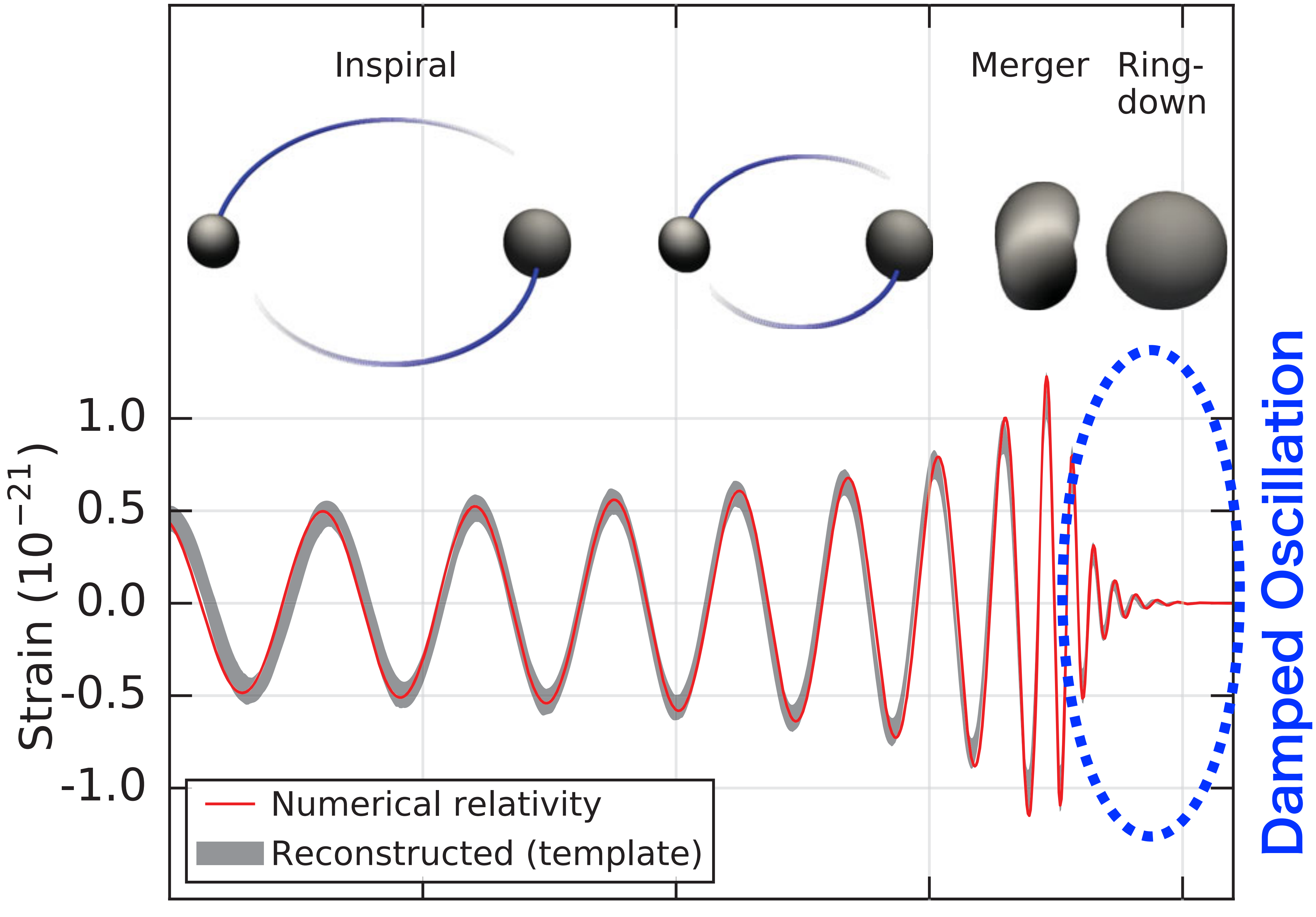
11/26/2019@IPMU

A new approach to black hole quasinormal modes

Yasuyuki Hatsuda
(Rikkyo University)

arXiv:1906.07232[gr-qc]
+(Ongoing work)

GW from Binary BHs



[Abbott et al., PRL 116, 061102 (2016)]

This damped oscillation in ringdown phase should be explained by QuasiNormal Modes of final black hole

Because of the black hole no-hair theorem, QNMs depend on mass and spin

This allows us to know
mass and spin of final BH
by comparing observation
and theoretical prediction

Why QNM?

- Direct connection with GW observation
- AdS/CFT
- Interesting in mathematical physics

Contents

1. Black hole perturbation and QNMs
2. A refined way
3. A new perspective (ongoing)

Contents

Linear Perturbation

- **Metric variation**

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

- **Connection**

$$\delta\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}\bar{g}^{\alpha\beta}(\bar{\nabla}_{\mu}\delta g_{\nu\beta} + \bar{\nabla}_{\nu}\delta g_{\mu\beta} - \bar{\nabla}_{\beta}\delta g_{\mu\nu})$$

- **Ricci tensor**

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu}$$

$$\delta R_{\mu\nu} = \bar{\nabla}_{\alpha}\delta\Gamma^{\alpha}{}_{\mu\nu} - \bar{\nabla}_{\nu}\delta\Gamma^{\alpha}{}_{\mu\alpha}$$

BH Perturbation

- **Background: Schwarzschild BH**

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu$$

$$= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$f(r) = 1 - \frac{2M}{r} \qquad (2M = 1)$$

- **Vacuum solution**

$$\bar{R}_{\mu\nu} = 0$$

$$R_{\mu\nu} = 0$$

$$\delta R_{\mu\nu} = 0$$

BH Perturbation

- **This condition leads to differential equations for 10 unknown functions $\delta g_{\mu\nu}$**
- **One can reduce the degree of freedom by fixing gauge symmetry**

