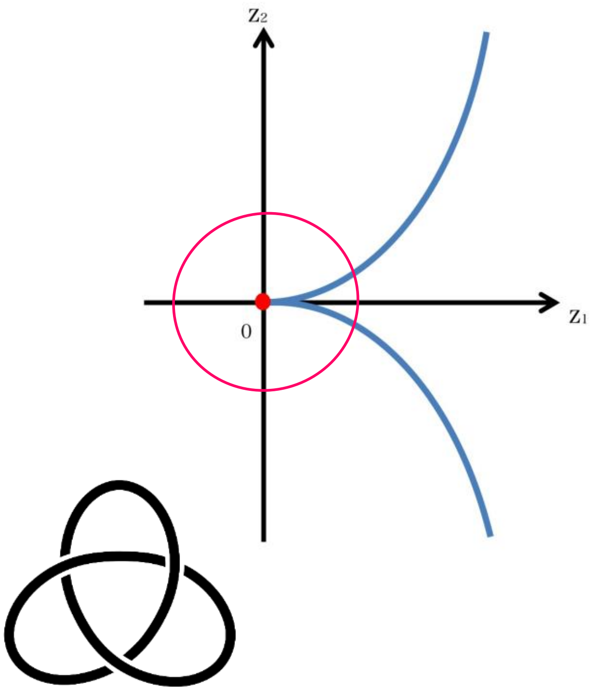
- Cancer?

Singularity theory is used in
CT, MRI machine in hospitals.

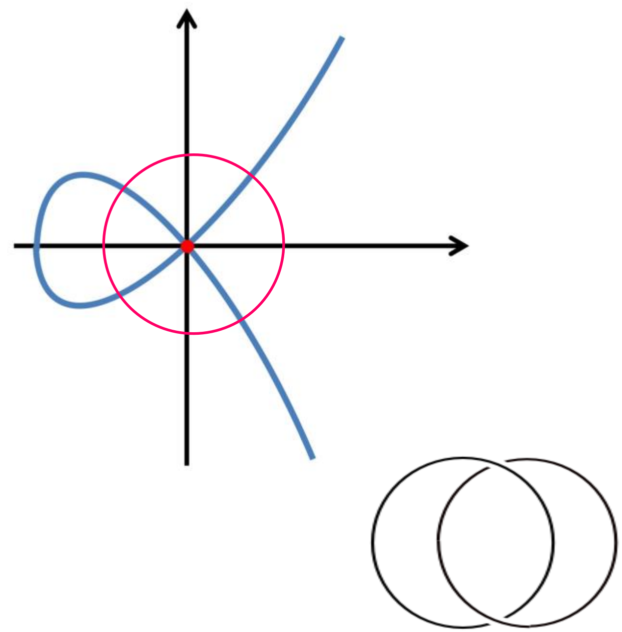
Singularity

- Research on singularities (real 2-dim → complex 2-dim)

