Volume Conjecture and Topological Recursion

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Papers:

R.H.Dijkgraaf and H.F., Fortsch.Phys.**57**(2009),825-856, arXiv:0903.2084 [hep-th]

R.H.Dijkgraaf, H.F. and M.Manabe, to appear.

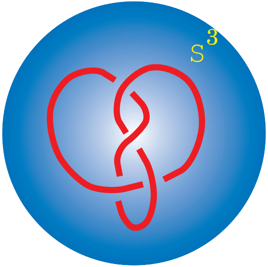
1. Introduction

Asymptotic analysis of the knot invariants is studied actively in the knot theory.

Volume Conjecture [Kashaev][Murakami^2]

Asymptotic expansion of the colored Jones polynomial for knot $\mathbf{K}$

$\Rightarrow$ The geometric invariants of the knot complement $\mathbb{S}^3 \backslash \mathbf{K}$.



Recent years the asymptotic expansion is studied to higher orders.

$\mathbf{S_k(u)}$: Perturbative invariants, $\mathbf{q = e^{2\hbar}}$ [Dimofte-Gukov-Lenells-Zagier]

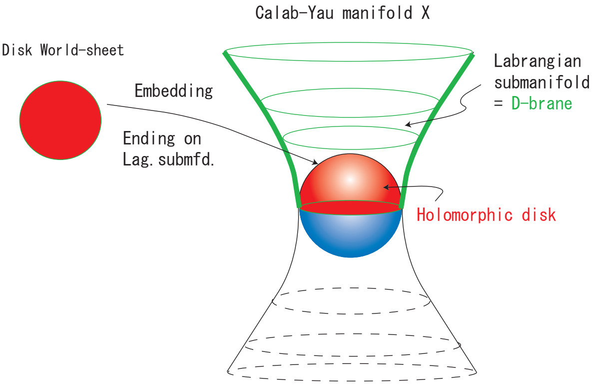
$$\mathbf{J_n(K; q) = \exp\left[\frac{1}{\hbar}S_0(u) + \frac{\delta}{2}\log\hbar + \sum_{k=0}^{\infty}\hbar^k S_{k+1}(u)\right]}, \quad \mathbf{u = 2\hbar n - 2\pi i.}$$

Topological Open String

Topological B-model on the local Calabi-Yau $\mathbf{X}^\vee$

$$\mathbf{X}^\vee = \left\{(\mathbf{z}, \mathbf{w}, \mathbf{e}^{\mathbf{p}}, \mathbf{e}^{\mathbf{x}}) \in \mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{C}^* | \mathbf{H}(\mathbf{e}^{\mathbf{p}}, \mathbf{e}^{\mathbf{x}}) = \mathbf{z}\mathbf{w}\right\}.$$

D-brane partition function $\mathbf{Z}_{\mathbf{D}}(\mathbf{u}_{\mathbf{i}})$



Topological Recursion [Eynard-Orantin]

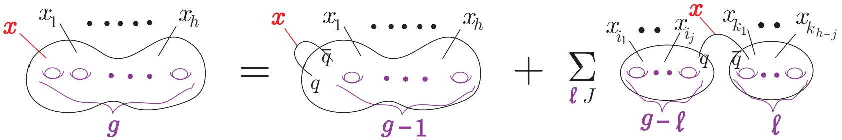
Eynard and Orantin proposed a spectral invariants for the spectral curve $\mathcal{C}$

$$\mathcal{C} = \{(x, y) \in (\mathbb{C}^*)^2 | H(y, x) = 0\}.$$

- Symplectic structure of the spectral curve $\mathcal{C}$,
- Riemann surface $\Sigma_{g,h}$ = World-sheet.

$\Rightarrow$ Spectral invariant $\mathcal{F}^{(g,h)}(u_1, \cdots, u_h)$ u_i:open string moduli

Eynard-Orantin's topological recursion is applicable.



$\Sigma_{g,h+1}$ $\Sigma_{g-1,h+2}$ $\Sigma_{\ell,k+1}$ $\Sigma_{g-\ell,h-k}$

Spectral invariant = D-brane free energy in top. string [BKMP]

Correspondences

Heuristically we discuss a relation between the perturbative invariants $\mathbf{S_k(u)}$ and the free energies $\mathcal{F}^{(g,h)}(u_1, \cdots, u_h)$ á la BKMP. [Dijkgraaf-F.]

