Galois Theory in Gauge Theory, through Seven Loops

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Phys. Rev. Lett. 117, 241601 (2016), with
S. Caron-Huot, L. Dixon, and M. von Hippel

JHEP 1908 (2019) 016 and JHEP 1909 (2019) 061, with
S. Caron-Huot, L. Dixon, F. Dulat, M. von Hippel, and G. Papathanasiou

CARLSBERG FOUNDATION

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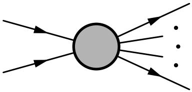
Outline

- Scattering Amplitudes: Motivation and Interest
- Bootstrapping Scattering Amplitudes
  - Planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory
  - Symmetries and analytic properties
  - The MHV sector and generalized polylogarithms
  - Kinematic limits as boundary data
- The Coaction and Cosmic Galois Theory
  - Reformulating our physical constraints
  - Extended Steinmann and the coaction principle
  - Six-particle scattering through seven loops
  - An all-loop example: the Ω integrals
- Conclusions and Further Directions

Scattering Amplitudes

Our ability to calculate scattering amplitudes directly impacts our ability to make predictions in particle physics experiments

- Difficult to calculate amplitudes to the desired levels of precision using Feynman diagrams
- Many of the amplitudes relevant for hard scattering processes at the LHC not known analytically to two loops



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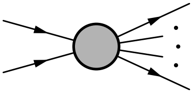


We still don't understand much of the mathematical structure underlying scattering amplitudes

- Scattering amplitudes aren't as complicated as the Feynman diagrams traditionally used to compute them [Parke, Taylor]

$$\left|\mathcal{A}_n(p_1^-, p_2^-, p_3^+, \dots, p_n^+)\right|^2 \propto \sum_{\sigma \in S_n} \frac{(p_1 \cdot p_2)^4}{(p_{\sigma_1} \cdot p_{\sigma_2})(p_{\sigma_2} \cdot p_{\sigma_3}) \cdots (p_{\sigma_n} \cdot p_{\sigma_1})}$$

- At loop level, the coaction proves an important simplification tool [Goncharov, Spradlin, Vergu, Volovich] [Duhr, Gangl, Rhodes]

...Abundant with Mathematical Structure

Nontrivial connections have been made between scattering amplitudes and diverse areas of mathematics in recent years

- Polylogarithms and their generalizations
- Symbols and Coactions
- Motivic Galois Theory
- Positroids, Grassmannians, and Cluster Algebras
- Simplicial Volumes
- Twisted de Rham Theory
- Graph theory

$\vdots$

Bootstrapping Amplitudes in Planar $\mathcal{N} = 4$

Scattering Amplitudes

This hidden structure is easiest to first discover in planar $\mathcal{N} = 4$ SYM

SUSY Ward identities	$\Rightarrow$	relate amplitudes with different helicity structure
Conformal symmetry	$\Rightarrow$	no running of the coupling or UV divergences
Planar limit	$\Rightarrow$	trivial color structure
AdS/CFT	$\Rightarrow$	dual to string theory on $AdS_5 \times S^5$

Much of what we learn here also augments our understanding of QCD

Planar Limit and Dual Conformal Symmetry

An additional simplification occurs in the planar limit, where $N_c \to \infty$ for fixed $g^2 = g^2_{\mathrm{YM}} N_c/(16\pi^2)$

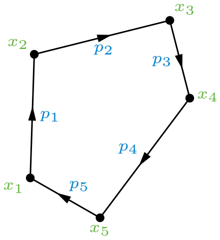
- The suppression of non-planar graphs allows us to endow the scattering particles with an ordering
- This ordering gives rise to a natural set of dual coordinates

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu$$

- The coordinates x_i^μ can be thought of as labelling the cusps of a light-like polygonal Wilson loop in the dual theory, which respects a superconformal symmetry in this dual space

  [Alday, Maldacena] [Drummond, Korchemsky, Sokatchev]