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$

$$10 \to 6$$

BH Perturbation

- **The basic strategy is to separate (time+radial)-directions and angular parts**

$$\delta g_{\mu\nu} = h_{\mu\nu}(t,r) y_{\mu\nu}(\theta,\phi)$$

Combinations of spherical harmonics

Regge-Wheeler Gauge

Regge & Wheeler 1957

$$\delta g_{\mu\nu} = \delta g_{\mu\nu}^{\mathrm{odd}} + \delta g_{\mu\nu}^{\mathrm{even}}$$

$$(10 = 3 + 7) \qquad \theta \to \pi - \theta$$

$$\mathbf{6 = 2 + 4} \qquad \phi \to \pi + \phi$$

Each parity part can be treated separately

$$\delta g_{\mu\nu}^{\mathrm{odd}} = \begin{pmatrix} 0 & 0 & 0 & h_0(t,r) \\ 0 & 0 & 0 & h_1(t,r) \\ 0 & 0 & 0 & 0 \\ h_0(t,r) & h_1(t,r) & 0 & 0 \end{pmatrix} \sin\theta \frac{\partial Y_{l0}}{\partial\theta}$$

Regge-Wheeler Equation

- **Substitute it into** $\delta R_{\mu\nu} = 0$

$$u(t,r) := r^{-1} f(r) h_1(t,r)$$

$$\left[-\frac{\partial^2}{\partial t^2} + f(r)\frac{\partial}{\partial r}\left(f(r)\frac{\partial}{\partial r}\right) - V(r)\right] u(t,r) = 0$$

$$V(r) = f(r)\left[\frac{l(l+1)}{r^2} - \frac{3}{r^3}\right]$$

Even parity sector

- The computation in the even parity sector is much more complicated
- Nevertheless the final result also leads a single partial differential equation that has the same form as the RW equation with a different potential

Zerilli 1970

Boundary Conditions

- **Change of variable**

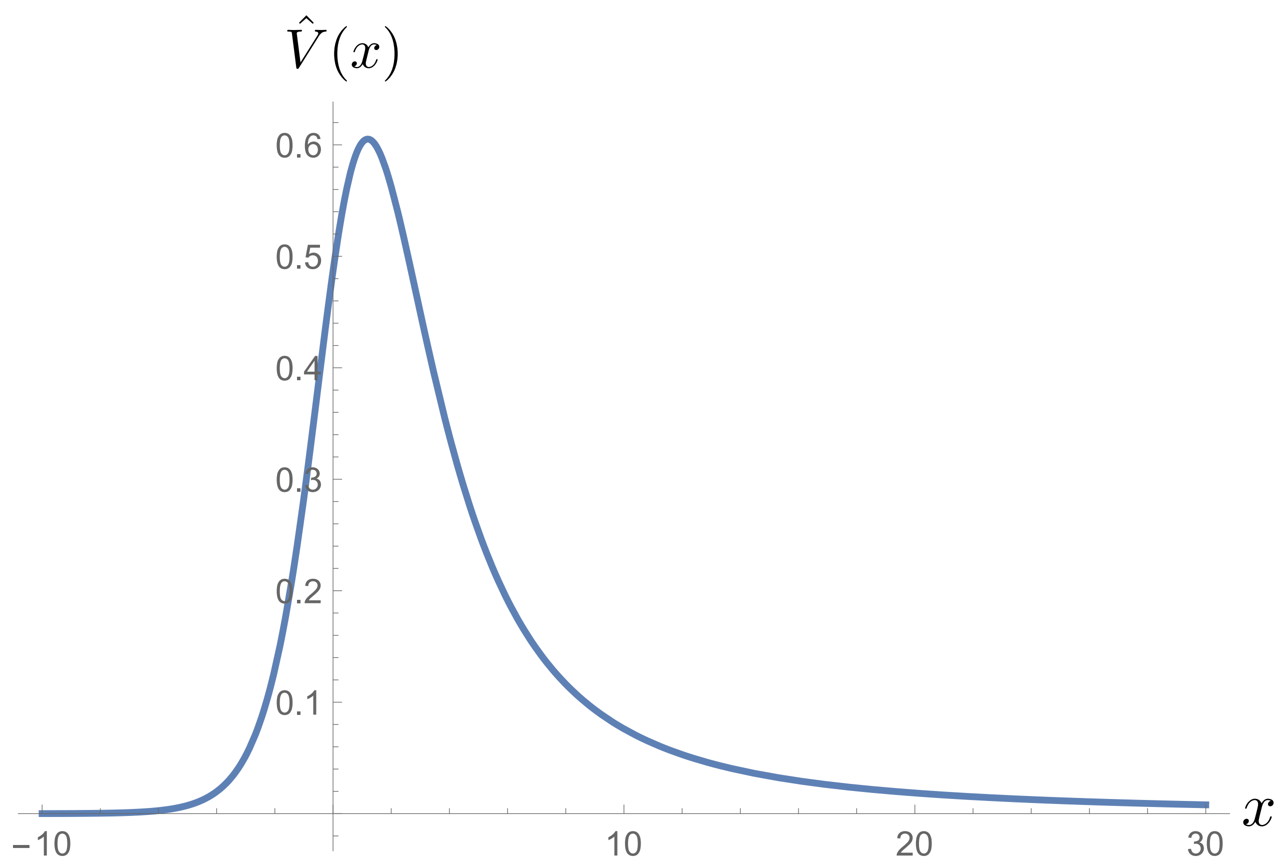
$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-\hat{V}(x)\right]\hat{u}(t,x)=0$$

