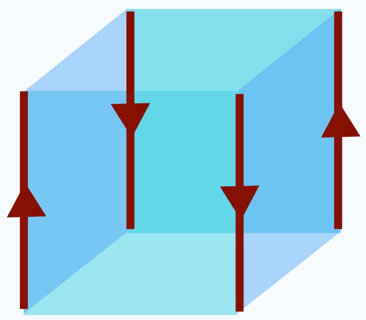
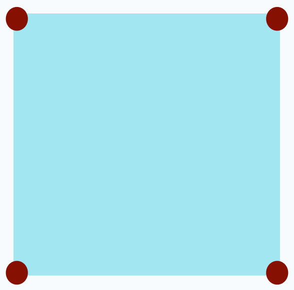
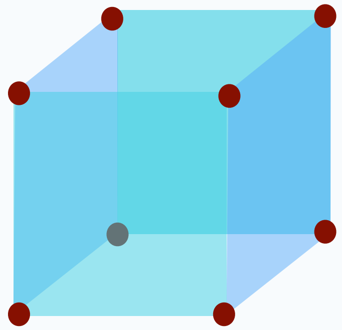
Some aspects of Higher-Order Topological Phases of Matter







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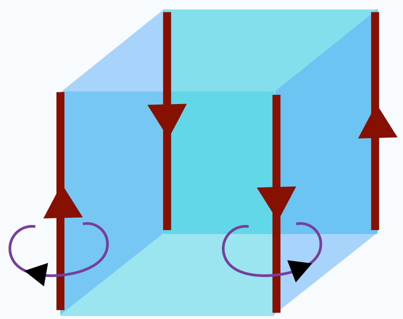
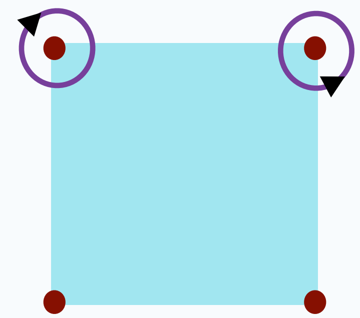
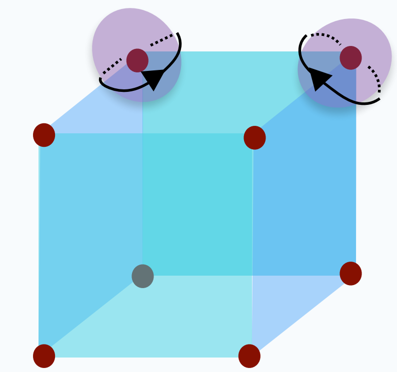
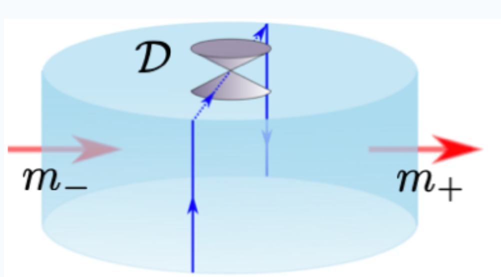
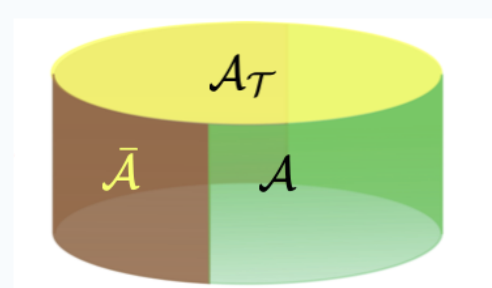
Outline

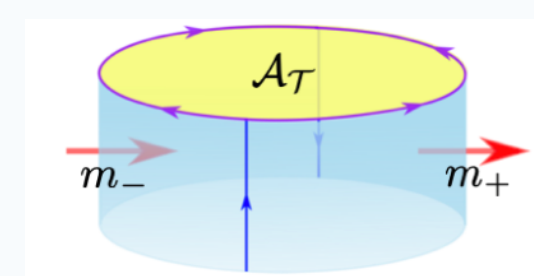
- Constructing (free-fermionic) models of HOTPs:

- Symmetric surface topological order for HOTPs:







Collaborators

Reference: arXiv 1905.11421

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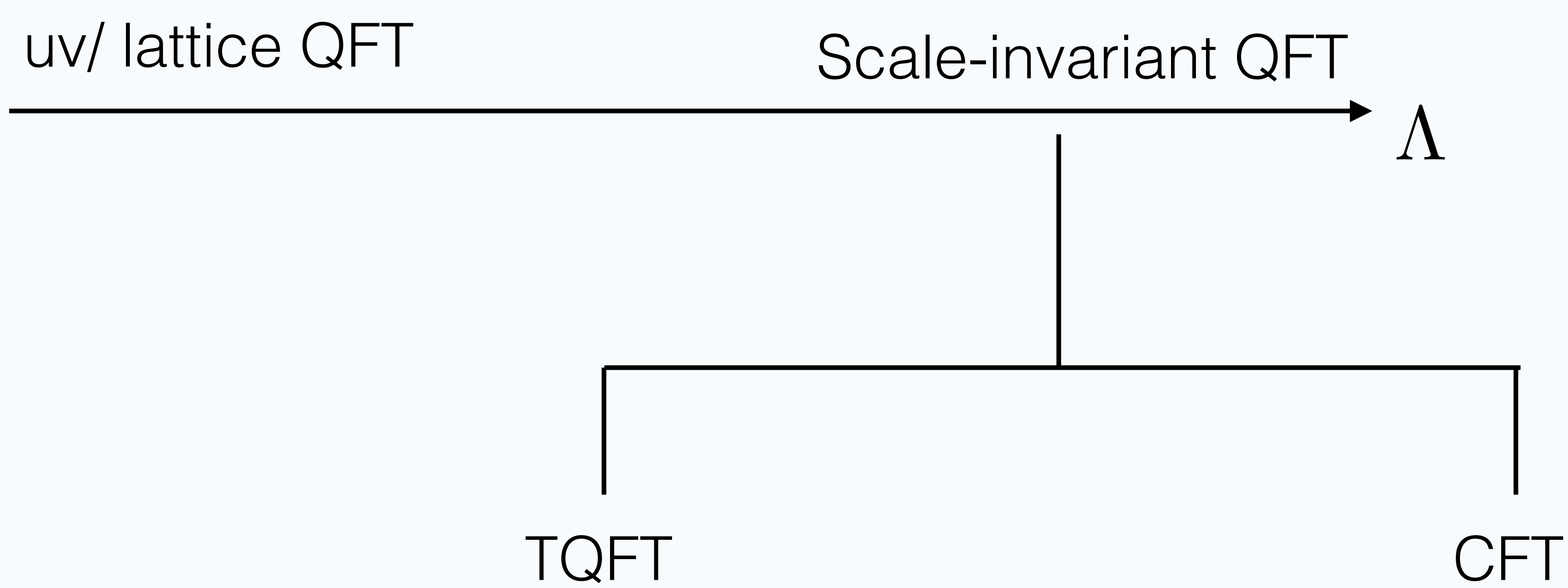
Classification of gapped phases (without crystalline symmetries)



- At long distances, gapped systems described by TQFTs.
- TQFTs can be enriched by other properties eg. global symmetries, fermionic/bosonic, long/short-range entanglement etc.
- {Classification of (★)-gapped phases}={Classification of (★) TQFTs}
- There has been progress in classifying (★) TQFTs. In comparison, much less known about "crystalline"-TQFTs.

Short-range entangled gapped phases
(without crystalline symmetries)

- Equivalence classes of gapped, (symmetric, fermionic/bosonic) systems with unique ground state.

- Described by invertible (★)- TQFTs:

$$|\mathcal{Z}(M, A, \dots)| = 1 \,.$$

- Can be `diagnosed' by topological response action:

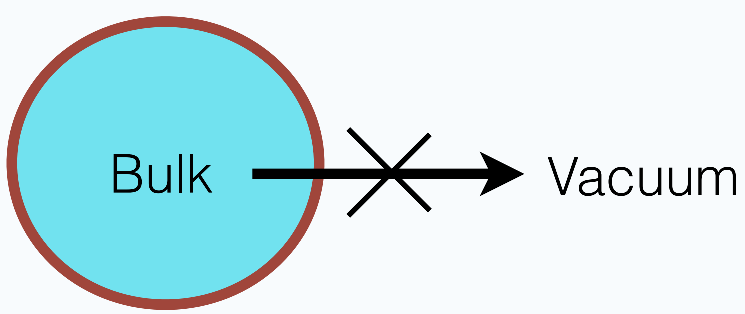
$$\mathcal{Z}(M, A, \dots) = e^{2\pi i \int_M \omega(A, \dots)} .$$

- Mathematically classified by group cohomology and (★)-cobordism group.

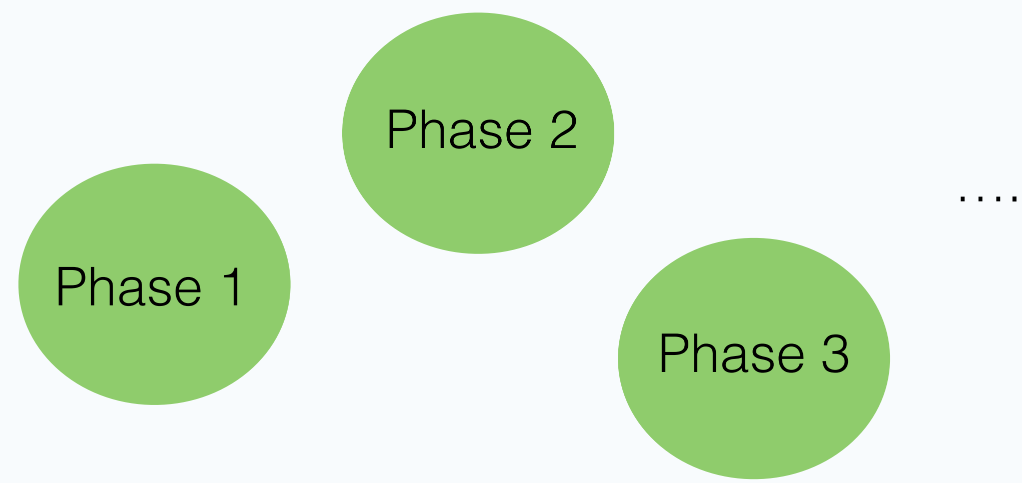
* Freed-Hopkins, Kapustin, Wen,…

Short-range entangled gapped phases
(without crystalline symmetries)

- SRE Topological Phases: Path connected components in the space of (symmetric) short-range entangled gapped systems.