- This strongly constrains the kinematic dependence of the amplitude



Helicity and Infrared Structure

- The infrared-divergent part of these amplitudes is accounted for at all particle multiplicity by the 'BDS ansatz' [Bern, Dixon, Smirnov]
- In the dual theory, the BDS ansatz solves an anomalous conformal Ward identity that determines the Wilson loop up to a function of dual conformal invariants [Drummond, Henn, Korchemsky, Sokatchev]
- Dual conformal invariants can first be formed in six-particle kinematics, so the four- and five-particle amplitudes are entirely described by the BDS ansatz

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$$\mathcal{A}_4 = \mathcal{A}_4^{\text{BDS}} \qquad \mathcal{A}_5 = \mathcal{A}_5^{\text{BDS}}$$

$$\mathcal{A}_n = \underbrace{\mathcal{A}_n^{\text{BDS}}}_{\text{IR structure}} \times \underbrace{\exp(R_n) \times \overbrace{\left(1 + \mathcal{P}_n^{\text{NMHV}} + \mathcal{P}_n^{\text{N}^2\text{MHV}} + \cdots + \mathcal{P}_n^{\overline{\text{MHV}}}\right)}^{\text{helicity structure}}}_{\text{finite function of dual conformal invariants}}$$

- In certain cases, enough is known about R_n and $\mathcal{P}_n^{\text{N}^k\text{MHV}}$ to 'bootstrap' these functions to high loop order

The functions R_6 and $\mathcal{P}_6^{\text{NMHV}}$

- In general, we can construct dual conformally invariant cross ratios out of combinations of Mandelstam invariants

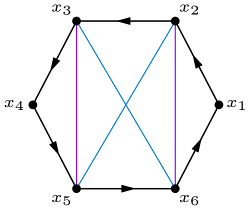
$$x_{ij}^2 \equiv (x_i - x_j)^2 = (p_i + p_{i+1} + \cdots + p_{j-1})^2 \equiv s_{i,\ldots,j-1}$$

  that remain invariant under the dual inversion generator

$$I(x_i^{\alpha\dot{\alpha}}) = \frac{x_i^{\alpha\dot{\alpha}}}{x_i^2} \qquad \Rightarrow \qquad I(x_{ij}^2) = \frac{x_{ij}^2}{x_i^2 x_j^2}$$

- For six particles, three dual conformal invariants can be formed

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad v = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2}, \quad w = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2}$$



Physical Branch Cuts

- Massless scattering amplitudes in the Euclidean region only have branch cuts where one of the Mandelstam invariants $s_{i,\dots,k}$ vanishes
- At six points, this immediately implies that R_6 and $\mathcal{P}_6^{\text{NMHV}}$ can only develop branch cuts where u, v, and w vanish or become infinite

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- However, after analytically continuing out of the Euclidean region, further discontinuities can appear—this turns out to happen where u, v, or w approach 1, or where y_u, y_v, or y_w vanish, where

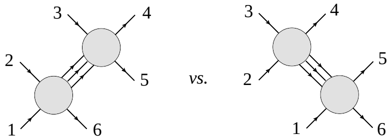
$$y_u = \frac{1+u-v-w-\sqrt{(1-u-v-w)^2-4uvw}}{1+u-v-w+\sqrt{(1-u-v-w)^2-4uvw}},$$

$$y_v = [y_u]_{u\to v\to w\to u}, \qquad y_w = [y_u]_{u\to w\to v\to u}$$

- This is believed to be true to all loop orders, and is consistent with all known six-particle amplitudes and an all-loop analysis of the Landau equations [Prlina, Spradlin, Stanojevic]

The Steinmann Relations

- Additional restrictions come from the Steinmann relations, which tell us that amplitudes cannot have double discontinuities in partially overlapping channels [Steinmann] [Cahill, Stapp]



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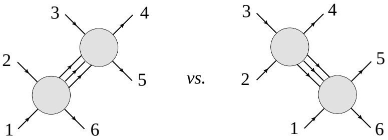

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- It turns out this strongly constrains the form of the six-point amplitude [Caron-Huot, Dixon, von Hippel, AJM]
- However, to see this one must normalize the amplitude appropriately. . .