$$\frac{\partial}{\partial x}:=f(r)\frac{\partial}{\partial r}$$

$$(x=r+\log(r-1))$$

Horizon $x=-\infty$

Infinity $x=+\infty$



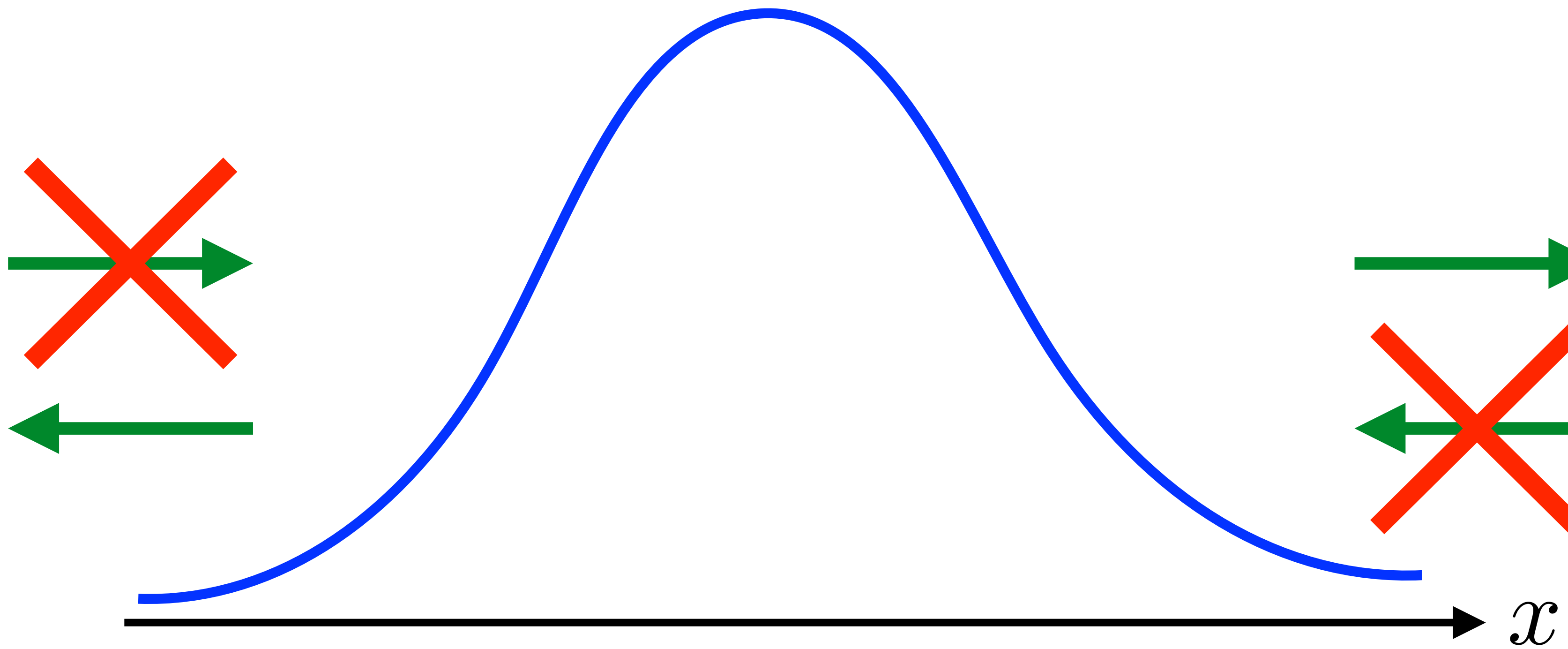
Boundary Conditions

- **Asymptotic behaviors**

$$\hat{u}(t,x) \sim A_{\rm R} e^{-i\omega(t-x)} + A_{\rm L} e^{-i\omega(t+x)} \qquad (x \to +\infty)$$

$$\hat{u}(t,x) \sim B_{\rm R} e^{-i\omega(t-x)} + B_{\rm L} e^{-i\omega(t+x)} \qquad (x \to -\infty)$$

- **B.C.**



Such a boundary condition is realized by special values of ω

$$\omega_n = \omega_n^{\mathrm{Re}} - i\omega_n^{\mathrm{Im}} \quad (\omega_n^{\mathrm{Im}} > 0)$$

$$e^{-i\omega_n t} = e^{-\omega_n^{\mathrm{Im}} t} e^{-i\omega_n^{\mathrm{Re}} t}$$

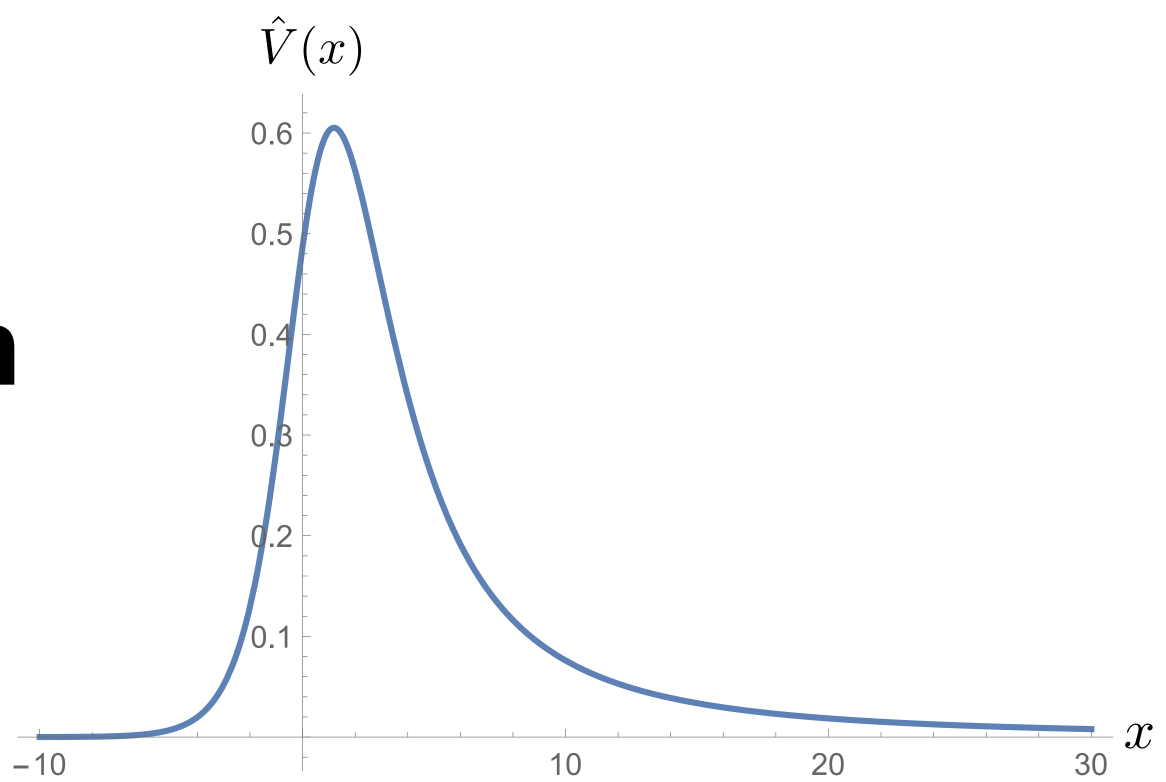
Damped Oscillation

WKB Approximation

$$\left[-\epsilon^2 \frac{d^2}{dx^2} + \hat{V}(x)\right]\psi(x) = \omega^2 \psi(x)$$