(Real space)



(Space of gapped SRE symmetric models)

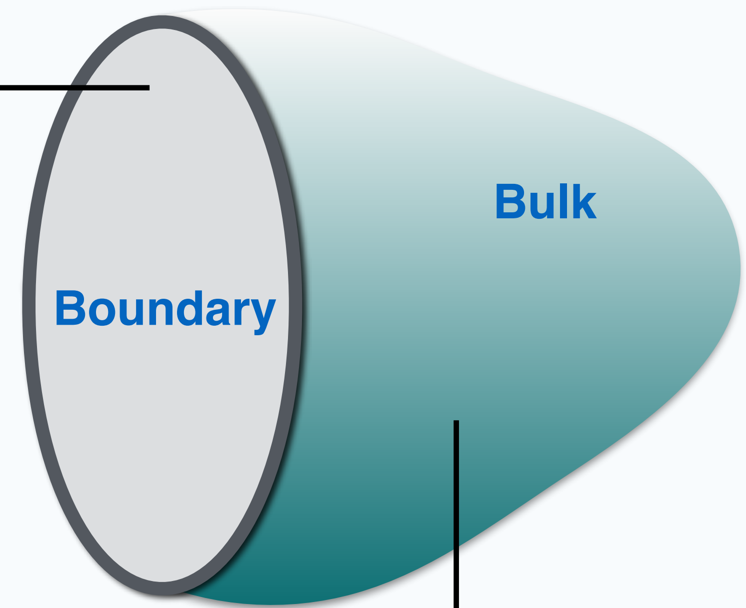
- **Path:** Adiabatic deformation or codimension-1 boundary.

- **Anomalous surface theories:** Labelled precisely by bulk topological response.

Bulk-Boundary correspondence

D-1 dimensional surface:

Anomalous i.e
global symmetry cannot be
implemented consistently.



D dimensional bulk:

Characterized by
topological response/ index.

Bulk-Boundary correspondence

- **Anomalous surface theories:** Labelled precisely by bulk topological response.

$$\frac{\mathcal{Z}_{\partial}(\partial M, A+\delta A, B+\delta B)}{\mathcal{Z}_{\partial M}(\partial M, A, B)} = e^{2\pi i \int_{\partial M} \nu(A,\delta A;B,\delta B)}.$$

where, $$\int_M [\omega(A+\delta A, B+\delta B) - \omega(A,B)] = \int_{\partial M} \nu(A,\delta A; B, \delta B).$$

- Anomaly → D-1 edge cannot be simultaneously gapped, short-range entangled and symmetric.

- Three minimal options:

  1. Symmetry broken.
  2. Symmetric and gapless.
  3. Symmetric and long-range entangled.

3+1d Electronic Topological Insulators

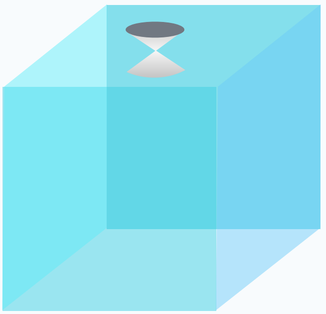
Symmetry: $\mathsf{U}(1) \rtimes \mathbb{Z}_2^{\mathsf{T}}$

Classification: $\mathbb{Z}_2$

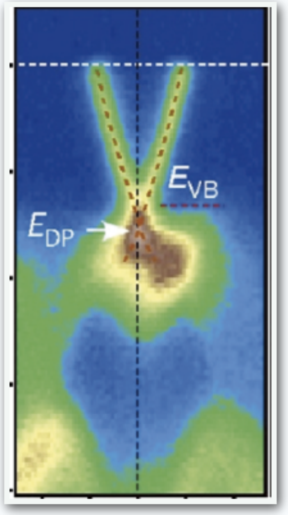
Bulk topological response: $S_{\text{em-resp}}[A] = \frac{\theta}{8\pi^2} \int_{\mathcal{M}} F \wedge F \,,$ $\qquad \theta = 0 \text{ or } \pi$

Boundary terminations for TIs:

(I) Gapless:



Single Dirac Cone

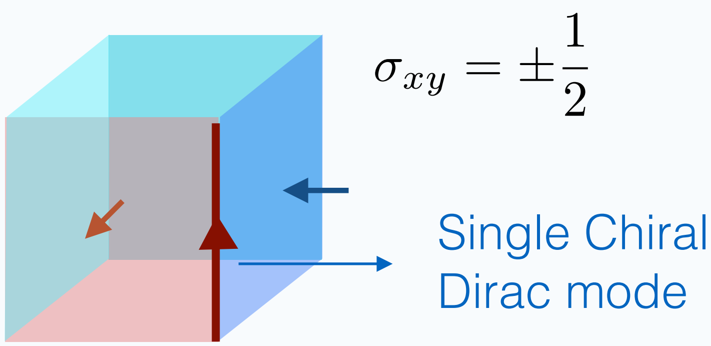


Partition function of a single gapless Dirac fermion cannot be made both real and gauge invariant without including a `bulk' contribution!

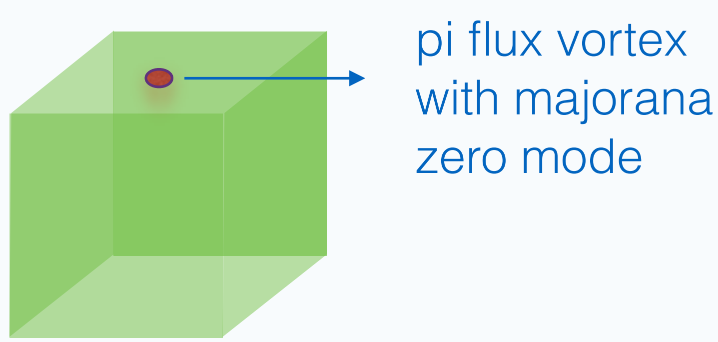
* Taylor Hughes, X-L Qi, SC Zhang (2008); Witten RMP (2016); Alvarez-Gaume Della Pietra, Moore (1985)

Boundary terminations for TIs (contd.):

(II) Symmetry broken:



Time reversal breaking (ferromagnetic coating)



U(1) breaking (s-wave pairing)

(III) Anomalous Surface Topological Order (STO):

➡ By physical requirements the topological order needs to have:

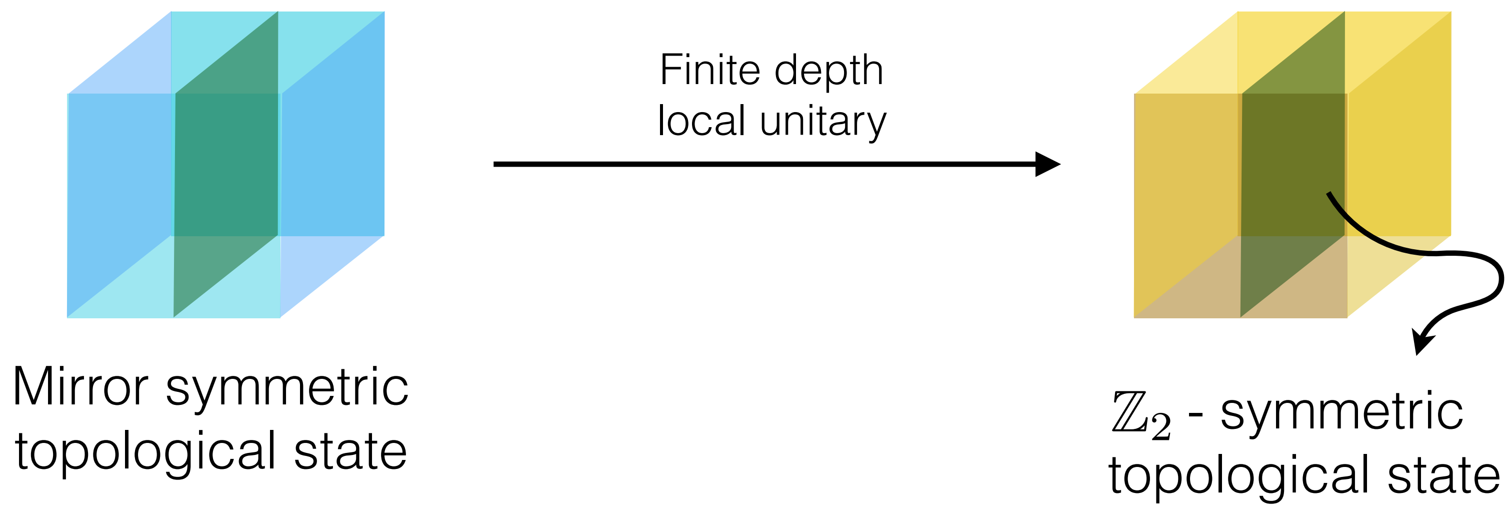
- A local fermion.
- Chiral central charge $c_- = 1/2.$
- Hall conductance $\sigma_{xy} = 1/2.$

➡ Minimal realization known as T-Pfaffian: $\text{T-Pfaffian} = \left[\mathrm{U}(1)_8 \times \overline{\text{Ising}}\right] / \mathbb{Z}_2$

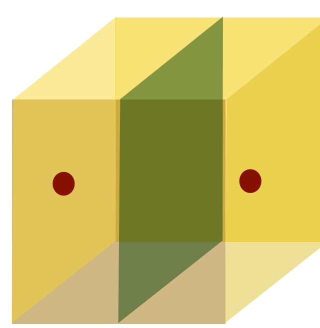
* **Fu-Kane, Bonderson-Nayak-Qi; Chen-Fidkoski-Vishwanath; Metlitski et al; Tachikawa-Yonekura**

Crystalline topological phases; General strategy:

- Reduce real space wavefunction into elementary building (cells) with only onsite symmetry.



- Can always add blocks related by spatial symmetry.