Infrared Subtraction

- Steinmann-satisfying functions don't form a ring—products of functions with incompatible branch cuts break this property

$$\mathcal{A}_n^{\text{BDS}} \sim \exp(A_n^{(1)})$$

- Therefore, we instead normalize by a 'BDS-like' ansatz that depends on only two-particle Mandelstam invariants

$$\mathcal{A}_n^{\text{BDS}} \times \exp(R_n) \to \rho \times \mathcal{A}_n^{\text{BDS-like}} \times \mathcal{E}_n^{\mathsf{MHV}}$$
$$\mathcal{A}_n^{\text{BDS}} \times \exp(R_n) \times \mathcal{P}_n^{\text{N}^k\text{MHV}} \to \rho \times \mathcal{A}_n^{\text{BDS-like}} \times \mathcal{E}_n^{\text{N}^k\text{MHV}}$$

  where a transcendental constant ρ can also appear

- This only scrambles the Steinmann relations involving two-particle invariants, which are obfuscated in massless kinematics anyways

$\bar{Q}$ Constraint

- The derivative of the amplitude is also heavily constrained by dual superconformal symmetry [Caron-Huot, He]
- For instance, in the MHV sector, we have that

$$\text{coeff}_{\frac{1}{u}}\,(dR_6) + \text{coeff}_{\frac{1}{1-u}}\,(dR_6) = 0$$

- This follows from the action of the dual superconformal group on the n-point BDS-subtracted $\mathrm{N}^k\mathrm{MHV}$ component amplitude $\mathcal{R}_{n,k} \equiv \mathcal{A}_n^{\mathrm{N}^k\mathrm{MHV}} / \mathcal{A}_n^{\mathrm{BDS}}$:

$$\begin{aligned}\bar{Q}_a^A \mathcal{R}_{n,k} &= \sum_{i=1}^{n} \chi_i^A \frac{\partial}{\partial Z_i^a} \mathcal{R}_{n,k} \\ &\propto g^2\ \mathrm{Res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left(d^{2|3} \mathcal{Z}_{n+1}\right)_a^A \left[\mathcal{R}_{n+1,k+1} - \mathcal{R}_{n,k} \mathcal{R}_{n+1,k}^{\mathrm{tree}}\right] \\ &\qquad + \text{cyclic}\end{aligned}$$

Thus, the kinematic dependance and analytic structure of the amplitude is highly constrained...

...but how do we put this all together?

Polylogarithms

- Loop-level contributions to MHV and NMHV amplitudes are conjectured to be generalized polylogarithms of uniform transcendental weight $2L$—namely, functions satisfying

$$dF = \sum_i F^{s_i} d\log s_i$$

  for some set of 'symbol letters' $\{s_i\}$, where F^{s_i} is a generalized polylogarithm of weight $2L-1$
- The symbol letters $\{s_i\}$ are algebraic functions of kinematic invariants
- Examples of such functions (and their special values) include $\log(z)$, $i\pi$, $\mathrm{Li}_m(z)$, and ζ_m. The classical polylogarithms $\mathrm{Li}_m(z)$ involve only the symbol letters $\{z, 1-z\}$

$$\mathrm{Li}_1(z) = -\log(1-z), \quad \mathrm{Li}_m(z) = \int_0^z \frac{\mathrm{Li}_{m-1}(t)}{t} dt$$

Polylogarithms

- The discontinuity structure of the six-particle amplitude tells us its symbol alphabet should be given by

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

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- By sequentially imposing these known properties of the amplitude, we find a function space of increasingly small size. For instance, at four loops:

Imposed Constraints	Number of Functions
Generalized polylogarithms with the correct kinematic dependence	1,675,553
That have branch cuts only in physical channels	6,916
That satisfy the Steinmann relations	839