$$\hat{u}(t,x) = e^{-i\omega t}\psi(x)$$

We can use results in scattering problem



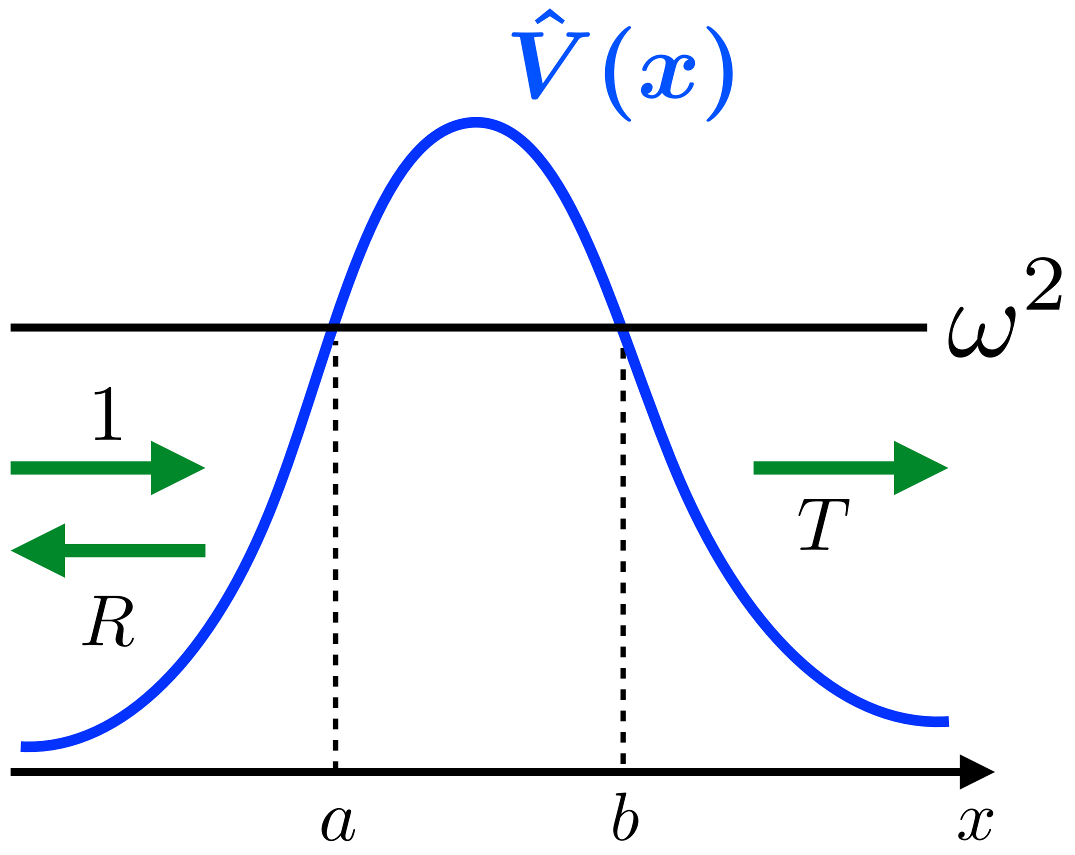
Scattering Problem

Reflection and Transmission

$$R = \frac{e^{-i\delta}}{(1 + e^{-2w(a,b)})^{1/2}}$$

$$T = \frac{e^{-w(a,b)-i\delta}}{(1 + e^{-2w(a,b)})^{1/2}}$$

$$w(a,b) := \frac{1}{\epsilon}\int_a^b \sqrt{\hat{V}(x) - \omega^2} dx$$



Quantization Condition

If the denominator of R and T vanishes, then the BC of QNM is realized (resonance)

$$2w(a,b) = \frac{2}{\epsilon}\int_a^b \sqrt{\hat{V}(x) - \omega^2}dx = 2\pi i\left(n + \frac{1}{2}\right)$$

Froeman et al. 1992

Technical Problems

- It is not easy to compute higher order corrections to quantization condition (I'll be back later)
- There are some other numerical methods