3D Geometry	Topological Open String
Character variety	Spectral curve
$\{(\ell, m) \in \mathbb{C}^* \times \mathbb{C}^* \| \tilde{A}_K(\ell, m) = 0\}$	$\{(e^p, e^x) \in \mathbb{C}^* \times \mathbb{C}^* \| H(e^p, e^x) = 0\}$
$u = \log m$: Holonomy	u: Open string moduli
Neumann-Zagier fn. $H(u)/2$	Disk Free Energy $\bar{\mathcal{F}}^{(0,1)}(u)$
Reidemeister Torsion $\mathcal{T}(M; u)$	Annulus Free Energy $\bar{\mathcal{F}}^{(0,2)}(u)$
$q = e^{2\hbar}$	$q = e^{g_s}$

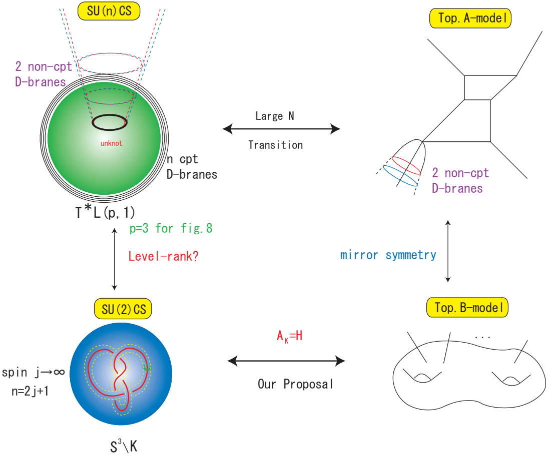
In this talk we will explore the following relation:

$$\boxed{S_k(u) \leftrightarrow F_k(u) = 2^{k-2} \sum_{2g+h=k+1,\ h\geq 0} \frac{1}{h!} \bar{\mathcal{F}}^{(g,h)}(u).}$$

Motivation of Our Research

- Realization of 3D quantum gravity in top. string
- Dual description of quantum CS theory as free boson on character variety
- Large **n** duality not for rank but for level

  $\Rightarrow$ Novel class of duality



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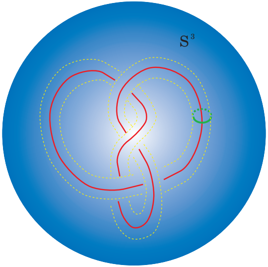
Volume Conjecture [Kashaev][Murakami[2]]

In 1997 Kashaev proposed a striking conjecture on the asymptotic expansion of the colored Jones polynomial $\mathbf{J_n(K;q)}$.

$$2\pi \lim_{n\to\infty} \frac{\log |J_n(K; e^{2\pi i/n})|}{n} = \mathrm{Vol}(\mathbb{S}^3\backslash K).$$



$$\boxed{\text{Knot complement} = \mathbb{S}^3\backslash \mathbf{N(K)}}$$

$\mathbf{N(K)}$: Tubular neighborhood of a knot $\mathbf{K}$.

The hyperbolic knot complement admits a hyperbolic structure.

Generalized Volume Conjecture

In 2003, Gukov generalized the volume conjecture to 1-parameter version.

$$(u+2\pi i)\lim_{n\to\infty}\frac{\log J_n(K; e^{(u+2\pi i)/n})}{n} = H(u), \quad u\in\mathbb{C}.$$

$H(u)$: Neumann-Zagier's potential function

$$\frac{\partial H(u)}{\partial u} = v + 2\pi i.$$

u and v satisfies an algebraic equation.

$$A_K(\ell, m) = 0, \quad \ell = e^v, \quad m = e^u.$$

$A_K(\ell, m)$: A-polynomial for a knot K. incomplete

$\Rightarrow$ Up to linear term of u and v, the Neumann-Zagier potential $H(u)$ yields to

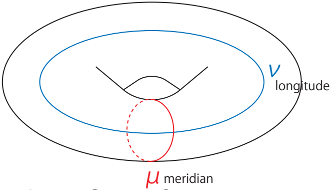
$$H(u) = \int_{2\pi i}^{u+2\pi i} du\, v(u) + \text{linear terms}.$$

AJ conjecture and higher order terms

In 2003 Garoufalidis proposed a conjecture on **q**-difference equation for the colored Jones polynomial. (Quantum Riemann Surface)

$$\mathsf{A}_K(\hat{\ell},\hat{m};q)J_n(K;q)=0,\quad \mathsf{A}_K(\ell,m;q=1)=(\ell-1)\mathsf{A}_K(\ell,m).$$

$$\hat{\ell}f(n)=f(n+1),\quad \hat{m}f(n)=q^{n/2}f(n),\quad \hat{\ell}\hat{m}=q^{1/2}\hat{m}\hat{\ell}.$$