Kinematic Limits as Boundary Data

We can then match a general ansatz of such polylogarithms to the amplitude's known behavior in various kinematic limits

- Collinear Factorization
- Multi-Regge Limits
- Near-Collinear OPE Expansion
- Multi-Particle Factorization
- Self-Crossing Limit

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These constraints are sufficient to uniquely determine the six-particle amplitude through seven loops

- Through five loops, collinear factorization and multi-Regge factorization are sufficient
- At six loops and seven loops, the near-collinear OPE is needed to fix a single residual ambiguity in the MHV sector
- There is a computational barrier (not a principled one) to going to higher loops

The Coaction and Cosmic Galois Theory

The Coaction on Polylogarithms

- Generalized polylogarithms are endowed with a coaction that maps functions to a tensor space of lower-weight functions

$$\mathcal{H}_w \xrightarrow{\Delta} \bigoplus_{p+q=w} \mathcal{H}_p \otimes \mathcal{H}_q^{\mathfrak{dr}}$$

- The location of branch cuts is encoded in the first component of the coaction
- The derivatives of a function are encoded in the second component of the coaction
- If we iterate this map $w-1$ times we arrive at a function's 'symbol', in terms of which all identities reduce to familiar logarithmic identities

$$\Delta_{1,\dots,1}\mathrm{Li}_m(z) = -\log(1-z) \otimes \log z \otimes \cdots \otimes \log z$$

The Coaction on Polylogarithms

The branch-cut conditions and Steinmann relations can be succinctly phrased using this formalism

- In the Euclidean region, only the symbol letters u, v, and w can appear in the first entry

$$\Delta_{1,w-1}F = \log u \otimes {}^{u}F + \log v \otimes {}^{v}F + \log w \otimes {}^{w}F$$

- The symbol of the amplitude cannot involve first and second symbol letters that correspond to disallowed branch cuts

$$\cancel{\log\left(\tfrac{u}{vw}\right) \otimes \log\left(\tfrac{w}{uv}\right)} \otimes \cdots \qquad \cancel{\log\left(\tfrac{u}{vw}\right) \otimes \log\left(\tfrac{v}{uw}\right)} \otimes \cdots$$

$$\frac{u}{vw} \sim s_{234}^2, \qquad \frac{v}{wu} \sim s_{345}^2, \qquad \frac{w}{uv} \sim s_{123}^2$$

- The last entry of the coproduct is restricted by the $\bar{Q}$ constraint

$$\Delta_{w-1,1}F = F^{u} \otimes \log\left(\frac{u}{1-u}\right) + \ldots$$

The *Extended* Steinmann Relations

In fact, the symbols of BDS-like normalized amplitudes exhibit an even more surprising property: the Steinmann relations are obeyed by *all* adjacent entries of the symbol [Caron-Huot, Dixon, von Hippel, AJM, Papathanasiou]

$$\cdots \otimes \cancel{\log\left(\frac{u}{vw}\right) \otimes \log\left(\frac{w}{uv}\right)} \otimes \cdots \qquad \cdots \otimes \cancel{\log\left(\frac{u}{vw}\right) \otimes \log\left(\frac{v}{uw}\right)} \otimes \cdots$$

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- The same constraint can also be seen to hold in the seven-particle amplitude through four loops, and all two-loop MHV amplitudes

  [Dixon, Drummond, Harrington, AJM, Papathanasiou, Spradlin]

  [Caron-Huot] [Golden, AJM, Spradlin, Volovich]
- This restriction (and the symbol letters in planar $\mathcal{N} = 4$) have an intriguing interpretation in terms of cluster algebras

  [Golden, Goncharov, Spradlin, Vergu, Volovich] [Drummond, Foster, Gürdoğan]

Cosmic Galois Theory

The coaction on generalized polylogarithms is also dual to the action of the ‘cosmic Galois group’

- The cosmic Galois group extends the classical Galois theory to the study of periods—integrals of rational functions over rational domains
- Thus, we can explore the stability of amplitudes and integrals under the action of this Galois group