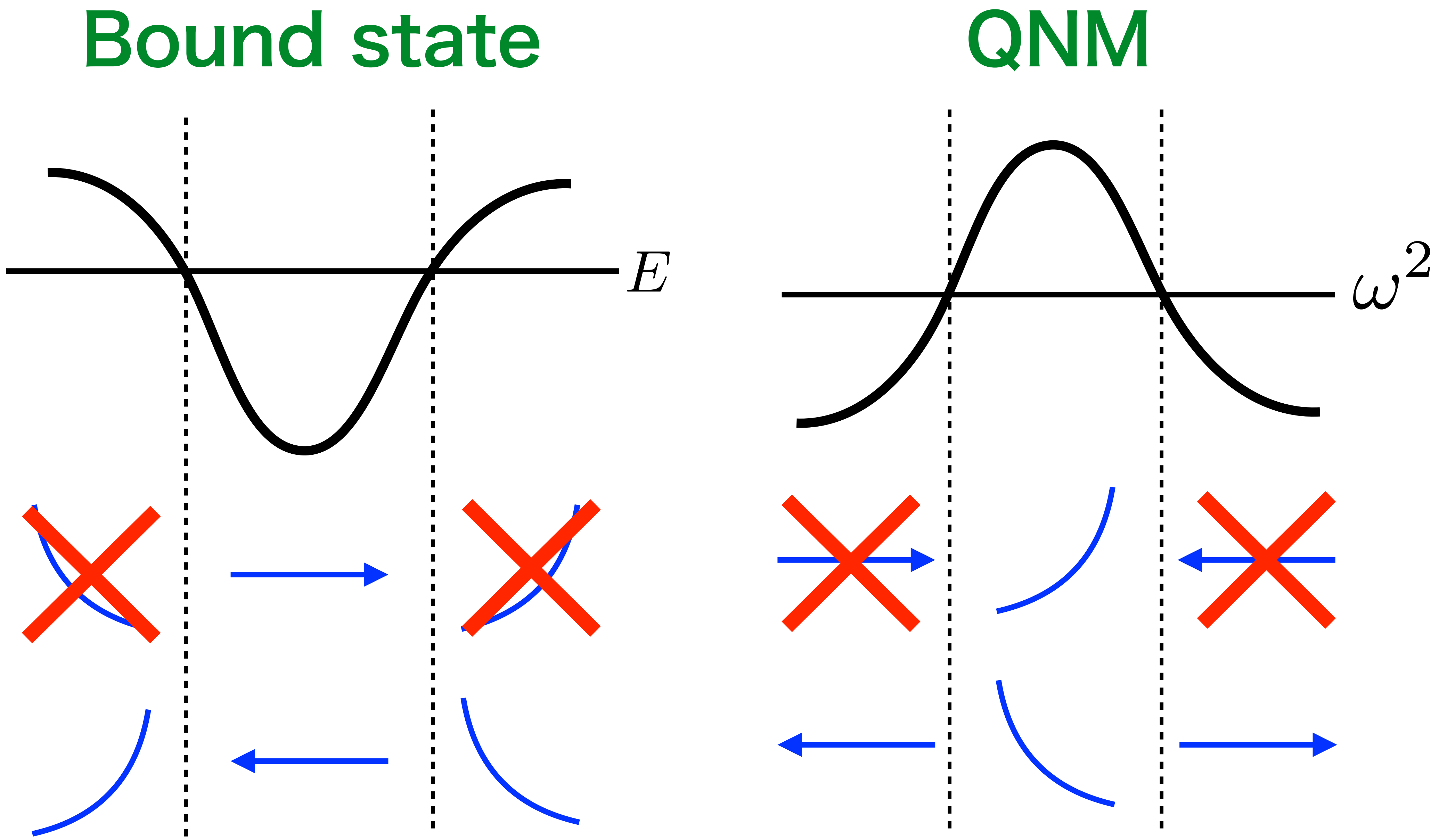
Accuracy ⟷ Generality

- I will show an accurate and widely applicable way

Contents

1. Black hole perturbation and QNMs
2. A refined way
3. A new perspective (ongoing)

Analogy



These are very similar

Mapping to Bound States

Blome & Mashhoon 1984
YH, arXiv:1906.07232

$$\left[-\epsilon^2 \frac{d^2}{dx^2} + \hat{V}(x)\right] \psi(x) = \omega^2 \psi(x)$$

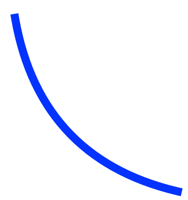
$$\hbar := i\epsilon \,, \qquad E := -\omega^2$$

$$\left[-\hbar^2 \frac{d^2}{dx^2} - \hat{V}(x)\right] \psi(x) = E\psi(x)$$

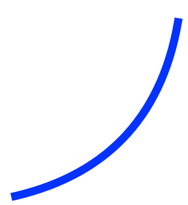
The potential is inverted

Mapping to Bound States

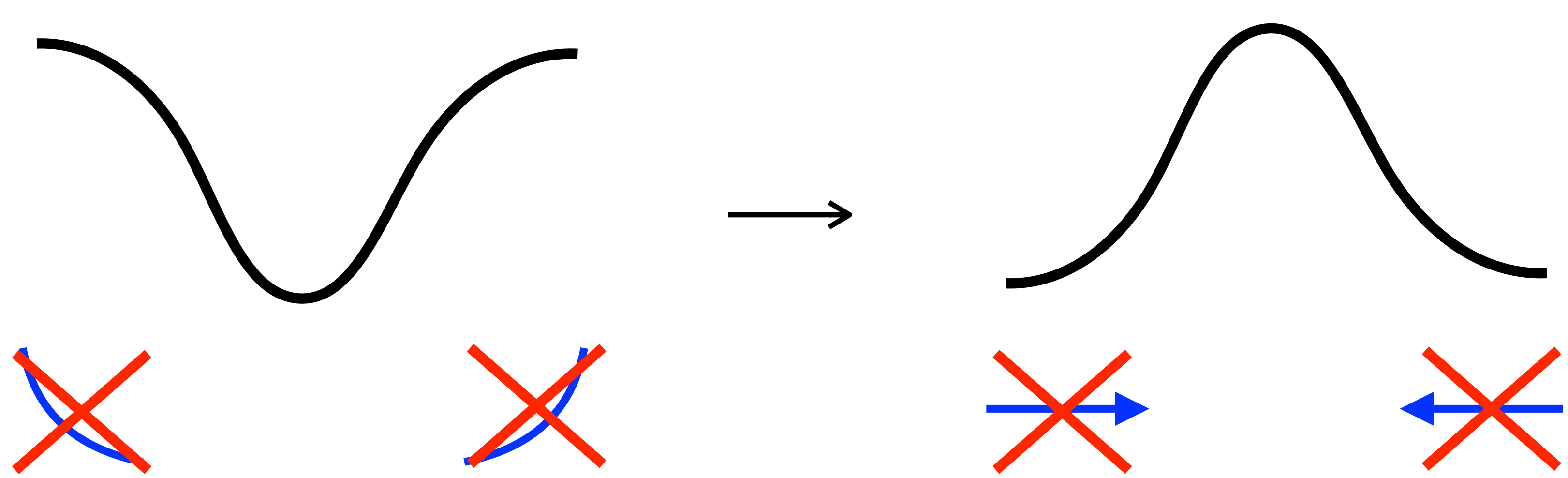
- **Asymptotic behaviors**



$$\exp\left(-\frac{\sqrt{-E}x}{\hbar}\right) \to \exp\left(+\frac{i\omega x}{\epsilon}\right)$$



$$\exp\left(+\frac{\sqrt{-E}x}{\hbar}\right) \to \exp\left(-\frac{i\omega x}{\epsilon}\right)$$