- Classification of (spatial)-symmetric (cell)-decompositions.
  (e.g Generalized Homology theory)

* Shiozaki-Xiong-Gomi, Sao-Huang-Fu-Hermele, Else-thorngren, Freed-Hopkins

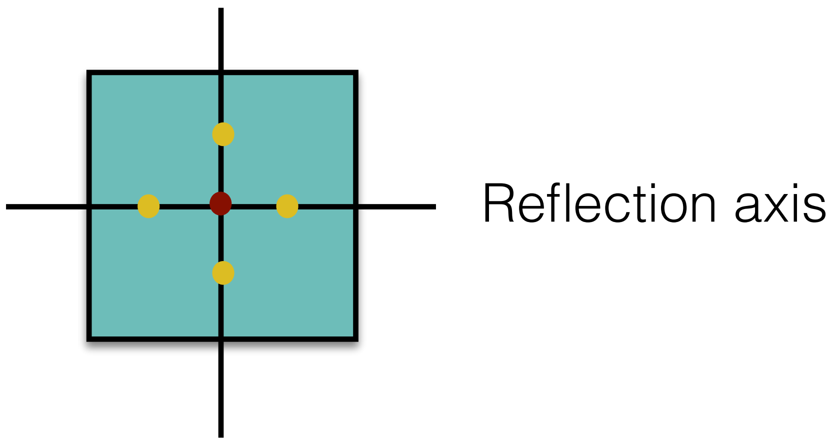
Crystalline topological phases; A 2D example:

- Point group D_4:

A single 0-cell at the origin with $\mathbb{Z}_2 \times \mathbb{Z}_2$ onsite symmetry.

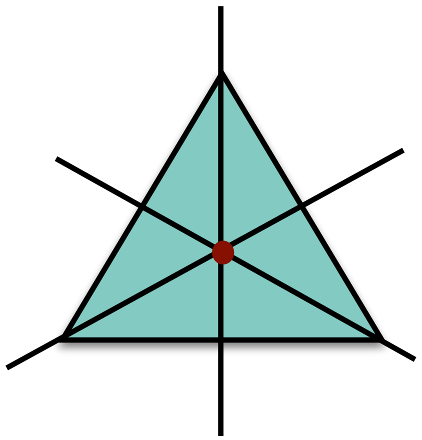
No non-trivial 1-cells.

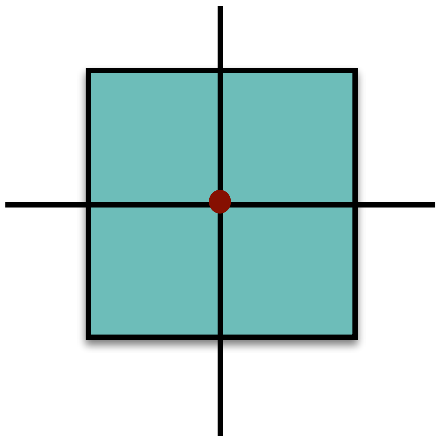
$H^1(\mathbb{D}_4, \mathsf{U}(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Classification is $\mathbb{Z}_2 \times \mathbb{Z}_2$.



- Point group C_n:

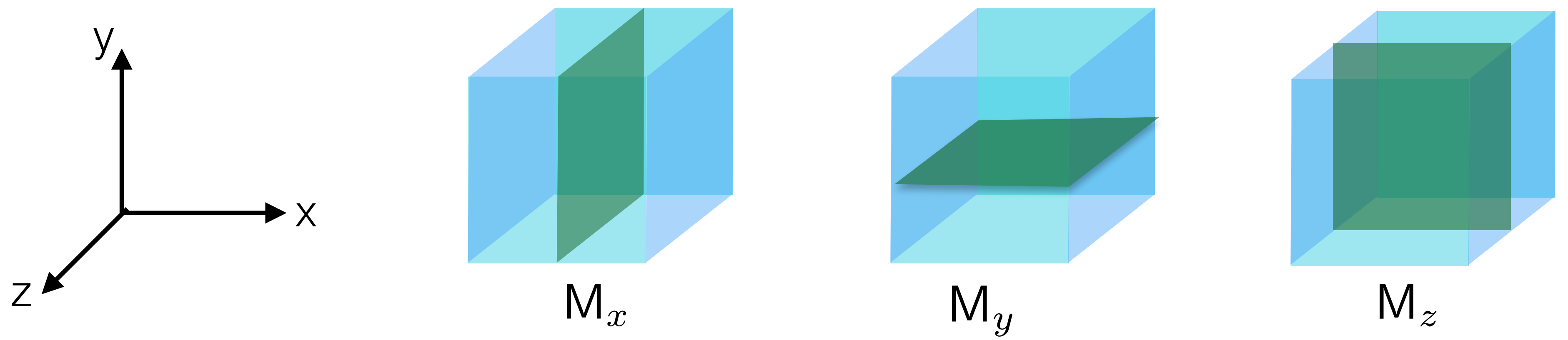
$H^1(\mathbb{Z}_n, \mathsf{U}(1)) = \mathbb{Z}_n \Rightarrow$ classification is $\mathbb{Z}_n$.





Crystalline topological phases; A 3D example:

- Point group D_{2h} : Generated by three mutually perpendicular mirrors, $\mathsf{M}_x, \mathsf{M}_y, \mathsf{M}_z$.



- A single 0-cell at the origin classified by $H^1(\mathsf{D}_{2h}, \mathsf{U}(1)) = \mathbb{Z}_2^3$.

- Three (x two) 1-cells along the coordinate axis, each classified by:

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathsf{U}(1)) = \mathbb{Z}_2 \qquad \text{(Haldane chain)}$$

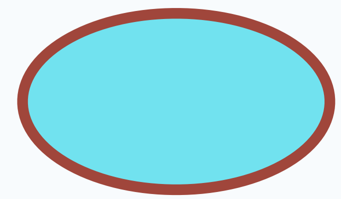
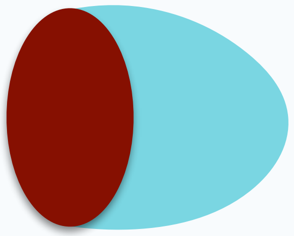
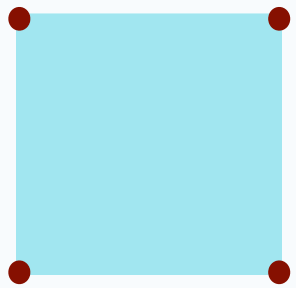
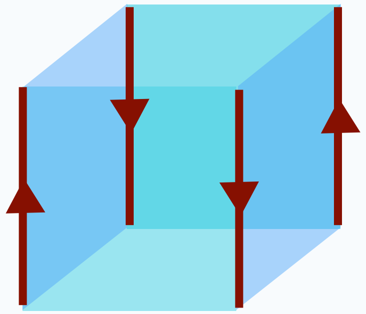
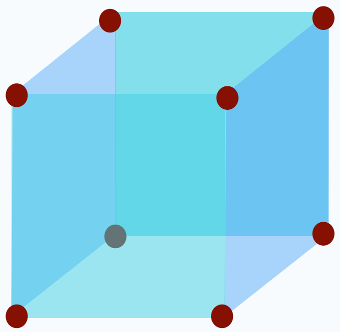
- Three (x four) 2-cells perpendicular to coordinate axis, each classified by:

$$H^3(\mathbb{Z}_2, \mathsf{U}(1)) = \mathbb{Z}_2$$

- Classification of crystalline SPTs with D_{2h} symmetry is $\mathbb{Z}_2^9$.

* Shiozaki-Xiong-Gomi, Sao-Huang-Fu-Hermele, Else-thorngren, Freed-Hopkins

Higher-Order Topological Phases: Bulk Boundary correspondence

p^{th} Order Topological phases	D=2	D=3
• 1st Order Topological Phases: eg. TIs, TSCs, SPTs		
• 2nd Order Topological Phases*:		
• 3rd Order Topological Phases* :		

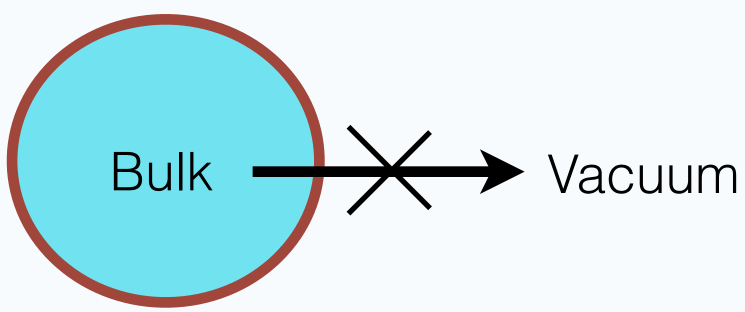
p>1 require spatial symmetries that map one surface to another.

-Gapless

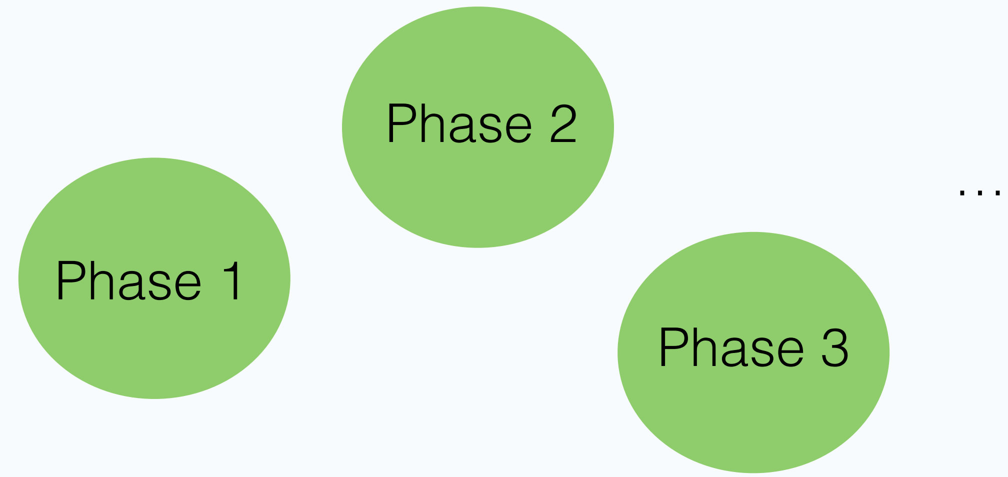
-Gapped

* Benalcazar et al; Schindler et al; Brouwer et al; Khalaf; …

- 1st Order Topological Phases: Path connected components in the space of (symmetric) short-range entangled gapped systems.

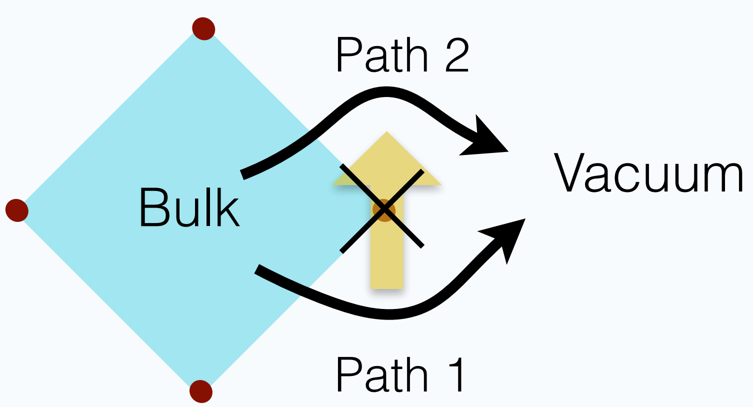


(Real space)



(Space of gapped SRE symmetric models)

- 2nd Order Topological Phases: Obstructed paths between paths. First (equivariant) Homotopy.



