A new approach to black hole quasinormal modes

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+(Ongoing work)
The basic features of GW150914 point to it being a gravitational wave signal from the coalescence of two black holes. The signal is characterized by a chirp that indicates the inspiral of two orbiting black holes, followed by a merger where the two black holes form a single black hole. The subsequent ringdown is a damped oscillation that results from the black hole relaxing to a final stationary Kerr configuration.

The gravitational wave strain from this event can be approximated using the analytical expression:

\[ h(t) = \frac{16 \pi^2}{9 c^3} \frac{M}{r^3} \cos(4 \pi f t) \]

where \( h(t) \) is the gravitational wave strain, \( M \) is the total mass of the binary system, \( r \) is the distance to the source, and \( f \) is the frequency of the gravitational wave. The waveform is sensitive to the chirp mass, which is a combination of the masses of the black hole components.

The waveform is observed to be consistent with the gravitational wave signal from the merger of two black holes, with the observation time and frequency range consistent with the calculated waveform. The detection of GW150914 was a significant milestone in gravitational wave astronomy, providing direct evidence for the existence of black holes and opening a new window into the universe.
This damped oscillation in ringdown phase should be explained by \textbf{QuasiNormal Modes} of final black hole
Because of the black hole no-hair theorem, QNMs depend on mass and spin.
This allows us to know mass and spin of final BH by comparing observation and theoretical prediction.
Why QNM?

- Direct connection with GW observation
- AdS/CFT
- Interesting in mathematical physics
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1. Black hole perturbation and QNMs
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Linear Perturbation

- **Metric variation**
  \[ g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \]

- **Connection**
  \[ \delta \Gamma^\alpha_{\mu\nu} = \frac{1}{2} \bar{g}^{\alpha\beta} (\bar{\nabla}_\mu \delta g_{\nu\beta} + \bar{\nabla}_\nu \delta g_{\mu\beta} - \bar{\nabla}_\beta \delta g_{\mu\nu}) \]

- **Ricci tensor**
  \[ R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu} \]
  \[ \delta R_{\mu\nu} = \bar{\nabla}_\alpha \delta \Gamma^\alpha_{\mu\nu} - \bar{\nabla}_\nu \delta \Gamma^\alpha_{\mu\alpha} \]
BH Perturbation

- **Background: Schwarzschild BH**

\[
    ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu
\]

\[
    = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

\[
    f(r) = 1 - \frac{2M}{r} \quad (2M = 1)
\]

- **Vacuum solution**

\[
    \bar{R}_{\mu\nu} = 0
\]

\[
    R_{\mu\nu} = 0
\]

\[
    \delta R_{\mu\nu} = 0
\]
BH Perturbation

• This condition leads to differential equations for 10 unknown functions $\delta g_{\mu\nu}$

• One can reduce the degree of freedom by fixing gauge symmetry

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x)$$

$$10 \rightarrow 6$$
BH Perturbation

• The basic strategy is to separate (time+radial)-directions and angular parts

\[ \delta g_{\mu\nu} = h_{\mu\nu}(t, r)y_{\mu\nu}(\theta, \phi) \]

Combinations of spherical harmonics
Regge-Wheeler Gauge

Regge & Wheeler 1957

\[ \delta g_{\mu\nu} = \delta g_{\mu\nu}^{\text{odd}} + \delta g_{\mu\nu}^{\text{even}} \]

\[ (10 = 3 + 7) \quad \theta \rightarrow \pi - \theta \]

\[ 6 = 2 + 4 \quad \phi \rightarrow \pi + \phi \]

Each parity part can be treated separately

\[ \delta g_{\mu\nu}^{\text{odd}} = \left( \begin{array}{cccc}
0 & 0 & 0 & h_0(t, r) \\
0 & 0 & 0 & h_1(t, r) \\
0 & 0 & 0 & 0 \\
h_0(t, r) & h_1(t, r) & 0 & 0 \\
\end{array} \right) \sin \theta \frac{\partial Y_{l0}}{\partial \theta} \]
Regge-Wheeler Equation

• Substitute it into $\delta R_{\mu\nu} = 0$

$$u(t, r) := r^{-1} f(r) h_1(t, r)$$

$$\left[ -\frac{\partial^2}{\partial t^2} + f(r) \frac{\partial}{\partial r} \left( f(r) \frac{\partial}{\partial r} \right) - V(r) \right] u(t, r) = 0$$

$$V(r) = f(r) \left[ \frac{l(l+1)}{r^2} - \frac{3}{r^3} \right]$$
Even parity sector

- The computation in the even parity sector is much more complicated.
- Nevertheless, the final result also leads to a single partial differential equation that has the same form as the RW equation with a different potential.