Mapping to Bound States

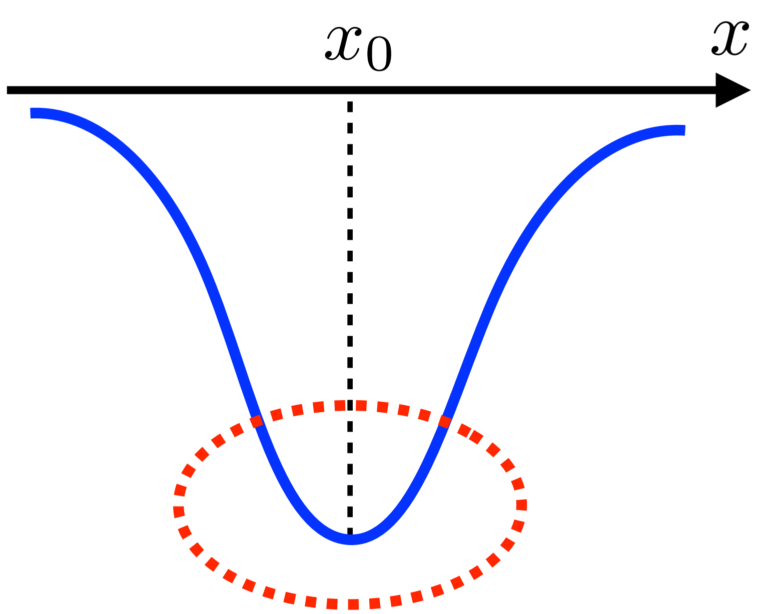
- **Idea:** Analytic continuation of the Planck parameter

$$(\omega_n^{\mathrm{QNM}})^2 = -E_n^{\mathrm{BS}}(\hbar = i)$$

- The bound state problem is however **not** solved **analytically**
- I use **perturbation theory**

Perturbation Theory

$$\left[-\hbar^2 \frac{d^2}{dx^2} - \hat{V}(x)\right]\psi(x) = E\psi(x)$$



$$-\hat{V}(x) = V_0 + \sum_{k=2}^{\infty} V_k (x - x_0)^k$$

$$x - x_0 = \sqrt{\hbar} q$$

$$\left(-\frac{1}{2}\frac{d^2}{dq^2} + \frac{V_2}{2} q^2 + V_{\text{int}}(q)\right)\Psi(q) = \epsilon\Psi(q) \qquad \epsilon = \frac{E - V_0}{2\hbar}$$

$$V_{\text{int}}(q) = \frac{1}{2}\sum_{k=3}^{\infty} \hbar^{k/2-1} V_k q^k$$

Anharmonic oscillator

Perturbation Theory

- **Spectrum**

$$\epsilon_n = \epsilon_n^{(0)} + \hbar \epsilon_n^{(1)} + \hbar^2 \epsilon_n^{(2)} + \cdots$$

- **There is a great MATHEMATICA package** Sulejmanpasic & Ünsal 2016

- **This perturbative series turns out to be a divergent series**

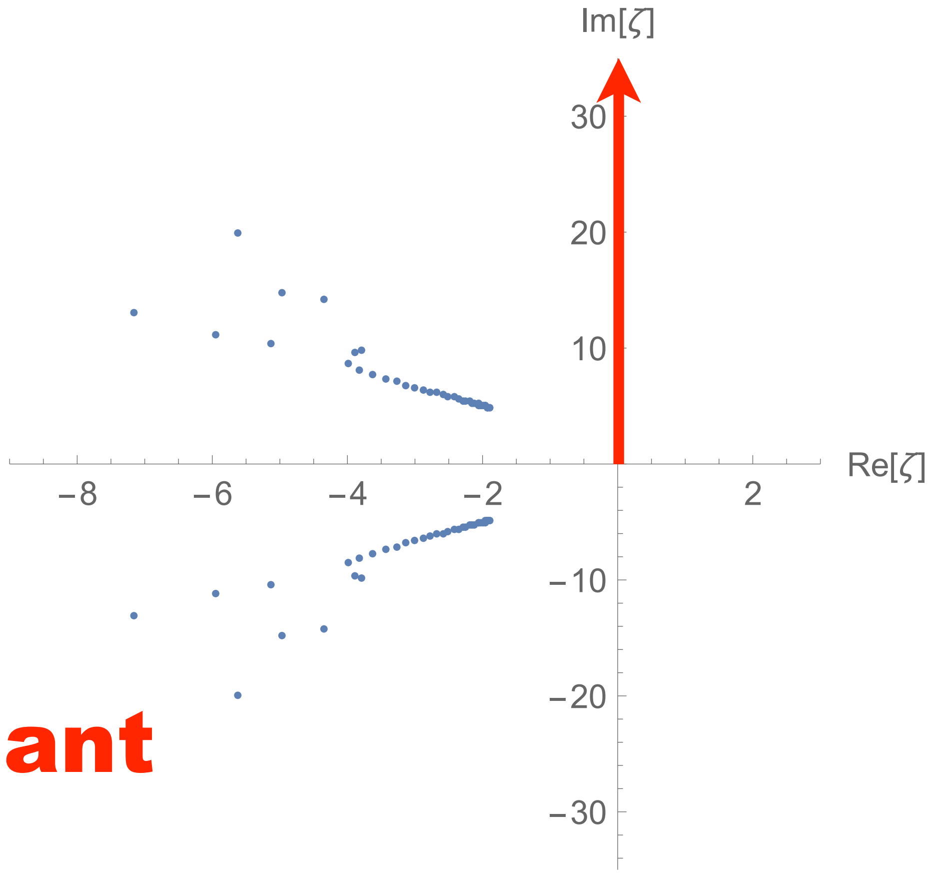
Borel(-Pade) Summation

$$\mathcal{B}[\epsilon_n^{\mathrm{pert}}](\zeta) := \sum_{k=0}^{\infty} \frac{\epsilon_n^{(k)}}{k!} \zeta^k$$

$$\epsilon_n^{\mathrm{pert,Borel}}(\hbar) = \int_0^{\infty} d\zeta\, e^{-\zeta} \mathcal{B}^{\mathcal{C}}[\epsilon_n^{\mathrm{pert}}](\hbar\zeta)$$