(Real space)

Path 2= $g \triangleright$ (Path 1), i.e related by spatial symmetry.

*** Shiozaki-Xiong-Gomi, Gaiotto-Freyd**

Free-fermion Higher-Order Topological Phases :

- A simple construction: use spatial symmetries to localize topological defects on high-symmetry corners or hinges.

- Known classification of point and line defects in Altland-Zirnbauer classes.

Classification of point defects

AZ class	Type	Classification
AIII	Dirac zero-mode	$\mathbb{Z}$
BDI	Majorana zero-mode	$\mathbb{Z}$
CII	Chiral Maj. Kramers doublet	$2\mathbb{Z}$
D	Majorana zero-mode	$\mathbb{Z}_2$
DIII	Majorana Kramers doublet	$\mathbb{Z}_2$

Classification of line defects

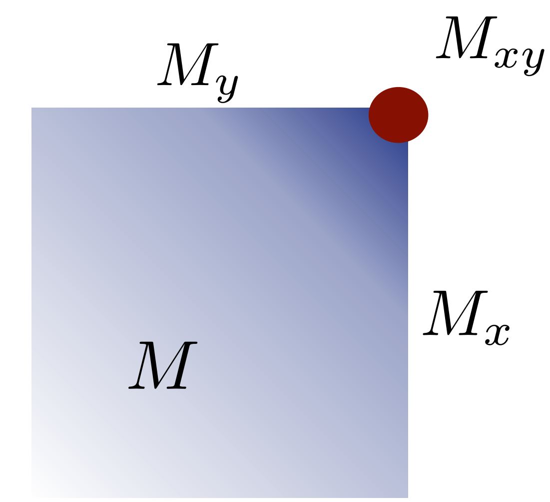
AZ class	Type	Classification
A	Chiral Dirac	$\mathbb{Z}$
D	Chiral Majorana	$\mathbb{Z}$
DIII	Helical Majorana	$\mathbb{Z}_2$
AII	Helical Dirac	$\mathbb{Z}_2$
C	Chiral Dirac	$2\mathbb{Z}$

*** Teo-Kane, …**

Free-fermion Higher-Order Topological Phases :

$$S = \int_{M\times\mathbb{R}_\tau} \Psi^\dagger \left[i\partial_\tau - i\sum_{\mathrm{i}} \alpha^i \partial_i - \mathcal{M}(x) \right] \Psi.$$

$\mathcal{M}(x)$ → Symmetric Dirac Mass



Boundary conditions (codimension-1):

$$i\Psi^\dagger \alpha^i \delta\Psi\Big|_{M_i\times\mathbb{R}_\tau} = 0 \qquad \Rightarrow \Psi\Big|_{M_i\times\mathbb{R}_\tau} \in \mathrm{im}(\mathcal{P}_i)$$

where

- $\mathcal{P}_i \alpha^i \mathcal{P}_i = 0.$
- $[\mathrm{g}, \mathcal{P}_i] = 0$ for all $\mathrm{g} \in \mathsf{G}$ such that $g \triangleright M_i = M_i.$
- $\mathrm{g}\mathcal{P}_x\mathrm{g}^{-1} = \mathcal{P}_y$ if $g \triangleright M_x = M_y.$

Boundary conditions (codimension-2): $\Psi\Big|_{M_{xy}\times\mathbb{R}_\tau} \in \mathrm{im}(\mathcal{P}_x) \cap \mathrm{im}(\mathcal{P}_y).$

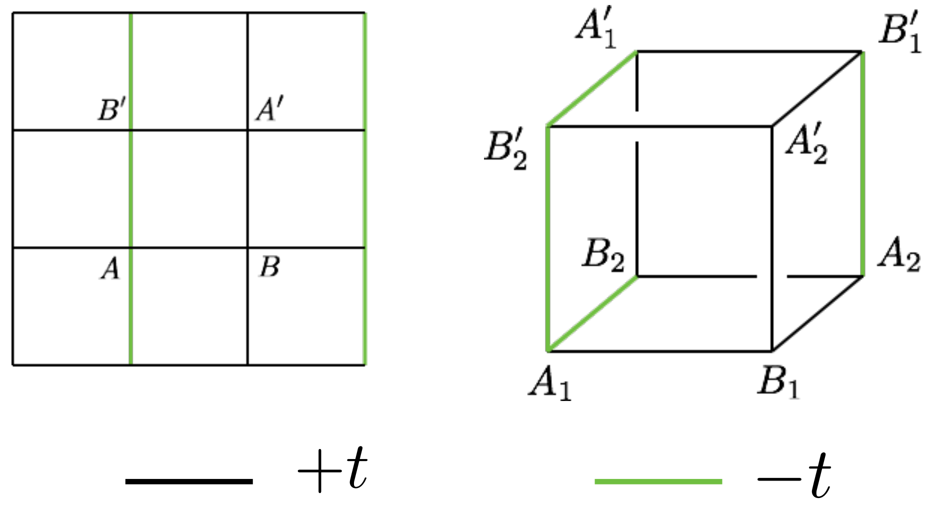
Observables can be restricted to various boundaries and corners, for example $\mathcal{H}_{M_i} = \mathcal{P}_i \mathcal{H}_M \mathcal{P}_i.$

Classification of topological defects in Dirac models → Classification of hinge and corner modes.

*** Po-Watanabe-Khalaf-Vishwanath, Teo-Kane, …** *** In preparation: AT-Grushin-Hughes-Neupert**

A simple example:

- Consider the π-flux model in 2D and 3D (Class AIII).



$$H_0 = -\sum_{\langle ss' \rangle} \left[t_{ss'} c_s^\dagger c_{s'} + \text{h.c} \right] = \sum_k c_k^\dagger H_k^0 c_k$$

where,

$$H_k^0 = -2t \sum_{i=1}^{\text{D}} \cos(k_i) \Gamma^i.$$

Which describes a Dirac semi-metal with chiral (sublattice) symmetry.

- The Dirac point can be gapped by Valence-bond mass terms:

$$\triangle H_k = -t' \sum_{i=1}^{\text{D}} \sin(k_i) \Gamma^{i+\text{D}}.$$

- Low energy Dirac model:

$$H_k \simeq \sum_{i=1}^{\text{D}} \left[k_i \Gamma^i + \mathsf{m}_i \Gamma^{i+\text{D}} \right],$$

with chiral symmetry $\left\{ H_k, \Gamma^{2\text{D}+1} \right\} = 0.$

- Low energy Dirac model: $H_k \simeq \sum_{i=1}^{\mathrm{D}} \left[k_i \Gamma^i + \mathsf{m}_i \Gamma^{i+\mathrm{D}}\right]$, with chiral symmetry $\{H_k, \Gamma^{2\mathrm{D}+1}\} = 0$.
- $\mathcal{M} := \sum_i \mathsf{m}_i \Gamma^{i+\mathrm{D}}$ transforms as an $\mathsf{O}(\mathrm{D})$ vector.
- What spatial symmetries can localize defects at high-symmetry corners?

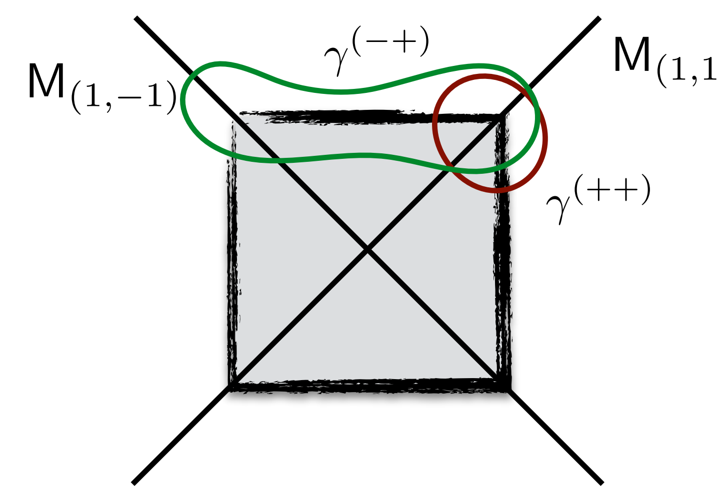
D=2:

- Two anti commuting mirrors: $\mathsf{M}_{(1,\pm1)} : \Psi(x,y,t) \to \widehat{\mathsf{M}}_{(1,\pm1)} \Psi(\pm y, \pm x, t)$

  where $\widehat{\mathsf{M}}_{(1,\pm1)} : (\Gamma^3, \Gamma^4) \to (\pm\Gamma^4, \pm\Gamma^3)$.

- Let $\mathcal{M}(x) = \mathsf{m}\left[\cos(\Theta)\Gamma^3 + \sin(\Theta)\Gamma^4\right]$,

$$\mathsf{N}_{\mathrm{w}}^{(++)} := \frac{1}{2\pi} \int_{\gamma^{(++)}} \mathrm{d}\Theta = (2n+1) \in \mathbb{Z}_{\mathrm{odd}}$$



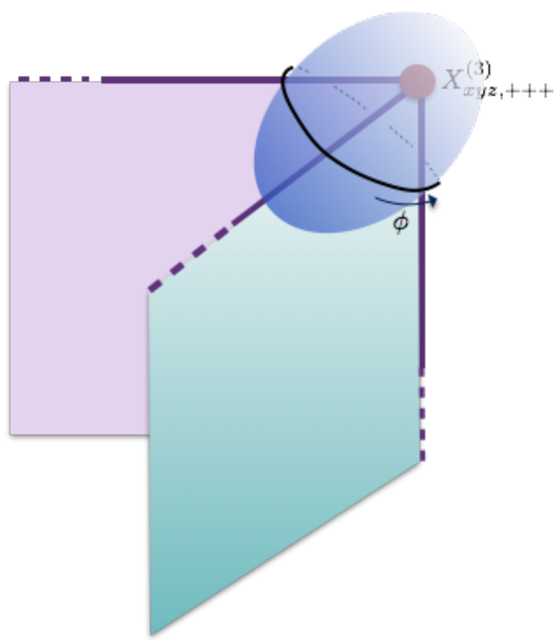
- Alternating winding number (charges): $\mathsf{N}_{\mathrm{w}}^{(++)} + \mathsf{N}_{\mathrm{w}}^{(-+)} = 0$.
- Topological response theory: $S_{\mathrm{eff}}[A] = \frac{1}{4\pi} \int \mathrm{d}\Theta \wedge \mathrm{d}A$. ➡ Vortex traps 1/2 quantum charge.