Commutation relation of the Chern-Simons gauge theory

$$\rho(\mu)=\mathrm{P}\exp\left[\oint_{\mu}A\right],\quad \rho(\nu)=\mathrm{P}\exp\left[\oint_{\nu}A\right],$$

$$\{A^a_\alpha(x),A^b_\beta(y)\}=\frac{2\pi}{k}\delta^{ab}\epsilon_{\alpha\beta}\delta^2(x-y).$$

Meridian μ and longitude ν intersect at one point.

$$\hat{\ell}\hat{m}=q^{1/2}\hat{m}\hat{\ell}\Rightarrow[\hat{u},\hat{v}]=\frac{2\pi}{k}.\quad(\theta=v\,du,\quad \omega=d\theta.)$$

q-difference Equation for Fig.8 Knot

Example: Figure 8 knot $\mathbf{4_1}$ [Garoufalidis]

$$A_{4_1}(\hat{\ell},\hat{m};q) = \frac{q^5\hat{m}^2(-q^3+q^3\hat{m}^2)}{(q^2+q^3\hat{m}^2)(-q^5+q^6\hat{m}^4)} - \frac{(q^2-q^3\hat{m}^2)(q^8-2q^9\hat{m}^2+q^{10}\hat{m}^2-q^9\hat{m}^4+q^{10}\hat{m}^4-q^{11}\hat{m}^4+q^{10}\hat{m}^6-2q^{11}\hat{m}^6+q^{12}\hat{m}^8)}{q^5\hat{m}^2(q+q^3\hat{m}^2)(q^5-q^6\hat{m}^4)}\hat{\ell} + \frac{(-q+q^3\hat{m}^2)(q^4+q^5\hat{m}^2-2q^6\hat{m}^2-q^7\hat{m}^4+q^8\hat{m}^4-q^9\hat{m}^4-2q^{10}\hat{m}^6+q^{11}\hat{m}^6+q^{12}\hat{m}^8)}{q^4\hat{m}^2(q^2+q^3\hat{m}^2)(-q+q^6\hat{m}^4)}\hat{\ell}^2 + \frac{q^4\hat{m}^2(-1+q^3\hat{m}^2)}{(q+q^3\hat{m}^2)(q-q^6\hat{m}^4)}\hat{\ell}^3.$$

AJ conjecture for Wilson loop

$$W_n(K;q) := J_n(K;q)W_n(U;q), \quad \tilde{A}_K(\hat{\ell},\hat{m};q)W_n(K;q)=0.$$

q-difference equation is factorized.

$$\tilde{A}_{4_1}(\hat{\ell},\hat{m};q) = (q^{1/2}\hat{\ell}-1)\hat{A}_{4_1}(\hat{\ell},\hat{m};q),$$

$$\hat{A}_{4_1}(\hat{\ell},\hat{m};q) = \frac{q\hat{m}^2}{(1+q\hat{m}^2)(-1+q\hat{m}^4)} - \frac{(-1+q\hat{m}^2)(1-q\hat{m}^2-(q+q^3)\hat{m}^4-q^3\hat{m}^6+q^4\hat{m}^8)}{q^{1/2}\hat{m}^2(-1+q\hat{m}^4)(-1+q^3\hat{m}^4)}\hat{\ell} + \frac{q^2\hat{m}^2}{(1+q\hat{m}^2)(-1+q^3\hat{m}^4)}\hat{\ell}^2.$$

Perturbative Invariants

WKB expansion of the Wilson loop expectation value:

$$W_n(K;q) = \exp\left[\frac{1}{\hbar}S_0(u) + \frac{\delta}{2}\log\hbar + \sum_{k=1}^{\infty}\hbar^{k-1}S_k(u)\right],$$

$$q := e^{2\hbar}, \quad q^n = m = e^u.$$

Applying this expansion into q-difference equation, one finds a hierarchy of differential equations :

$$\hat{A}_K(\ell, m; q) = \sum_{k=0}^{d}\sum_{k=0}^{\infty}\ell^j\hbar^k a_{j,k}(m).$$

$$\sum_{j=0}^{d} e^{jS_0'} a_{j,0} = 0, \quad \leftarrow \quad \mathrm{A-polynomial}$$

$$\sum_{j=0}^{d} e^{jS_0'}\left[a_{j,1} + a_{j,0}\left(\frac{1}{2}j^2 S_0'' + jS_1'\right)\right] = 0,$$

$$\sum_{j=0}^{d} e^{jS_0'}\left[a_{j,2} + a_{j,1}\left(\frac{1}{2}j^2 S_0'' + jS_1'\right) + a_{j,0}\left(\frac{1}{2}(\frac{1}{2}j^2 S_0'' + jS_1')^2 + \frac{1}{6}j^3 S_0''' + \frac{1}{2}j^2 S_1'' + jS_2'\right)\right] = 0,$$

$$\cdots.$$

Computational Results top1 top2

Solving $\mathbf{q}$-difference equation, one obtains the expansion of the expectation value of the Wilson loop around a non-trivial flat connection.