↓ **Use Pade approximant**

$$\epsilon_n^{\mathrm{pert},[M/N]}(\hbar) := \int_0^{\infty} d\zeta\, e^{-\zeta} \mathcal{B}^{[M/N]}[\epsilon_n^{\mathrm{pert}}](\hbar\zeta)$$



$$(\omega_n^{[M/N]})^2 := -(V_0 + 2i\epsilon_n^{\mathrm{pert},[M/N]}(i))$$

Result

YH, arXiv:1906.07232

Quasinormal Frequencies

l	n	$\omega_n^{[100/100]}$	$\lvert\omega_n^{[100/100]} - \omega_n^{[99/99]}\rvert$
2	0	$0.74734336883608367159 - 0.17792463137787139656i$	6.4×10^{-24}
	1	$0.693421993758327 - 0.547829750582470i$	1.3×10^{-16}
	2	$0.602106909 - 0.956553967i$	8.0×10^{-11}
3	0	$1.19888657687498014548 - 0.18540609588989520794i$	4.8×10^{-42}
	1	$1.16528760606659886123 - 0.56259622687008808936i$	5.0×10^{-34}
	2	$1.10336980155690263277 - 0.95818550193392446993i$	8.6×10^{-26}
4	0	$1.61835675506447828139 - 0.18832792197784649881i$	7.0×10^{-53}
	1	$1.59326306406900503032 - 0.56866869880968143729i$	1.2×10^{-44}
	2	$1.54541906521341859968 - 0.95981635024232615560i$	8.0×10^{-37}

Remarkable accuracy!

Easy to apply to other cases

Contents

1. Black hole perturbation and QNMs

2. A refined way

3. **A new perspective (ongoing)**

Conjecture

QNMs for spherically symmetric BHs are related to supersymmetric gauge theories

Example

Regge-Wheeler equation for Schwarzschild BH

=

Quantum Seiberg-Witten curve for Nf=3 SYM

Regge-Wheeler Equation

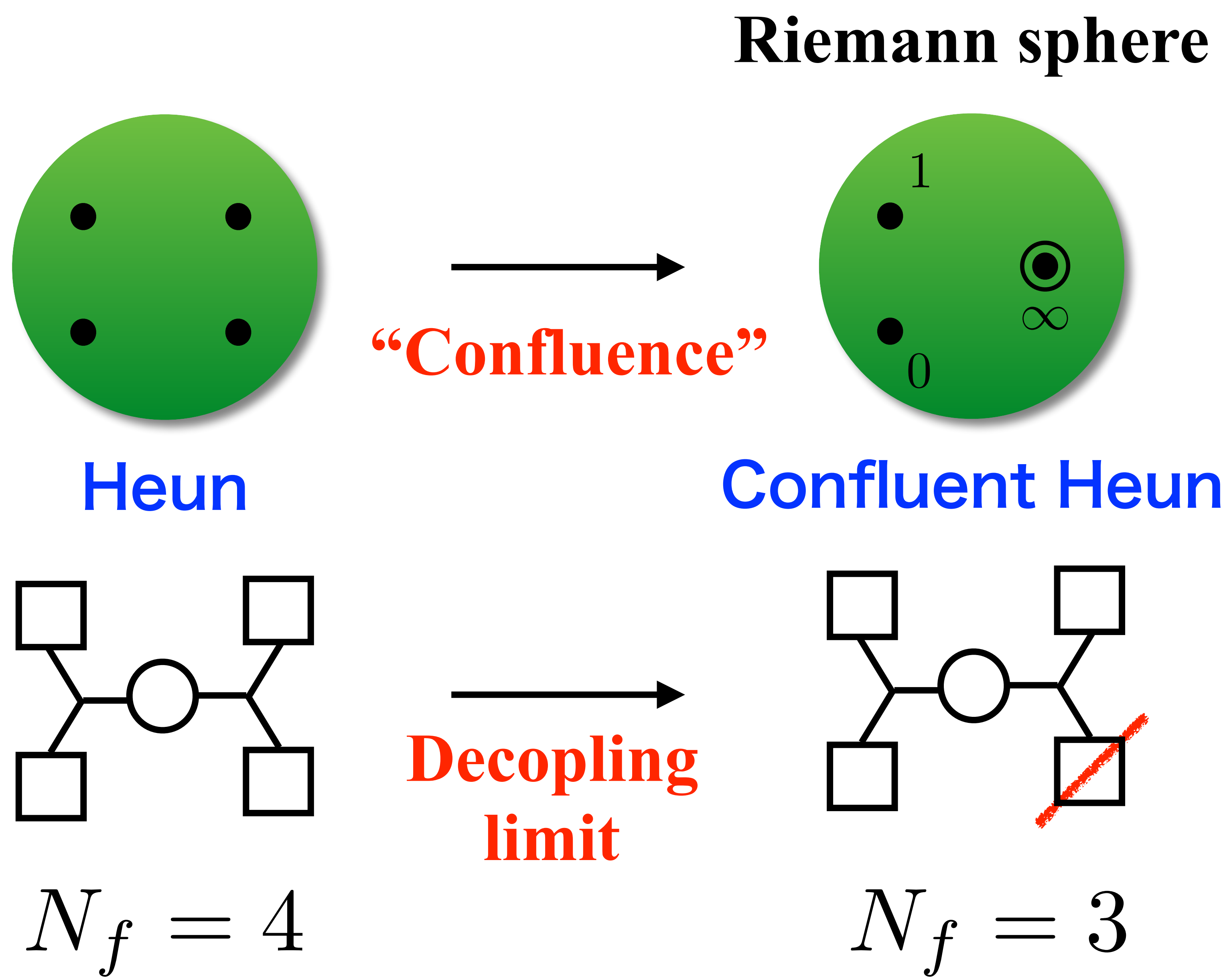
$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr}+\omega^2-V(r)\right]\phi(r)=0$$

$$f(r)=1-\frac{1}{r}, \quad V(r)=f(r)\left[\frac{l(l+1)}{r^2}-\frac{3}{r^3}\right]$$

$r=0,1$ **Regular singularities**

$r=\infty$ **Irregular singularity**

Singularities



Gaiotto 2009

Seiberg-Witten Curve

Hanany & Oz 1995

$$K(p) - \frac{\Lambda^{1/2}}{2}(e^{ix/2}K_+(p)e^{ix/2} + e^{-ix/2}K_-(p)e^{-ix/2}) = 0$$

$$K(p) = p^2 - u + \frac{\Lambda}{4}\left(p + \frac{m_1 + m_2 + m_3}{2}\right)$$

$$K_+(p) = (p + m_1)(p + m_2) \qquad K_-(p) = p + m_3$$

Symplectic form $\Omega = dp \wedge dx$

Quantum SW Curve

We regard x and p as canonical variables, and replace them noncommuting operators

$$[x, p] = i\hbar$$

$$p = -i\hbar\partial_x$$

SW curve → 2nd order ODE

Quantum SW Curve

By the following identification between SW and BH parameters,

$$\frac{2}{\sqrt{\Lambda}} e^{-ix} = r\,, \qquad \Lambda = 8i\omega$$

$$u = \frac{1}{4} + l(l+1) - 2\omega^2,$$

$$m_1 = 2 + i\omega, \quad m_2 = -2 + i\omega, \quad m_3 = i\omega$$

we get the RW equation!