Zerilli 1970
Boundary Conditions

- Change of variable

\[
\left[ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - \hat{V}(x) \right] \hat{u}(t, x) = 0
\]

\[
\frac{\partial}{\partial x} := f(r) \frac{\partial}{\partial r}
\]

\[
(x = r + \log(r - 1))
\]

Horizon \( x = -\infty \)

Infinity \( x = +\infty \)
Boundary Conditions

• Asymptotic behaviors

\[ \hat{u}(t, x) \sim A_R e^{-i\omega(t-x)} + A_L e^{-i\omega(t+x)} \quad (x \to +\infty) \]

\[ \hat{u}(t, x) \sim B_R e^{-i\omega(t-x)} + B_L e^{-i\omega(t+x)} \quad (x \to -\infty) \]

• B.C.
Such a boundary condition is realized by special values of $\omega$

$$\omega_n = \omega_n^\text{Re} - i\omega_n^\text{Im} \quad (\omega_n^\text{Im} > 0)$$

$$e^{-i\omega_n t} = e^{-\omega_n^\text{Im} t} e^{-i\omega_n^\text{Re} t}$$

Damped Oscillation
WKB Approximation

\[
\left[ -\epsilon^2 \frac{d^2}{dx^2} + \hat{V}(x) \right] \psi(x) = \omega^2 \psi(x)
\]

\[
\hat{u}(t, x) = e^{-i\omega t} \psi(x)
\]

We can use results in scattering problem
Scattering Problem

Reflection and Transmission

\[
R = \frac{e^{-i\delta}}{(1 + e^{-2w(a,b)})^{1/2}}
\]

\[
T = \frac{e^{-w(a,b) - i\delta}}{(1 + e^{-2w(a,b)})^{1/2}}
\]

\[
w(a, b) := \frac{1}{\varepsilon} \int_{a}^{b} \sqrt{\hat{V}(x) - \omega^2} \, dx
\]
Quantization Condition

If the denominator of R and T vanishes, then the BC of QNM is realized (resonance)

\[ 2\omega(a, b) = \frac{2}{\epsilon} \int_{a}^{b} \sqrt{\hat{V}(x) - \omega^2} \, dx = 2\pi i \left( n + \frac{1}{2} \right) \]

Froeman et al. 1992
Technical Problems

- It is not easy to compute higher order corrections to quantization condition (I’ll be back later)
- There are some other numerical methods

Accuracy ⇔ Generality

- I will show an accurate and widely applicable way
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These are very similar
Mapping to Bound States

\[ \left[ -\epsilon^2 \frac{d^2}{dx^2} + \hat{V}(x) \right] \psi(x) = \omega^2 \psi(x) \]

\[ \hbar := i\epsilon, \quad E := -\omega^2 \]

\[ \left[ -\hbar^2 \frac{d^2}{dx^2} - \hat{V}(x) \right] \psi(x) = E\psi(x) \]

The potential is inverted

Blome & Mashhoon 1984
YH, arXiv:1906.07232
Mapping to Bound States

- Asymptotic behaviors

\[
\exp\left(-\frac{\sqrt{-Ex}}{\hbar}\right) \rightarrow \exp\left(\frac{i\omega x}{\epsilon}\right)
\]

\[
\exp\left(\frac{\sqrt{-Ex}}{\hbar}\right) \rightarrow \exp\left(-\frac{i\omega x}{\epsilon}\right)
\]
Mapping to Bound States

- **Idea:** Analytic continuation of the Planck parameter

$$(\omega_n^{\text{QNM}})^2 = -E_n^{\text{BS}}(\hbar = i)$$

- The bound state problem is however not solved analytically

- I use perturbation theory
Perturbation Theory

\[
-\hbar^2 \frac{d^2}{dx^2} - \hat{V}(x) \psi(x) = E \psi(x)
\]

\[
-\hat{V}(x) = V_0 + \sum_{k=2}^{\infty} V_k (x - x_0)^k
\]

\[
x - x_0 = \sqrt{\hbar}q
\]

\[
\left(-\frac{1}{2} \frac{d^2}{dq^2} + \frac{V_2}{2} q^2 + V_{\text{int}}(q)\right) \Psi(q) = \epsilon \Psi(q) \quad \epsilon = \frac{E - V_0}{2\hbar}
\]

\[
V_{\text{int}}(q) = \frac{1}{2} \sum_{k=3}^{\infty} \hbar^{k/2-1} V_k q^k
\]

Anharmonic oscillator
Perturbation Theory

- Spectrum

\[ \varepsilon_n = \varepsilon_n^{(0)} + \hbar \varepsilon_n^{(1)} + \hbar^2 \varepsilon_n^{(2)} + \cdots \]

- There is a great MATHEMATICA package

- This perturbative series turns out to be a divergent series

Sulejmanpasic & Ünsal 2016
Borel(-Pade) Summation

\[ B[\varepsilon_n^{\text{pert}}](\zeta) := \sum_{k=0}^{\infty} \frac{\varepsilon_n^{(k)}}{k!} \zeta^k \]

\[ \varepsilon_n^{\text{pert,Borel}}(\bar{h}) = \int_0^\infty d\zeta \, e^{-\zeta} B^C[\varepsilon_n^{\text{pert}}](\bar{h}\zeta) \]

\[ \varepsilon_n^{\text{pert,}[M/N]}(\bar{h}) = \int_0^\infty d\zeta \, e^{-\zeta} B^{[M/N]}[\varepsilon_n^{\text{pert}}](\bar{h}\zeta) \]

\[ (\omega_n^{[M/N]})^2 := - (V_0 + 2i\varepsilon_n^{\text{pert,}[M/N]}(i)) \]
Figure 3. The pole distributions of the Padé approximants $B_{100/100}$ for $n = 0$ (left) and for $n = 1$ (right). There are no singularities on the positive imaginary axis in both cases. A cluster of poles implies a branch cut.