D=3:

- Space of chiral symmetric masses is $\mathbb{R}^3 - \{0\} \simeq S^2$.

- Spatial symmetries can localize a topological defect at the corners.

- Example: C_3 -rotation about the (111)-body diagonal such that $\widehat{\mathsf{C}}_3 : \begin{pmatrix} \Gamma^4 \\ \Gamma^5 \\ \Gamma^6 \end{pmatrix} \longmapsto \begin{pmatrix} \Gamma^5 \\ \Gamma^6 \\ \Gamma^4 \end{pmatrix}$.

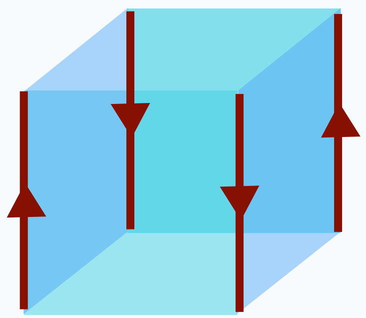


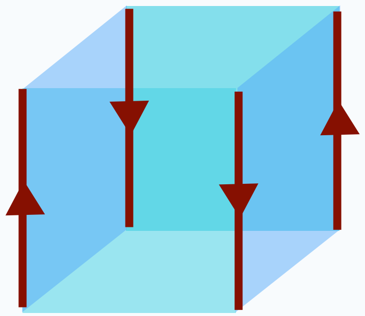
- Non-trivial winding number at the corners $\mathsf{N}_w^{(+++)} = 3n+1$.

- Topological response action: $S_{\text{resp}} = \dfrac{q}{\text{Vol}(S^2)} \displaystyle\int \mathrm{d}\mu_{S^2} \wedge \mathrm{d}A$, where q=1/3.

- More generally, one can follow a similar strategy to construct models within various Altland-Zirnbauer classes.

Surface Topological Order for Hinge Higher-Order Topological Phases





What are the other possible symmetric surface terminations of hinge HOTPs?

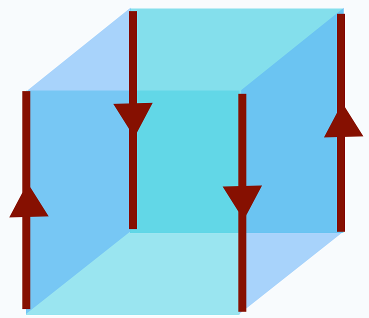
$\mathsf{C}_{2n}\mathcal{T}$ symmetric second-order topological phases:

Higher-order phase	Symmetry	Chiral Hinge mode	Surface pasting
Fermionic HOTI	$\mathsf{C}_{2n}\mathcal{T} \ltimes \mathsf{U}(1)$	Dirac $q = 1;\ c_- = 1$	IQHE
Fermionic HOTSC	$\mathsf{C}_{2n}\mathcal{T} \times \mathbb{Z}_2^f$	Majorana $c_- = 1/2$	$\mathsf{p} \pm i\mathsf{p}$
Bosonic HOSPT	$\mathsf{C}_{2n}\mathcal{T}$	Bosonic $c_- = 8$	$\mathbb{E}_8$ phase

$\mathbb{Z}_2$ classified.

We ask the following questions:

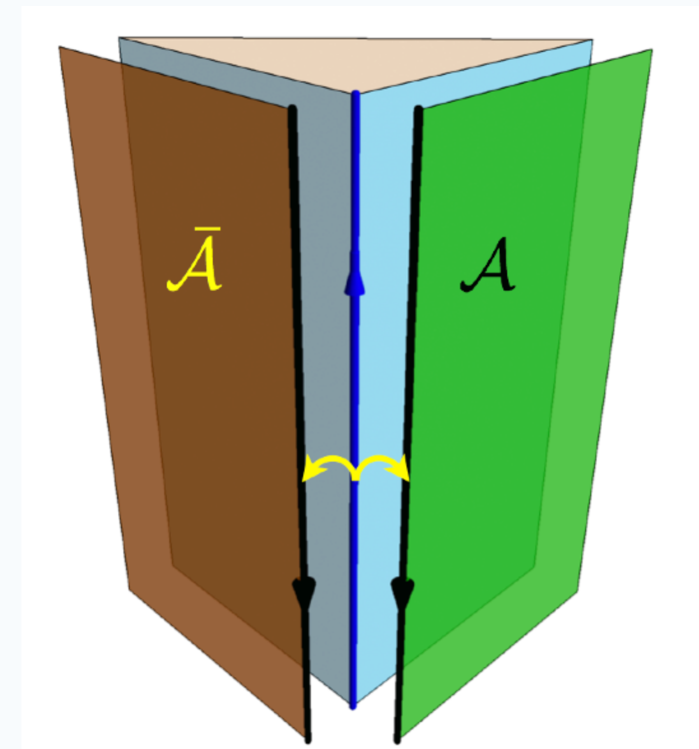
- What are the possible surface terminations for HOTPs?
- Is there a generalized notion of anomalies for HOTPs?



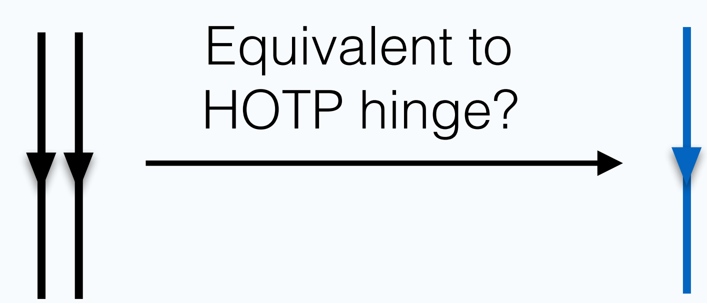
In this work, we focus on hinge HOTPs.

General strategy to “unhinge” HOTPs

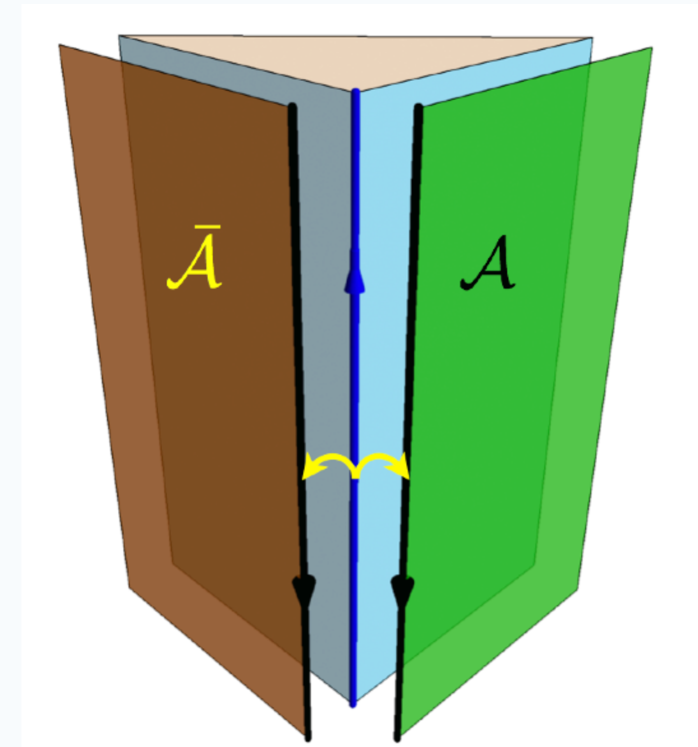
- Start from hinge HOTP.



- Introduce $C_{2n}\mathcal{T}$ symmetric topological order on the surface.
- Properties of surface topological order (STO) can be read-off from properties of hinge it needs to absorb.
- Look for symmetric gapping channels.
(Haldane gapping criteria and anyon condensation)

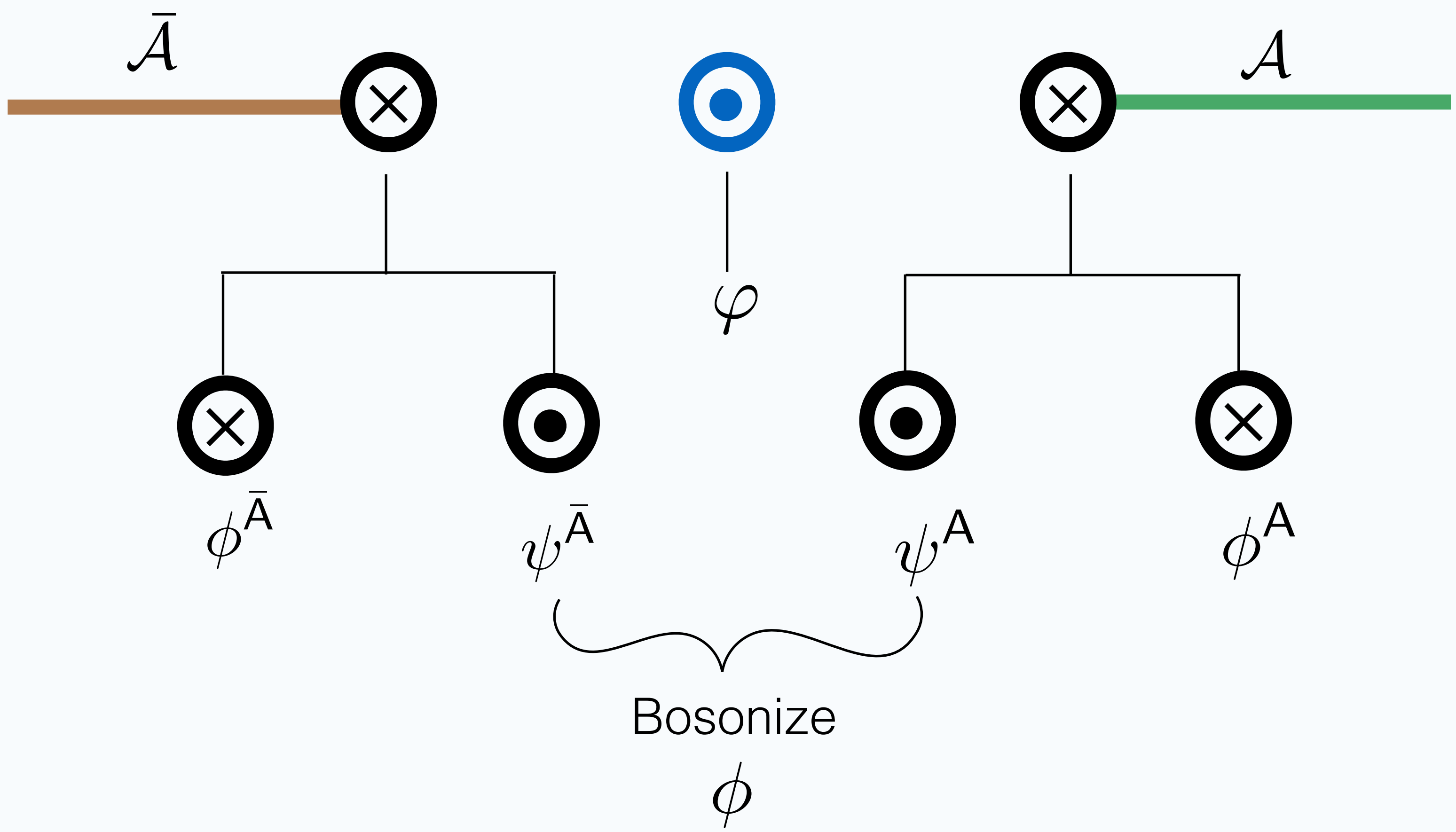


Unhinging the hinge HOTI



- Properties of $\mathcal{A}$:
  - ➡ Chiral central charge, $c_{-} = 1/2.$ Therefore non-abelian!
  - ➡ Hall conductance, $\sigma_{xy} = 1/2.$
- Same constraints as STO for TI, therefore we can use T-Pfaffian.