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Specifically, we ask: does the space of Steinmann hexagon functions $\mathcal{H}^{\text{hex}}$ satisfy a ‘coaction principle’? [Schnetz] [Brown]

$$\Delta \mathcal{H}^{\text{hex}} \subset \mathcal{H}^{\text{hex}} \otimes \mathcal{H}$$

- This can be formulated in terms of the action of the cosmic Galois group C as

$$C \times \mathcal{H}^{\text{hex}} \xrightarrow{?} \mathcal{H}^{\text{hex}}$$

The Coaction Principle

$$\Delta\mathcal{H}^{\text{hex}} \subset \mathcal{H}^{\text{hex}} \otimes \mathcal{H},$$

- Part of the content of this statement is that the coaction preserves the locations of branch cuts (which we already know is the case from general physical principles)
- However, more general transcendental constants also appear in this space
  - multiple zeta values
  - alternating sums
  - transcendental constants involving higher roots of unity
- These constants exhibit nontrivial structure under the coaction, which is not *a priori* constrained by physical principles
- This also relates back to the ambiguity in our infrared subtraction; does there exist a constant factor ρ such that the amplitude satisfies a coaction principle?

The coaction on MZVs

- For instance, we can consider our function space at $(u, v, w) = (1, 1, 1)$, where everything evaluates to MZVs
- To study the behavior of the MZVs under the coaction, it's convenient to map to an f-alphabet [Brown]
- In this setting one has natural derivations ∂_{2m+1} that act on the (f-alphabet representation of motivic) zeta values as

$$\partial_{2m+1}\zeta_{2n+1} = \delta_{m,n}$$

  and that satisfy the Leibniz rule—for example,

$$\partial_3(\zeta_7\zeta_3^2) = 2\zeta_7\zeta_3$$

- These operators act nontrivially on multiple zeta values, in a way that is easy to calculate using the f-alphabet
- There is no ∂_2, as the even zeta values are semi-simple elements of the coaction

The coaction principle at (1,1,1)

Weight	Multiple Zeta Values	Appear in $\mathcal{H}^{\text{hex}}\big|_{u,v,w\to 1}$
0	1	1
1		
2	ζ_2	ζ_2
3	ζ_3	
4	ζ_4	ζ_4
5	$\zeta_5,\ \zeta_3\zeta_2$	$5\zeta_5 - 2\zeta_3\zeta_2$
6	$\zeta_3^2,\ \zeta_6$	ζ_6
7	$\zeta_7,\ \zeta_5\zeta_2,\ \zeta_3\zeta_4$	$\zeta_5\zeta_2 - 7\zeta_7 + 3\zeta_3\zeta_4$
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Weight	Multiple Zeta Values	Appear in $\mathcal{H}^{\text{hex}}\big|_{u,v,w\to 1}$
0	1	1
1		
2	∂_7 ζ_2	ζ_2
✓ 3	ζ_3	
4	∂_5 ζ_4	ζ_4
✓ 5	$\zeta_5,\ \zeta_3\zeta_2$	$5\zeta_5 - 2\zeta_3\zeta_2$
✓ 6	$\zeta_3^2,\ \zeta_6$	∂_3 ζ_6
7	$\zeta_7,\ \zeta_5\zeta_2,\ \zeta_3\zeta_4$	$\zeta_5\zeta_2 - 7\zeta_7 + 3\zeta_3\zeta_4$
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$$\partial_3(\zeta_{5,3}) = 0, \qquad \partial_5(\zeta_{5,3}) = -5\zeta_3$$

$$\Downarrow$$

$$\partial_3(\zeta_{5,3} + 5\zeta_5\zeta_3 - \zeta_3^2\zeta_2) = 5\zeta_5 - 2\zeta_3\zeta_2$$

$$\partial_5(\zeta_{5,3} + 5\zeta_5\zeta_3 - \zeta_3^2\zeta_2) = 0$$

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- Unexplained dropouts were required at low weights for the coaction principle to be nontrivial
- Each zeta value that drops out seeds an infinite tower of constraints at higher loop orders, which we find are satisfied