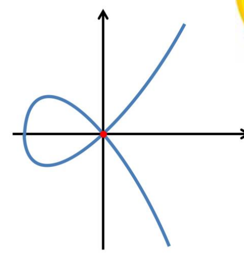
Cusp

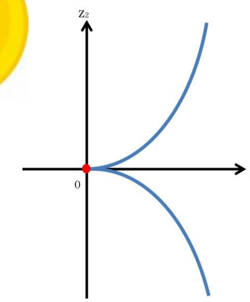


Node

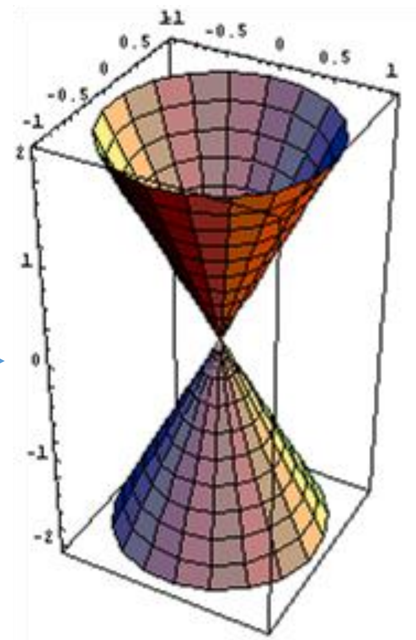


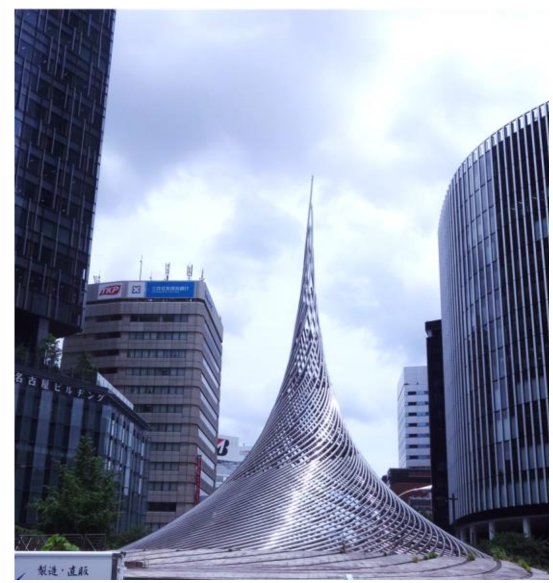
Resolution of singularity!