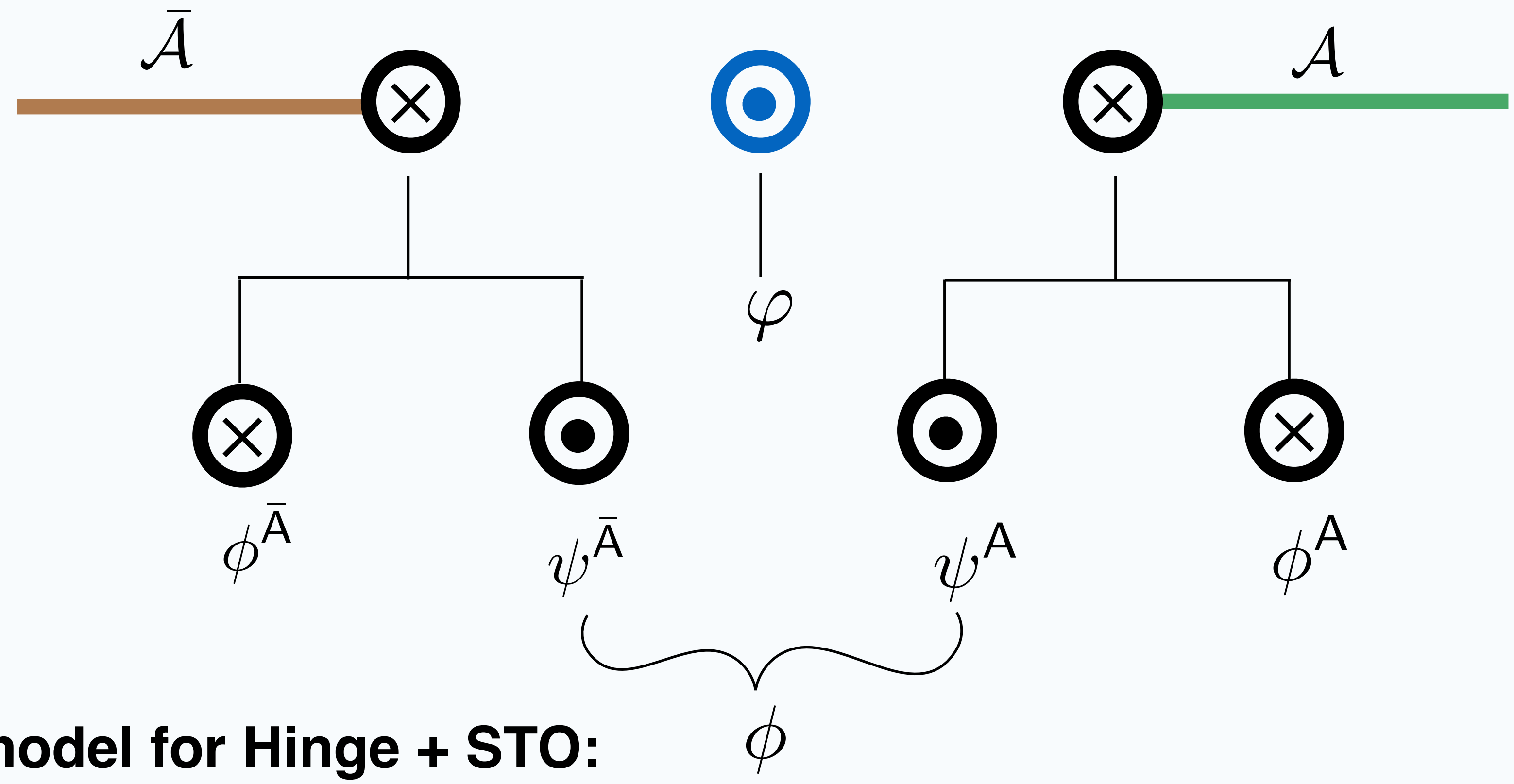
$$\text{T-Pfaffian} = \left[\mathsf{U}(1)_8 \times \overline{\text{Ising}}\right] / \mathbb{Z}_2$$



	$\lvert c \rvert$	$\lvert \sigma_{xy} \rvert$
φ	1	1
$\phi^{A/\bar{A}}$	1	1/2
$\psi^{A/\bar{A}}$	1/2	0
ϕ	1	0

- Effective model for Hinge + STO, multicomponent chiral Tomonaga-Luttinger liquid.

Unhinging the hinge HOTI



- **Effective model for Hinge + STO:**

$$\mathcal{L}_{\text{Hinge}} = \frac{1}{4\pi}\partial_x \Phi^T K \partial_t \Phi - \frac{V}{4\pi}\partial_x \Phi^T \partial_x \Phi \;+ \sum_I \lambda_I \cos[\ell_I^T \Phi + \alpha_I]$$

$$\Phi^T = [\phi, \phi^{\mathsf{A}}, \phi^{\bar{\mathsf{A}}}, \varphi]; \quad K = \text{diag}[1, -2, -2, 1]; \qquad q = [0, 1, 1, 1].$$

- **Haldane gapping criteria:**
  - ➡ Condensability: $\ell_I^T K^{-1} \ell_I = 0.$
  - ➡ Mutual locality: $\ell_I^T K^{-1} \ell_J = 0.$
  - ➡ No Spontaneous symmetry breaking: $\ell_I^T K^{-1} q = 0.$
  - ➡ Non-fractional: $\ell_i \in K\mathbb{Z}^4$

$$\ell_1 = (0, 4, 4, 4)^T$$

$$\ell_2 = (2, 2, -2, 0)^T$$

Unhinging other hinge HOTPs….

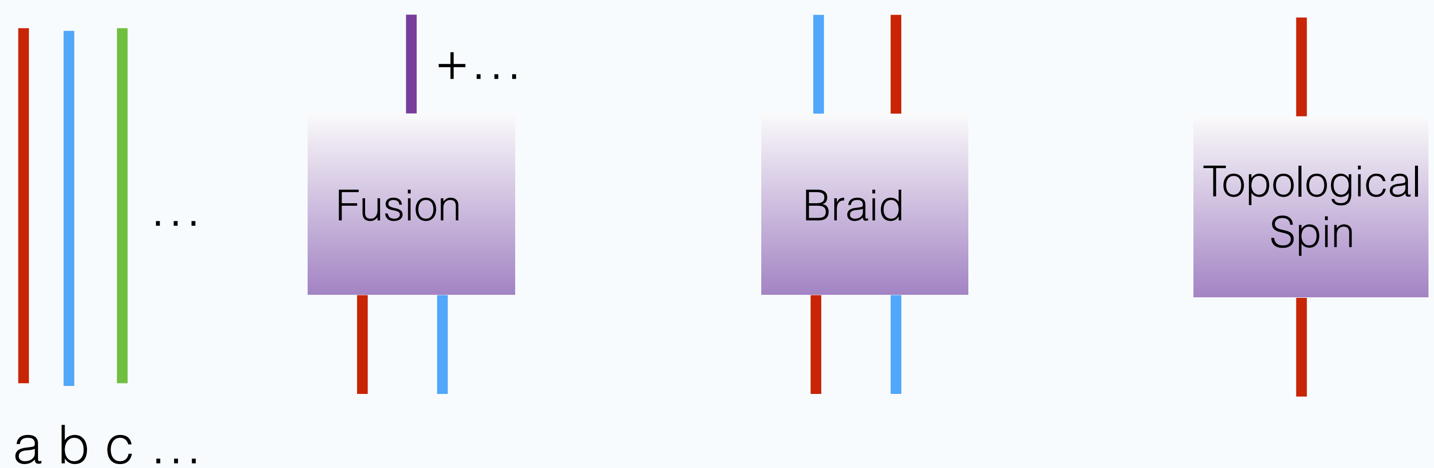
- Reformulation in terms of chiral Tomonaga-Luttinger liquid isn't always possible.

- Can use CFT techniques such as **edge condensation**.