General Kinematics

- Different spaces of constants appear in different limits
  - at $(\frac{1}{2},1,\frac{1}{2})$ the space of alternating sums is saturated
  - at $(\frac{1}{2},v \to 0,\frac{1}{2})$ dropouts are observed starting at weight 6
  - at $(u, v \to 0, u)|_{u\to\infty}$ dropouts are observed starting at weight 1 ($\log 2$ doesn't appear)
  - fourth roots and sixth roots of unity also appear

General Kinematics

- Different spaces of constants appear in different limits
  - at $(½,1,½)$ the space of alternating sums is saturated
  - at $(½,v \to 0,½)$ dropouts are observed starting at weight 6
  - at $(u, v \to 0, u)|_{u\to\infty}$ dropouts are observed starting at weight 1 ($\log 2$ doesn't appear)
  - fourth roots and sixth roots of unity also appear

Everywhere we have checked, the coaction principle is respected

- This requires choosing a nonzero value for ρ

$$\begin{aligned}\rho(g^2) = 1 &+ 8(\zeta_3)^2\, g^6 - 160\zeta_3\zeta_5\, g^8 \\ &+ \left[1680\zeta_3\zeta_7 + 912(\zeta_5)^2 - 32\zeta_4(\zeta_3)^2\right] g^{10} \\ &- \left[18816\zeta_3\zeta_9 + 20832\zeta_5\zeta_7 - 448\zeta_4\zeta_3\zeta_5 - 400\zeta_6(\zeta_3)^2\right] g^{12} \\ &+ \mathcal{O}(g^{14})\end{aligned}$$

- It would be interesting to know if there is a 'physical definition' of this constant. . .

Polylogarithms

We can add the power of these constraints to our four-loop example:

Imposed Constraints	Number of Functions
Generalized polylogarithms with the correct kinematic dependence	1,675,553
That have branch cuts only in physical channels	6,916
That satisfy the Steinmann condition in the second entry	839
That satisfy extended Steinmann and the coaction principle	372

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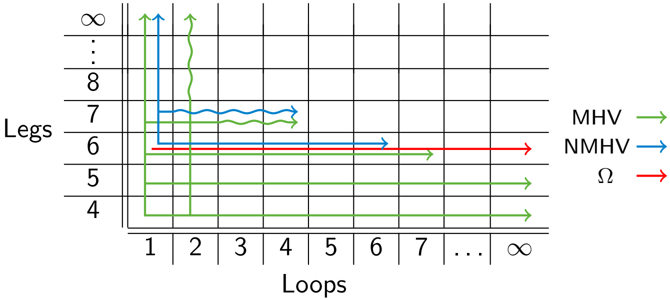
At five and six loops, we are left with (1,5) and (6,17) undetermined coefficients, respectively, in the (MHV, NMHV) sector

All of these ambiguities can be fixed by boundary data from the multi-Regge and near-collinear limits

Status of Loops and Legs in Planar $\mathcal{N} = 4$



[Bern, Caron-Huot, Dixon, Drummond, Duhr, Foster, Gürdoğan, He, Henn, von Hippel, Golden, Kosower, AJM, Papathanasiou, Pennington, Roiban, Smirnov, Spradlin, Vergu, Volovich, . . .]