- Figure eight knot: [DGLZ]

$$\ell(m)=\frac{1-2m^2-2m^4-m^6+m^8+(1-m^4)\sqrt{1-2m^2+m^4-2m^6+m^8}}{2m^4},$$

$$S_0'(u)=\log\ell(m),$$

$$S_1(u)=-\frac{1}{2}\log\left[\frac{\sqrt{\sigma_0(m)}}{2}\right],\quad \sigma_0(m):=m^{-4}-2m^{-2}+1-2m^2+m^4,$$

$$S_2(u)=\frac{-1}{12\sigma_0(m)^{3/2}m^6}(1-m^2-2m^4+15m^6-2m^8-m^{10}+m^{12}),$$

$$S_3(u)=\frac{2}{\sigma_0(m)^3m^6}(1-m^2-2m^4+5m^6-2m^8-m^{10}+m^{12}).$$

$\mathbf{S_1(u)}$ coincides with the Reidemeiser torsion. [Porti]

$$T(M;u)=\exp\left[-\frac{1}{2}\sum_{n=0}^{3}n(-1)^n\log\det{}'\Delta_n^{E_\rho}\right].$$

$\mathbf{E_\rho}$: flat line bdle, $\quad \mathbf{\Delta_n^{E_\rho}}$: Laplacian on $\mathbf{n}$-forms.

BKMP's Free Energy

Topological B-model amplitudes are computed in the similar way as the matrix models.

The general structure of the amplitudes is capctured by the symplectic structure of the spectural curve $\mathcal{C}$.

$$\mathcal{C} = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^* \times \mathbb{C}^* | \mathbf{H}(\mathbf{y}, \mathbf{x}) = 0\right\}.$$

- Free energies for closed world-sheet:
  Symplectic invariants $\mathcal{F}^{(g,0)}$
- Free energies for world-sheet with boundaries:
  Spectral invariants $\mathcal{F}^{(g,h)}(\mathbf{u}_1, \cdots, \mathbf{u}_h)$, $\mathbf{u}_i$: open string moduli

These free energies are integrals of the meromorphic forms $\mathbf{W}_h^{(g)}$.

[Bouchard-Klemm-Marino-Pasquetti]

$$\mathcal{F}^{(g,h)}(\mathbf{u}_1, \cdots, \mathbf{u}_h) = \int_{e^{u_1^*}}^{e^{u_1}} \mathbf{dx}_1 \cdots \int_{e^{u_h^*}}^{e^{u_h}} \mathbf{dx}_h\, \mathbf{W}_h^{(g)}(\mathbf{x}_1, \cdots, \mathbf{x}_h).$$

D-brane Partition Function

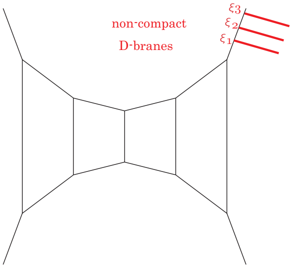
The D-brane partition function is defined by

$$Z_D(\xi_1, \cdots, \xi_n) = \sum_R Z_R \mathrm{Tr}_R V$$

$$\log Z_D = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \sum_{w_1,\cdots,w_h} \frac{1}{h!} g_s^{2g-2+h} F^{(g)}_{w_1,\cdots,w_h} \mathrm{Tr} V^{w_1} \cdots \mathrm{Tr} V^{w_h},$$

$$V = \mathrm{diag}(\xi_i, \cdots \xi_n)$$

ξ_i $(i = 1, \cdots n)$ are location of non-compact D-brane in X.