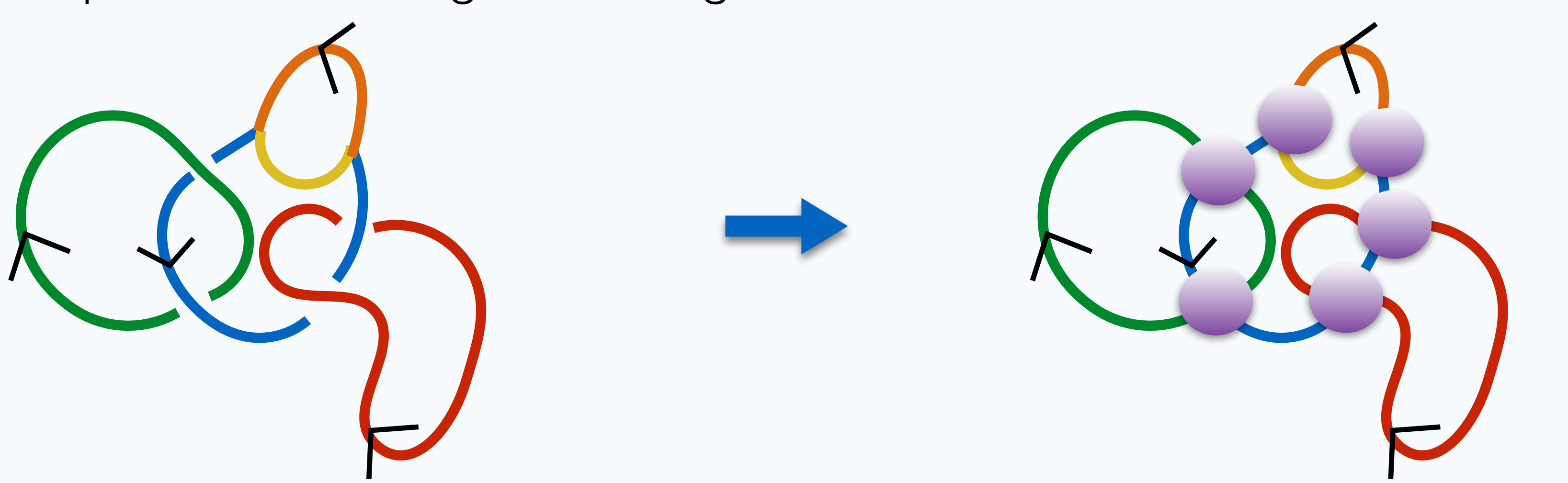
Algebraic formulation of $\mathcal{A}$ as a **Modular Tensor Category (MTC)**:

- ➡ **Particle types**, a,b,c…i.e **Anyon types** in the bulk TQFT and **conformal blocks** in edge CFT.
- ➡ **Fusion rules** a x b = ….
- ➡ **Braiding phases** and **topological spins** (`S' and `T' matrices).

} Several consistency conditions between this data.



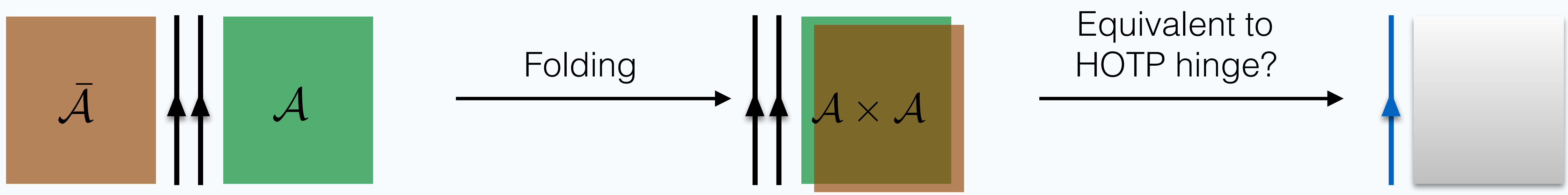
➡ Can compute Ribbon diagrams using MTC:



Ribbon diagram MTC computation sketch

* **Moore-Seiberg**

Unhinging via edge condensation



Edge condensation between $\mathcal{A}$ and $\bar{\mathcal{A}}$. $\simeq$ Anyon condensation in $\mathcal{A} \times \mathcal{A}$.

Anyon/edge condensation: Theoretical tool to study possible phase transitions.
More powerful than K-matrix Tomonaga-Luttinger liquid approach.

Procedure:

- Identify a set $\mathcal{B}$ of bosonic mutually local anyons that may condense.
- Two anyons a_1 and a_2 identified if $a_1 \in \mathcal{B} \times a_2$.
- An anyon a splits if $a \in \mathcal{B} \times a$.
- Anyons that braid non-trivially with $\mathcal{B}$ get confined.

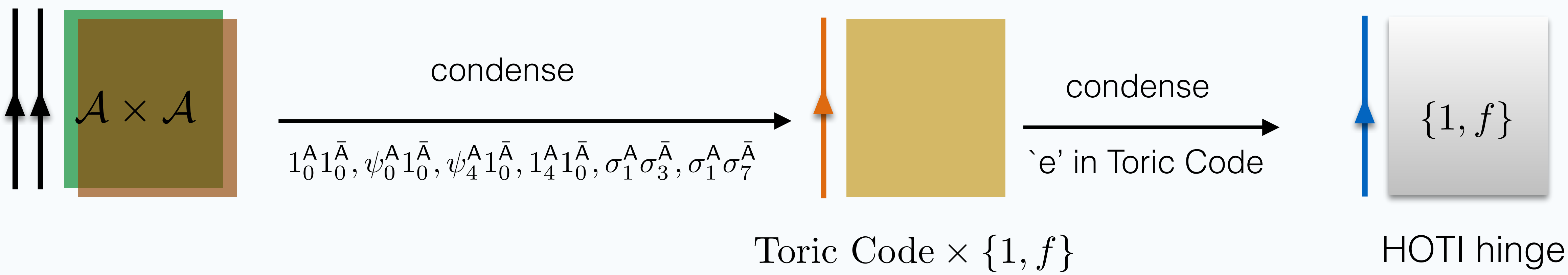
* **Bais-Slingerland, Kong-Wen, Neupert et al, …**

Unhinging the HOTI via edge condensation

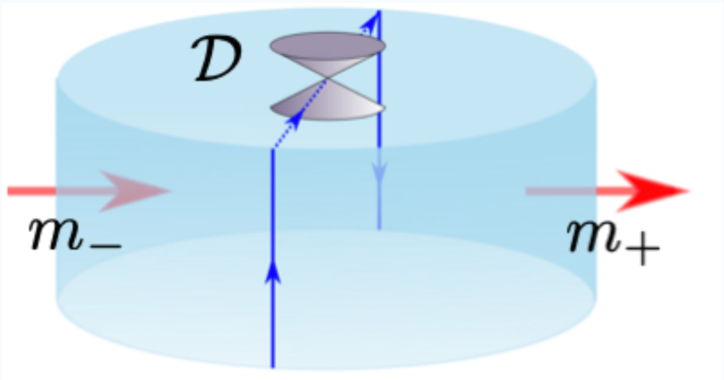
$\mathcal{A} \equiv \text{T}-\text{Pfaffian}$

$$\{1^{\mathsf{A}}_j, \psi^{\mathsf{A}}_j, \sigma^{\mathsf{A}}_j\} \times \{1^{\bar{\mathsf{A}}}_j, \psi^{\bar{\mathsf{A}}}_j, \sigma^{\bar{\mathsf{A}}}_j\} \longrightarrow \text{anyons in } \mathcal{A} \times \mathcal{A}$$

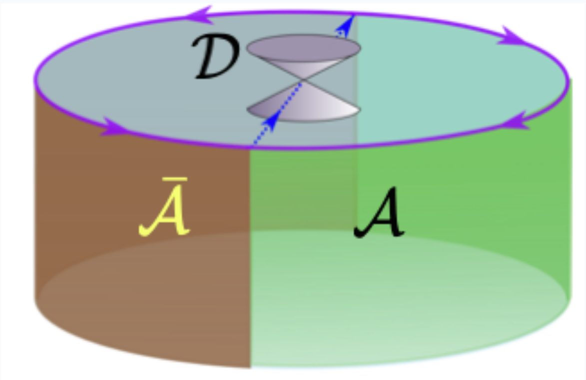
j=0,2,4,6 **j=1,3,5,7**



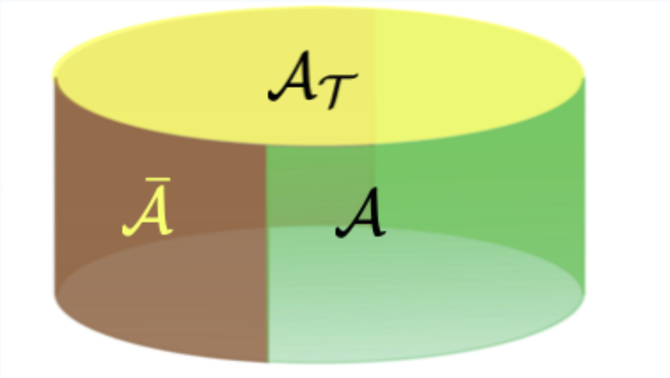
Other surface terminations



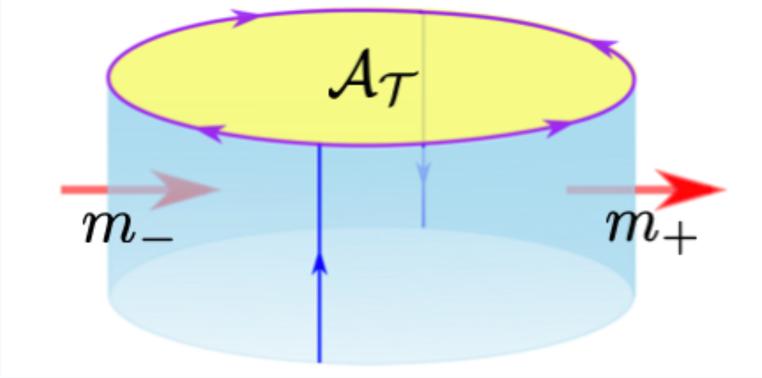
HOTI surface with no topological order.



Only side-surfaces gapped.



Completely gapped surface.



Higher-Order surface as a **beam splitter**

Anomaly: two equivalent statements

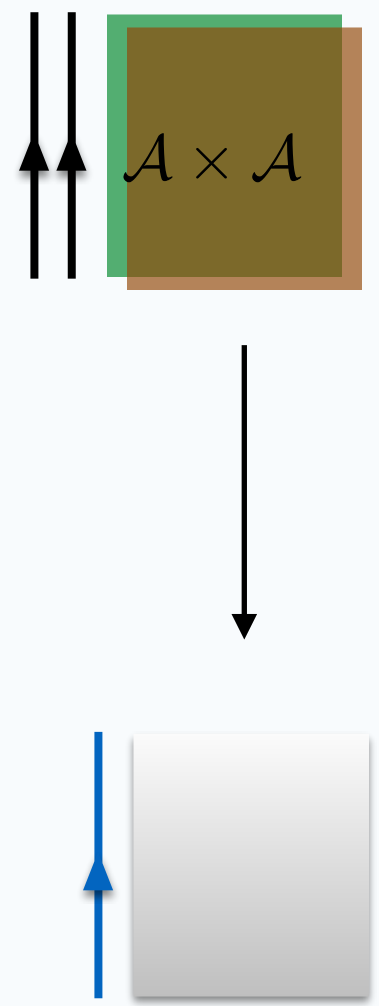
- The surface of a 3D HOTI cannot be gapped while preserving symmetry unless we introduce STO.
- The hinge pattern cannot be introduced in a purely 2D "hollow" theory, but will always involve fractionalized quasiparticles.