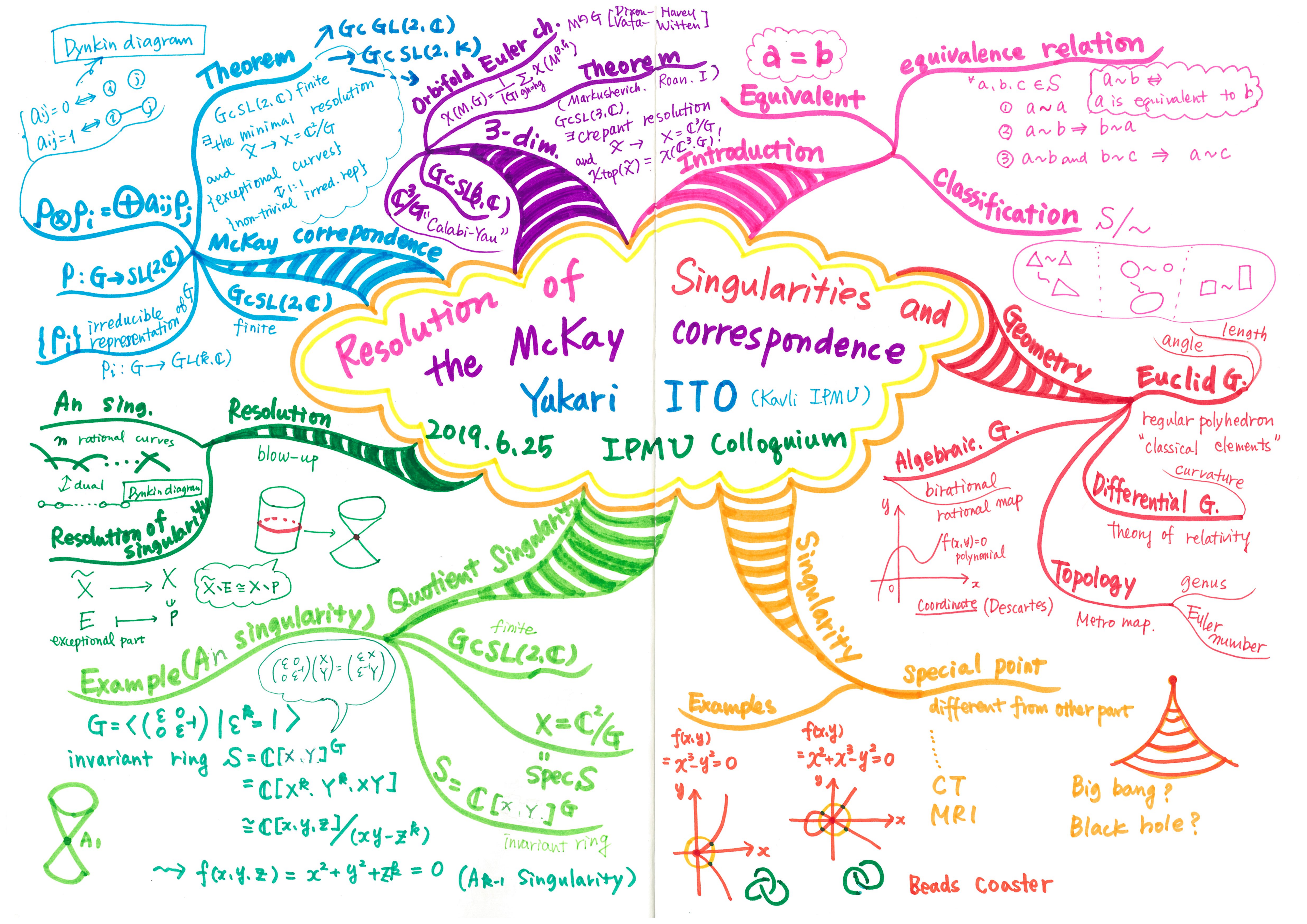


KOBE





NAGOYA


Resolution of Singularities and the McKay correspondence
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Dynkin diagram
$a_{ij}=0 \Leftrightarrow$ ⓘ ⓙ
$a_{ij}=1 \Leftrightarrow$ ⓘ—ⓙ
$\rho \otimes \rho_i = \bigoplus a_{ij} \rho_j$
Theorem
$G \subset SL(2,\mathbb{C})$ finite
∃ the minimal resolution
$\tilde{X} \to X = \mathbb{C}^2/G$
and {exceptional curves} ↕ 1:1 {non-trivial irred. rep}
→ $G \subset GL(2,\mathbb{C})$
→ $G \subset SL(2,K)$
McKay correspondence
$\rho : G \to SL(2,\mathbb{C})$
$G \subset SL(2,\mathbb{C})$ finite
$\{\rho_i\}$ irreducible representation of G
$\rho_i : G \to GL(k,\mathbb{C})$
Orbifold Euler ch.
M∩G [Dixon-Havey Vafa-Witten]
$\chi(M,G) = \frac{1}{|G|} \sum_{gh=hg} \chi(M^{g,h})$
Theorem
(Markushevich, Roan, I)
$G \subset SL(3,\mathbb{C})$
∃ crepant resolution
$\tilde{X} \to X = \mathbb{C}^3/G$!
and $\chi_{top}(\tilde{X}) = \chi(\mathbb{C}^3, G)$!
3-dim.
$G \subset SL(3,\mathbb{C})$
$\mathbb{C}^3/G$ "Calabi-Yau"
a = b
Equivalent
equivalence relation
∀a, b, c ∈ S
a ∼ b ⇔ a is equivalent to b
① a ∼ a
② a ∼ b ⇒ b ∼ a
③ a ∼ b and b ∼ c ⇒ a ∼ c
Introduction
Classification S/∼
△∼△ ○∼○ □∼▯
Geometry
Euclid G.
angle
length
regular polyhedron
"classical elements"
Algebraic G.
birational
rational map
f(x,y)=0 polynomial
Coordinate (Descartes)
Differential G.
curvature
theory of relativity
Topology
genus
Euler number
Metro map.
Singularity
Special point
different from other part
Examples
f(x,y) = $x^3 - y^2 = 0$
f(x,y) = $x^2 + x^3 - y^2 = 0$
CT
MRI
Big bang?
Black hole?
Beads coaster
An sing.
n rational curves
dual
Dynkin diagram
Resolution
blow-up
Resolution of singularity
$\tilde{X} \longrightarrow X$
$E \longmapsto P$
exceptional part
$\tilde{X} \setminus E \cong X \setminus P$
Quotient Singularity
Example (A_n singularity)
$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \varepsilon X \\ \varepsilon^{-1} Y \end{pmatrix}$
$G = \langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \mid \varepsilon^k = 1 \rangle$
invariant ring $S = \mathbb{C}[X,Y]^G$
$= \mathbb{C}[X^k, Y^k, XY]$
$\cong \mathbb{C}[x,y,z]/(xy - z^k)$
$\rightsquigarrow f(x,y,z) = x^2 + y^2 + z^k = 0$ (A_{k-1} Singularity)
A_1
finite
$G \subset SL(2,\mathbb{C})$
$X = \mathbb{C}^2/G$
Spec S
$S = \mathbb{C}[X,Y]^G$
invariant ring