BKMP's Free Energy and D-brane Partition Function

D-brane partition function is related with BKMP's free energies $\mathcal{F}^{(g,h)}(u_1, \cdots, u_h)$. [Marino]

Dictionary

$$\mathrm{Tr}V^{w_1} \cdots \mathrm{Tr}V^{w_h} \quad \leftrightarrow \quad x_1^{w_1} \cdots x_h^{w_h}, \quad x_i = e^{u_i}.$$

Identification of the free energy:

$$\mathcal{F}^{(g,h)}(u_1, \cdots u_h) = \sum_{w_1, \cdots, w_h} \frac{1}{h!} F^{(g)}_{w_1, \cdots w_h} x_1^{w_1} \cdots x_h^{w_h}.$$

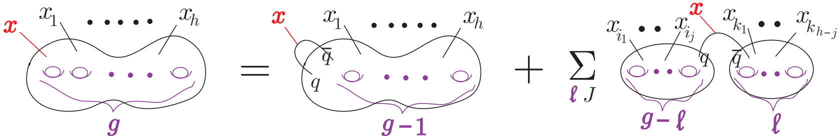
Topological Recursion

Eynard-Orantin's topological recursion $(2g + h \geq 3)$:

$$W^{(g)}_{h+1}(x, x_1, \cdots, x_h)$$
$$= \sum_{x_i} \mathrm{Res}_{q=q_i} \frac{dE_q(x)}{y(q) - y(\bar{q})} \Big[W^{(g-1)}_{h+2}(q, \bar{q}, x_1, \cdots, x_h)$$
$$+ \sum_{\ell=0}^{g} \sum_{J \subset H} W^{(g-\ell)}_{|J|+1}(q, p_J) W^{(\ell)}_{|H|-|J|+1}(\bar{q}, p_{H \setminus J}) \Big].$$

q, $\bar{q}$: points $x = q$ on the 1st sheet and 2nd sheet of the spectral curve

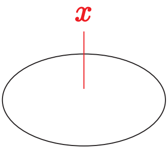
q_i: End points of cuts in the double covering of $\mathcal{C}$.



$dE_q(x)$: Meromorphic 1-form w/ properties.

- Simple pole at $x = q$ with residue $+1$
- Zero A-period on the spectral curve $\mathcal{C}$.

Disk invariant



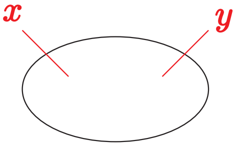
The initial condition for $\mathbf{W}_1^{(0)}(\mathbf{x})$:

$$W_1^{(0)}(x) = 0.$$

The top. string free energy is determined independently. [Aganagic-Vafa]

$$\mathcal{F}^{(0,1)}(u) = \int_{u_*}^{u} d\log x \ \log y(x), \quad H(y,x) = 0, \quad u := \log x.$$

Annulus invariant



The initial condition for $\mathbf{W}_2^{(0)}(\mathbf{x})$:

$$W_2^{(0)}(x,y) = B(x,y).$$

$\mathbf{B(x,y)}$: Bergmann kernel on the spectral curve

$$B(x,y) \overset{x\sim y}{\sim} \frac{dx\,dy}{(x-y)^2}, \quad \oint_{A_I} B(x,y) = 0, \quad \frac{1}{2}\int_q^{\bar{q}} B(x,p) = dE_q(p).$$

The top. string free energy is regularized. [Marino][F.-Mizoguchi]

$$\mathcal{F}^{(0,2)}(u_1,u_2) = \int_{x_1^*}^{x_1}\int_{x_2^*}^{x_2}\left[B(x,y) - \frac{1}{(x-y)^2}\right], \quad x_i = e^{u_i}$$

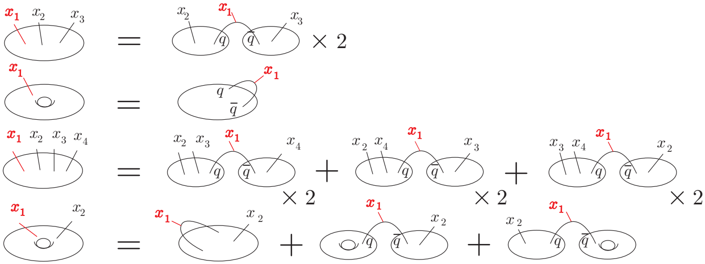
Topological Recursion in Lower Orders

$W_3^{(0)}(x_1,x_2,x_3)$:

$W_1^{(1)}(x_1)$:

$W_4^{(0)}(x_1,x_2,x_3,x_4)$:

$W_2^{(1)}(x_1,x_2)$:



$$W_3^{(0)}(x_1,x_2,x_3) = \sum_{q_i} \operatorname*{Res}_{q=q_i} \frac{dE_q(x_1)}{y(q)-y(\bar{q})} B(x_2,q)B(x_3,\bar{q})$$

$$W_1^{(1)}(x) = \sum_{q_i} \operatorname*{Res}_{q=q_i} \frac{dE_q(x)}{y(q)-y(\bar{q})} B(q,\bar{q}),$$

$$W_4^{(0)}(x_1,x_2,x_3,x_4) = \sum_{q_i} \operatorname*{Res}_{q=q_i} \frac{dE_q(x_1)}{y(q)-y(\bar{q})} \Big(B(x_2,\bar{q})W_3^{(0)}(x_3,x_4,q) + \mathrm{perm}(x_2,x_3,x_4)\Big),$$

$$W_2^{(1)}(x_1,x_2) = \sum_{q_i} \operatorname*{Res}_{q=q_i} \frac{dE_q(x_1)}{y(q)-y(\bar{q})} \Big(W_3^{(0)}(x_2,q,\bar{q}) + 2W_1^{(1)}(q)B(x_2,\bar{q})\Big).$$

2-cut Solutions

$$W_3^{(0)}(x_1,x_2,x_3) = \frac{1}{2}\sum_{q_i} M_i^2\sigma_i'\chi_i^{(1)}(x_1)\chi_i^{(1)}(x_2)\chi_i^{(1)}(x_3),$$

$$W_1^{(1)}(x) = \frac{1}{16}\sum_{q_i}\chi^{(2)}(x) + \frac{1}{4}\sum_{q_i}\left(\frac{G}{\sigma_i'} - \frac{\sigma_i''}{12\sigma_i'}\right)\chi_i^{(1)}(x),$$

$$W_4^{(0)}(x_1,x_2,x_3,x_4) = \frac{1}{4}\sum_{q_i}\Big\{3M_i^2\left(G + \frac{2}{3}\sigma_i'' + 3\sigma_i'\frac{M_i'}{M_i}\right)\chi_i^{(1)}(x_1)\chi_i^{(1)}(x_2)\chi_i^{(1)}(x_3)\chi_i^{(1)}(x_4)$$

$$+\sum_{j\neq i} M_iM_j\left(G + \frac{2f(q_i,q_j)}{(q_i-q_j)^2}\right)\left(\chi_i^{(1)}(x_1)\chi_i^{(1)}(x_2)\chi_j^{(1)}(x_3)\chi_j^{(1)}(x_4) + \mathrm{perm}(x_2,x_3,x_4)\right)$$

$$+3M_i^2\sigma_i'\left(\chi_i^{(1)}(x_1)\chi_i^{(1)}(x_2)\chi_i^{(1)}(x_3)\chi_i^{(2)}(x_4) + \mathrm{perm}(x_1,x_2,x_3,x_4)\right)\Big\},$$

$$W_2^{(1)}(x_1,x_2) = \frac{1}{32}\sum_{q_i}\Big\{\Big\{\frac{8G^2}{\sigma_i'^2} - \left(\frac{2\sigma_i''}{3\sigma_i'^2} - \frac{11M_i'}{\sigma_i' M_i}\right)G - \frac{\sigma_i''^2}{12\sigma_i'^2} - \frac{5\sigma_i'''}{18\sigma_i'} - \frac{7\sigma_i''M_i'}{6\sigma_i'M_i} + \frac{5M_i''}{2M_i} - \frac{3M_i'^2}{M_i^2}$$

$$+\sum_{j\neq i}\frac{M_i}{M_j\sigma_j'^2}\left[-\frac{\sigma_i'\sigma_j'}{3(q_i-q_j)^2} + \left(4G - \frac{2}{3}\sigma_j'' - \sigma_j'\frac{M_j'}{M_j}\right)\left(G + \frac{2f(q_i,q_j)}{(q_i-q_j)^2}\right)\right]\Big\}\chi_i^{(1)}(x_1)\chi_i^{(1)}(x_2)$$

$$+\sum_{j\neq i}\frac{4}{\sigma_i'\sigma_j'}\left(G + \frac{2f(q_i,q_j)}{(q_i-q_j)^2}\right)^2\chi_i^{(1)}(x_1)\chi_j^{(1)}(x_2) + \left(\frac{12G}{\sigma_i'} - \frac{\sigma_i''}{2\sigma_i'} + \frac{2M_i'}{M_i}\right)\left(\chi_i^{(1)}(x_1)\chi_i^{(2)}(x_2) + (x_1\leftrightarrow x_2)\right)$$

$$+3\chi_i^{(2)}(x_1)\chi_i^{(2)}(x_2) + 5\left(\chi_i^{(1)}(x_1)\chi_i^{(3)}(x_2) + (x_1\leftrightarrow x_2)\right)\Big\}.$$