Unhinging the HOTSC and HOSPT

HOTSC

$\mathcal{A} \equiv \mathsf{SO}(3)_6$ anyon model

Single Majorana mode



Bosonic HOSPT

$\mathcal{A} \equiv \mathsf{SO}(8)_1$ anyon model

c=8 chiral Boson

Unhinging the HOTSC

- HOTSc hinge mode: Single chiral Majorana (c=1/2).

$$\mathcal{A} \equiv \mathsf{SO}(3)_6 \text{ anyon model} \subset \mathsf{SU}(2)_6 \text{ anyon model}$$

- $\mathsf{SO}(3)_6$ Is obtained from $\mathsf{SU}(2)_6$ by discarding the half-integral representation.

- Anyon types: j={0,1,2,3}. `3' is a local fermion!

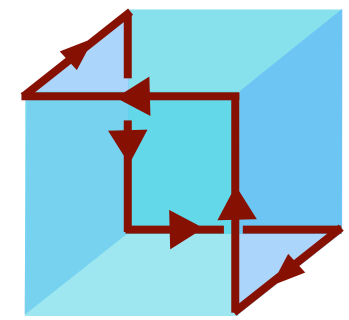
- Can condense $(00) \oplus (12) \oplus (21) \oplus (33)$ to obtain $\{1, \mathfrak{f}\}$.

- Lifting map:
$$\begin{aligned} 1 &\mapsto (00) + (33) + (12) + (21), \\ \mathfrak{f} &\mapsto (03) + (30) + (11) + (22). \end{aligned}$$

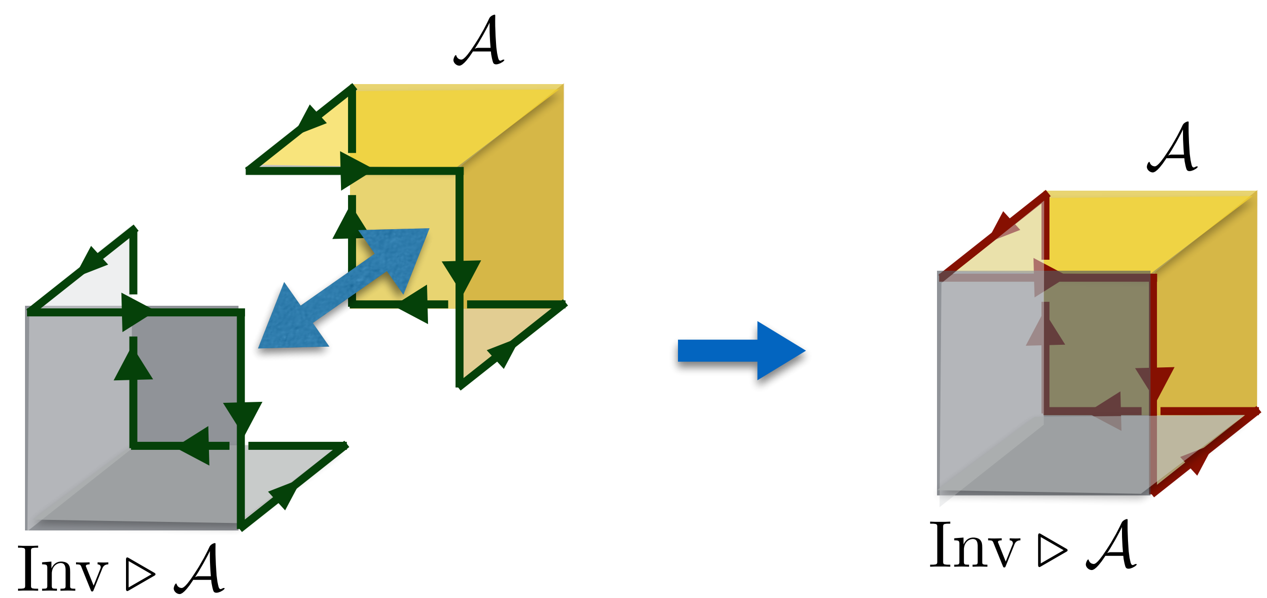
- Consequently, an HOTSC can be unhinged.

Inversion-symmetric HOTPs

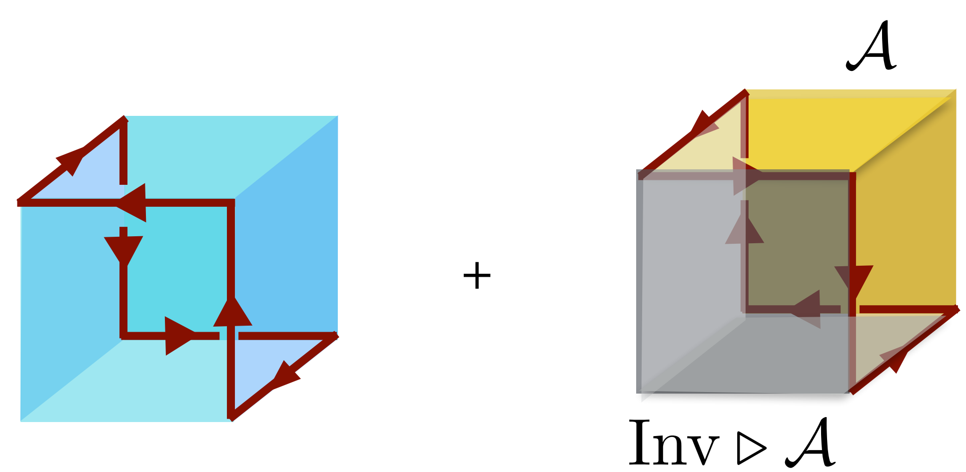
Inversion-symmetric HOTP with chiral hinge



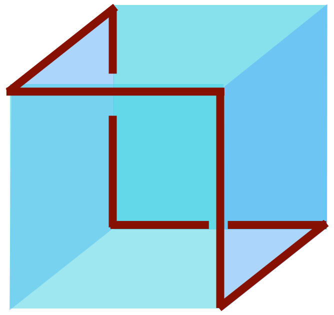
Inversion-symmetric STO



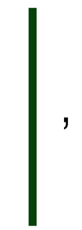
Inversion-symmetric HOTP with
fully gapped surface



Unhinging helical HOTPs

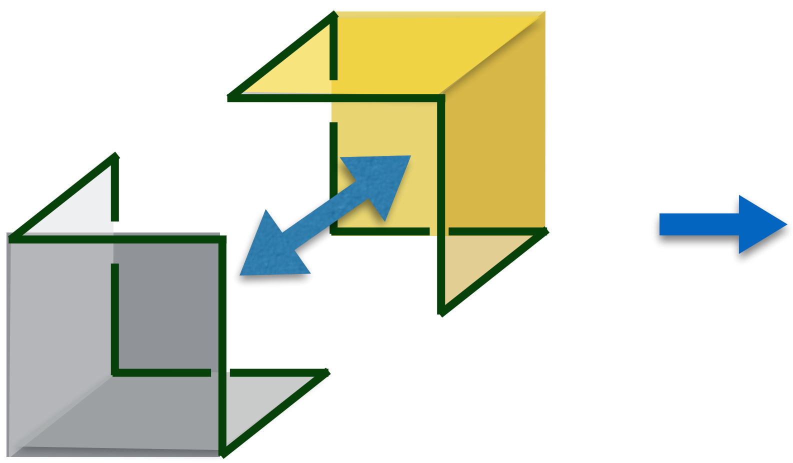


HOTP with helical hinges may appear in systems with inversion and time reversal symmetry.



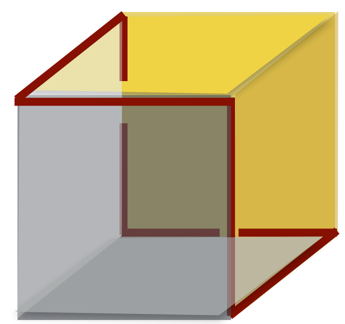
, - gapless helical modes.

$\mathcal{A} \times \mathcal{A}^{\mathcal{T}}$



$\text{Inv} \triangleright (\mathcal{A} \times \mathcal{A}^{\mathcal{T}})$

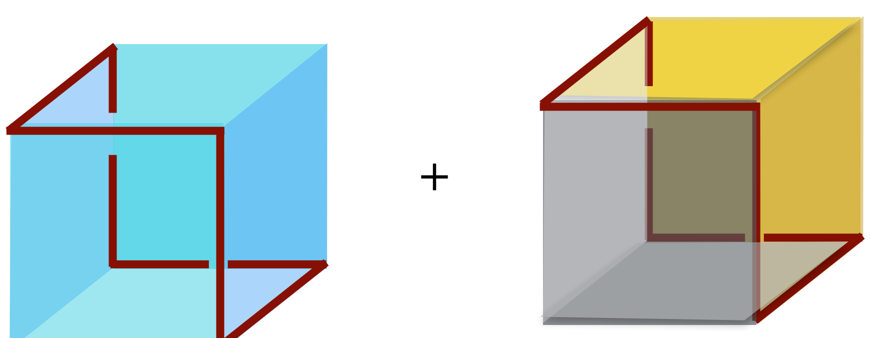
$\mathcal{A} \times \mathcal{A}^{\mathcal{T}}$



$\text{Inv} \triangleright (\mathcal{A} \times \mathcal{A}^{\mathcal{T}})$

Time reversal symmetry required to prevent helical hinges from gapping out.

$\mathcal{A} \times \mathcal{A}^{\mathcal{T}}$



$\text{Inv} \triangleright (\mathcal{A} \times \mathcal{A}^{\mathcal{T}})$

Inversion+Time reversal symmetric HOTP with fully gapped surfaces.

* Po-Watanabe-Vishwanath, Khalaf; Levin-Stern,…

* In preparation: Li-AT-Parmeswaran-Neupert

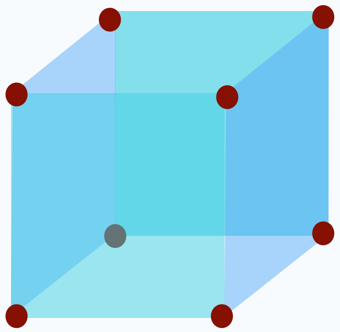
Future directions

➡ Building Hamiltonians from cell-decomposition data. Crystalline quantum lego!

➡ Making sense of higher homotopy for the space of gapped phases.

➡ Generalized notion of anomalies for hinge HOTPs.

➡ Surface topological order for 3^{rd}-order topological phases?



➡ Topological order enriched by spatial symmetries. Fractionalized higher-order topological phases.

Thank you for your attention!