Table 1. The Borel–Padé summations of the quasinormal frequencies in the odd-parity gravitational perturbations of the Schwarzschild black hole. We have computed the perturbative expansion of $\Xi_n$ up to the 200th order, and have used the (diagonal) Padé approximant of the Borel transform. We show only the reliable stable parts of the numerical values. These values are consistent with all the available data in the literature.

| $l$ | $n$ | $\omega_n^{[100/100]}$ | $|\omega_n^{[100/100]} - \omega_n^{[99/99]}|$ |
|-----|-----|----------------------|----------------------------------|
| 2   | 0   | 0.74734336883608367159 $- 0.17792463137787139656i$ | $6.4 \times 10^{-24}$ |
| 1   |     | 0.693421993758327 $- 0.547829750582470i$ | $1.3 \times 10^{-16}$ |
| 2   |     | 0.602106909 $- 0.956553967i$ | $8.0 \times 10^{-11}$ |
| 3   | 0   | 1.19888657687498014548 $- 0.18540609588989520794i$ | $4.8 \times 10^{-42}$ |
| 1   |     | 1.16528760606659886123 $- 0.56259622687008808936i$ | $5.0 \times 10^{-34}$ |
| 2   |     | 1.10336980155690263277 $- 0.95818550193392446993i$ | $8.6 \times 10^{-26}$ |
| 4   | 0   | 1.61835675506447828139 $- 0.18832792197784649881i$ | $7.0 \times 10^{-53}$ |
| 1   |     | 1.59326306406900503032 $- 0.56866869880968143729i$ | $1.2 \times 10^{-44}$ |
| 2   |     | 1.54541906521341859968 $- 0.95981635024232615560i$ | $8.0 \times 10^{-37}$ |

Remarkable accuracy! Easy to apply to other cases.
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Conjecture

QNMs for spherically symmetric BHs are related to supersymmetric gauge theories
Example

Regge-Wheeler equation for Schwarzschild BH

II

Quantum Seiberg-Witten curve for Nf=3 SYM
Regge-Wheeler Equation

\[
\left[ f(r) \frac{d}{dr} f(r) \frac{d}{dr} + \omega^2 - V(r) \right] \phi(r) = 0
\]

\[
f(r) = 1 - \frac{1}{r}, \quad V(r) = f(r) \left[ \frac{l(l+1)}{r^2} - \frac{3}{r^3} \right]
\]

\[
r = 0, 1 \quad \text{Regular singularities}
\]

\[
r = \infty \quad \text{Irregular singularity}
\]
Singularities

Heun $N_f = 4$

"Confluence"

Confluent Heun $N_f = 3$

Decoupling limit

Gaiotto 2009
Seiberg-Witten Curve

\[ K(p) - \frac{\Lambda^{1/2}}{2} \left( e^{ix/2} K_+(p) e^{ix/2} + e^{-ix/2} K_-(p) e^{-ix/2} \right) = 0 \]

\[ K(p) = p^2 - u + \frac{\Lambda}{4} \left( p + \frac{m_1 + m_2 + m_3}{2} \right) \]

\[ K_+(p) = (p + m_1)(p + m_2) \quad K_-(p) = p + m_3 \]

Symplectic form \[ \Omega = dp \wedge dx \]
Quantum SW Curve

We regard $x$ and $p$ as canonical variables, and replace them noncommuting operators

$$[x, p] = i\hbar$$

$$p = -i\hbar \partial_x$$

SW curve $\rightarrow$ 2nd order ODE
Quantum SW Curve

By the following identification between SW and BH parameters,

\[
\frac{2}{\sqrt{\Lambda}} e^{-ix} = r, \quad \Lambda = 8i\omega
\]

\[
u = \frac{1}{4} + l(l + 1) - 2\omega^2,
\]

\[
m_1 = 2 + i\omega, \quad m_2 = -2 + i\omega, \quad m_3 = i\omega
\]

we get the RW equation!
**Expectation**

**Quantization condition**

\[
\frac{1}{2\pi \epsilon} \int_{\gamma} p_0(x) dx = n + \frac{1}{2}
\]

\[
\frac{\partial F_0(a)}{\partial a} = n + \frac{1}{2}
\]

Probably **all the quantum corrections** are systematically included by **Nekrasov’s partition function**

Nekrasov & Shatashvili 2009
D-dim Schwarzschild

Regular: D-2

Irregular: 1

D-dim Schwarzschild-AdS

Regular: D+1

Kodama & Ishibashi 2003
Summary

• I proposed an efficient way to compute QNMs by perturbation theory

• This method is widely applicable

• “QNM/SYM” correspondence