Notations:

$$\sigma(x;q_i) := \sigma(x)/(x-q_i),\quad \sigma_i' := \sigma(q_i;q_i)\quad \sigma_i'' := 2\sigma'(q_i;q_i),\quad \sigma_i''' := 3\sigma''(q_i;q_i),$$

$$\chi_i^{(n)}(x) := \mathop{\mathrm{Res}}_{q=q_i}\left(\frac{dE_q(x)}{y(q)-y(\bar{q})}\frac{1}{(q-q_i)^n}\right),\quad M_i := M(q_i).$$

Our Set-up: 1

Character variety as spectral curve

We choose the character variety as the spectral curve.

character variety of knot $\mathbf{K}$.

$$\mathcal{C} = \left\{(\ell, \mathrm{m}) \in \mathbb{C}^* \times \mathbb{C}^* | \tilde{\mathrm{A}}_{\mathrm{K}}(\ell, \mathrm{m}) = 0\right\}, \quad \tilde{\mathrm{A}}_{\mathrm{K}}(\ell, \mathrm{m}^2) := \mathrm{A}_{\mathrm{K}}(\ell, \mathrm{m}).$$

$$\text{i.e. } \mathrm{H}(\mathrm{y}, \mathrm{x}) = \tilde{\mathrm{A}}_{\mathrm{K}}(\mathrm{y}, \mathrm{x}).$$

Location of D-brane

On the information of D-brane we identify

$$\mathrm{V} = \mathrm{diag}(\xi_1, \xi_2) \quad \leftrightarrow \quad \rho(\mu) = \mathrm{diag}(\mathrm{m}, \mathrm{m}^{-1}), \quad \mathrm{m} = e^{\mathrm{u}}.$$

Actually this choice of D-brane locus is computed as

$$\bar{\mathcal{F}}^{(\mathrm{g},\mathrm{h})}(\mathrm{u}) := \sum_{\text{All signs}} \mathcal{F}^{(\mathrm{g},\mathrm{h})}(\pm\mathrm{u}, \cdots, \pm\mathrm{u}).$$

Our Set-up: 2

Expansion Parameters

We identify the expansion parameters

$$2\hbar \leftrightarrow g_s.$$

Therefore we compare the free energies with a fixed Euler number.

Fixed Euler Number

We discuss the following correspondence:

$$S_k(u) \leftrightarrow F_k(u) := 2^{k-2} \sum_{2g+h=k+1} \frac{1}{h!} \bar{\mathcal{F}}^{(g,h)}(u)$$

Computational Results perturbative

In the following, we summarize the spectral invariants on the character variety for the figure eight knot.

- Disk invariant:

$$\mathbf{F_0(u) = \int_{u_*}^{u} d\log x \ \log y, \quad A_K(y,x) = 0.}$$

  This satisfies the Neumann-Zagier's relation up to constant shift.

$$\mathbf{\partial F_0(u)/\partial u = \log y = v, \ \Rightarrow \ H(u) = F_0(u) + (\text{linear terms}).}$$

  Essential $\mathbf{u}$-dependence is consistent with the perturbative invariant $\mathbf{S_0(u)}$.

- Annulus invariant:

$$\mathbf{\frac{1}{2!}\bar{\mathcal{F}}^{(0,2)}(x) = \log \frac{1}{\sqrt{\sigma(x)}},}$$

$$\mathbf{\sigma(x) = x^2 - 2x - 1 - 2x^{-1} + x^{-2}, \quad x = m^2.}$$

  Comparing with $\mathbf{F_1(u) = \bar{\mathcal{F}}^{(0,2)}(u)/2}$, we recover the subleading term of the perturbative invariant $\mathbf{S_1(u)}$.

- 2nd order term: perturbative

  The spectral invariants $\bar{\mathcal{F}}^{(0,3)}$ and $\bar{\mathcal{F}}^{(1,1)}$ are

- $$\frac{1}{3!}\bar{\mathcal{F}}^{(0,3)}(x) = -\frac{12w^2 - 12w + 7}{12\sigma(x)^{3/2}}, \quad w := \frac{x + x^{-1}}{2},$$
- $$\bar{\mathcal{F}}^{(1,1)}(x) = -\frac{8(1+6G)w^3 - 4(11+21G)w^2 + 30w + 87 + 27G}{180\sigma(x)^{3/2}}.$$