Expectation

Quantization condition

$$\frac{1}{2\pi\epsilon}\oint_\gamma p_0(x)dx = n + \frac{1}{2}$$

⬇

$$\frac{\partial F_0(a)}{\partial a} = n + \frac{1}{2}$$

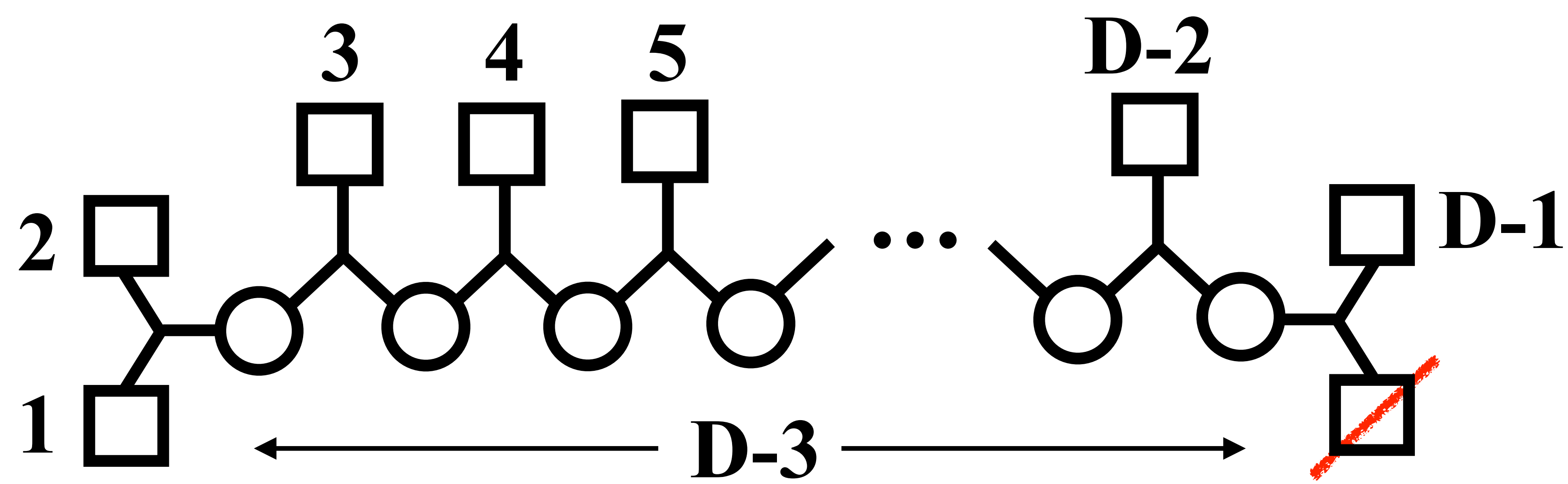
Probably all the quantum corrections are systematically included by Nekrasov's partition function

Nekrasov & Shatashvili 2009

D-dim Schwarzschild

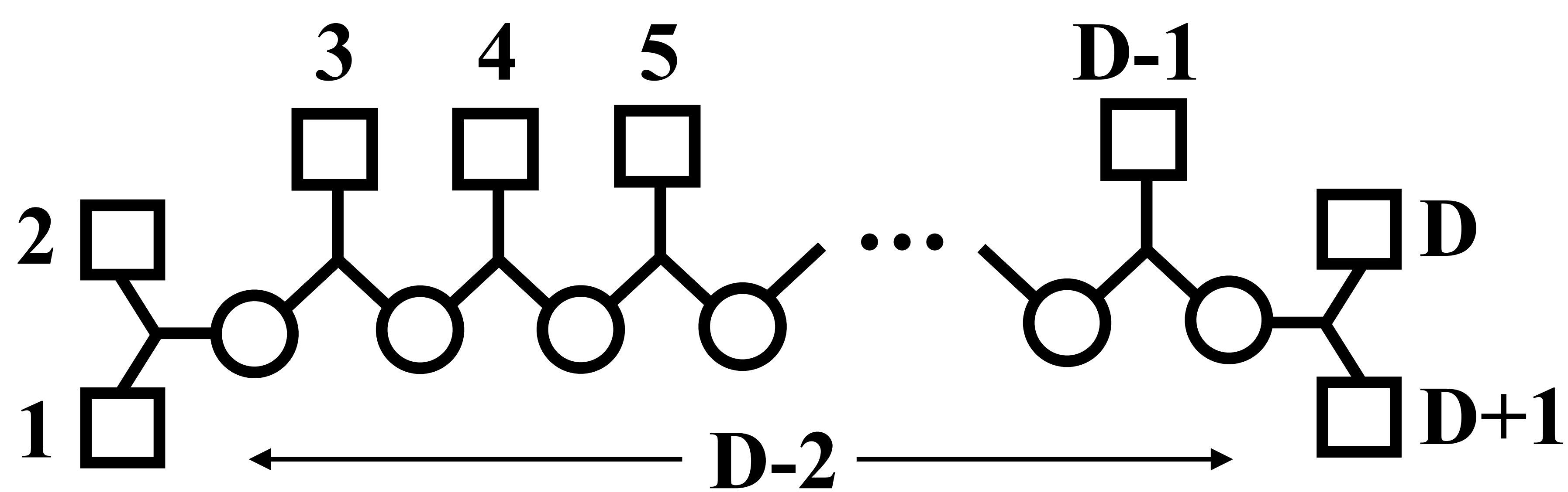
Kodama & Ishibashi 2003

Regular: D-2 Irregular: 1



D-dim Schwarzschild-AdS

Regular: D+1



Summary

- I proposed an efficient way to compute QNMs by perturbation theory
- This method is widely applicable
- "QNM/SYM" correspondence