- Unexpected and striking structure exists in the the direction of both higher loops and legs

$L \to \infty$: Cosmic Galois Coaction Principle

[Caron-Huot, Dixon, Dulat, von Hippel, AJM, Papathanasiou]

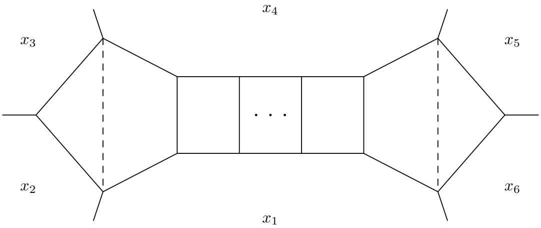
$n \to \infty$: Cluster-Algebraic Structure

[Golden, Goncharov, Spradlin, Vergu, Volovich] [Golden, Paulos, Spradlin, Volovich] [Golden, AJM, Spradlin, Volovich]

The Double Pentaladder Integrals

We also have some control over this space of functions to all loop orders [Caron-Huot, Dixon, von Hippel, AJM, Papathanasiou]

$\Omega^{(L)} \equiv$



- $\Omega^{(L)}$ contributes to the six-point amplitude in planar $\mathcal{N} = 4$ at all loops, as well as to non-supersymmetric amplitudes
- A related integral $\tilde{\Omega}^{(L)}$ can also be defined using a different numerator
- These integrals are related at adjacent loop orders by second-order differential equations [Drummond, Henn, Trnka]

The Double Pentaladder Integrals

- These differential equations can be solved at finite coupling in terms of Mellin integrals over hypergeometric functions

$$\Omega = \sum_L (-g^2)^L \Omega^{(L)} \qquad \tilde{\Omega} = \sum_L (-g^2)^L \tilde{\Omega}^{(L)}$$

- Ω and $\tilde{\Omega}$ naturally complete to a space involving two new integrals

$$\mathcal{O} = \frac{1}{g^2}\left(x\partial_x - y\partial_y\right)\Omega, \quad \mathcal{W} = \left(x\partial_x + y\partial_y\right)\Omega,$$

  which perturbatively evaluate to polylogarithms of odd weight

- The set of first-order differential equations that relate these four functions can be rearranged into a coaction

$$\Delta_{\bullet,1}\mathcal{V}_i = \mathcal{V}_j \otimes M_{ji}$$

  where $\mathcal{V}_i = \{\mathcal{W}, \Omega, \tilde{\Omega}_e, \mathcal{O}, \dots\}$, and M_{ji} is a matrix of logarithms

- Thus, these integrals satisfy a coaction principe by construction

Beyond Planar $\mathcal{N} = 4$

Coaction principles of this type have been observed in other settings

- Tree-level string theory amplitudes [Schlotterer, Stieberger]
- Feynman graphs in ϕ^4 theory [Panzer, Schnetz]
- The electron anomalous magnetic moment [Schnetz]

It is tempting to believe these coaction principles point to some (possibly graph-theoretic) symmetry respected by quantum field theory more generally

- A coaction can also be defined on the more complicated types of functions that appear in scattering amplitudes [Brown] [Broedel, Duhr, Dulat, Penante, Tancredi]
- However, things become more complicated when one loses purity and uniform transcendental weight. . .

Conclusions

- A large amount of information is encoded in the formal structure of amplitudes, much of which is still not well understood
- In particular, there exists surprising motivic structure in amplitudes that remains to be explained in terms of physical principles
  - a coaction principle seems to hold not only in the amplitudes of planar $\mathcal{N} = 4$ SYM theory, but also in other contexts
  - does the factor ρ admit a physical definition?
- Hopefully, better understanding these structures will provide us with new physical insights and calculational tools

Thanks!

The Steinmann Hexagon Space

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12
$L = 1$	1	3	4										
$L = 2$	1	3	6	10	6								
$L = 3$	1	3	6	13	24	15	6						
$L = 4$	1	3	6	13	27	53	50	24	6				
$L = 5$	1	3	6	13	27	54	102	118	70	24	6		
$L = 6$	1	3	6	13	27	54	105	199	269	181	78	24	6

The dimension of the space of coproduct weight-n first coproduct entries of the MHV and NMHV amplitudes at a given loop order L

Cosmic Galois Theory

In science you sometimes have to find a word that strikes, such as "catastrophe", "fractal", or "noncommutative geometry". They are words which do not express a precise definition but a program worthy of being developed.

- Pierre Cartier