  G: Constant in the Bergmann kernel on 2-cut curve

$$G = \frac{(q_1+q_2)(q_3+q_4) - 2(q_1q_2 + q_3q_4)}{12} - \frac{E(k)}{K(k)}(q_1-q_2)(q_3-q_4).$$

  The function $\mathbf{F_2}$ yields to

$$F_2 = -\frac{192 + 27G - 150w + 136w^2 - 84Gw^2 + 8w^3 + 48Gw^3}{180\sigma(x)^{3/2}}.$$

- 3rd order term:
  The spectral invariants $\bar{\mathcal{F}}^{(0,4)}$ and $\bar{\mathcal{F}}^{(1,2)}$ are

- $$\frac{1}{4!}\bar{\mathcal{F}}^{(0,4)}(x) = \frac{25 - 67w + 44w^2 + 24w^3 - 32w^4 + 16w^5}{12\sigma(x)^3},$$
- $$\frac{1}{2!}\bar{\mathcal{F}}^{(1,2)}(x) = \frac{1280w^6 - 9088w^5 + 13136w^4 + 22176w^3 - 17928w^2 - 26352w + 23193}{6480\sigma(x)^3} + G\frac{64w^4 - 232w^3 + 156w^2 + 378w - 243}{1080\sigma(x)^2} + G^2\frac{(4w-3)^2}{3600\sigma}.$$

Summing these contributions, we find $\mathbf{F_3}$.

$$F_3 = 2\left[\frac{1280w^6 - 448w^5 - 4144w^4 + 35136w^3 + 5832w^2 - 62532w + 36693}{6480\sigma(x)^3} + G\frac{64w^4 - 232w^3 + 156w^2 + 378w - 243}{1080\sigma(x)^2} + G^2\frac{(4w-3)^2}{3600\sigma}\right].$$

Change of $\mathbf{G}^n$

Unfortunately both of the contributions does not coincide because of the constant $\mathbf{G} \in \mathbb{C}$ in the Bergmann kernel. Bergmann kernel

$$\mathbf{G} = \frac{(q_1+q_2)(q_3+q_4) - 2(q_1q_2+q_3q_4)}{12} - \frac{E(k)}{K(k)}(q_1-q_2)(q_3-q_4).$$

But the coincidence is found by the following small changes.

1. Change 1:
   We discard the red part which consists of the elliptic functions.
   $$G^{(1)}_{\mathrm{reg}} = \frac{(q_1+q_2)(q_3+q_4) - 2(q_1q_2+q_3q_4)}{12}.$$
2. Change 2:
   We regularize $\mathbf{G}^2$ independent of $\mathbf{G}$.
   $$G^2 \to G^{(2)}_{\mathrm{reg}} = (G^{(1)}_{\mathrm{reg}})^2 - (1-k^2)(q_1-q_3)^2(q_2-q_4)^2,$$

$\boxed{\text{Conjecture}}$:
By changing $\mathbf{G}^n$ independently we will find

$$G^n \to G^{(n)}_{\mathrm{reg}}, \quad \Rightarrow \quad S_k(u) = F^{(\mathrm{reg})}_k(u).$$

Comparing Results perturbative

$$y(x) = \frac{1 - 2x - 2x^2 - x^3 + x^4 + (1 - x^2)\sqrt{1 - 2x + x^2 - 2x^3 + x^4}}{2x^2}$$

$$F_0 = \int d\log x \;\; \log y(x),$$

$$F_1 = \frac{1}{2}\log\frac{1}{\sqrt{-3 + 4w + 4w^2}}, \quad w = \frac{x + x^{-1}}{2},$$

$$F_2^{(\mathrm{reg})} = -\frac{192 + 27G_{\mathrm{reg}}^{(1)} - 150w + 136w^2 - 84G_{\mathrm{reg}}^{(1)}w^2 + 8w^3 + 48G_{\mathrm{reg}}^{(1)}w^3}{180\sigma(x)^{3/2}}.$$

$$F_3^{(\mathrm{reg})} = \frac{1280w^6 - 448w^5 - 4144w^4 + 35136w^3 + 5832w^2 - 62532w + 36693}{6480\sigma(x)^3}$$

$$+G_{\mathrm{reg}}^{(1)}\frac{64w^4 - 232w^3 + 156w^2 + 378w - 243}{1080\sigma(x)^2} + G_{\mathrm{reg}}^{(2)}\frac{(4w-3)^2}{3600\sigma}.$$

We find

$$S_0 = F_0 + \mathrm{linear}, \quad S_1 = F_1, \quad S_2 = F_2^{(\mathrm{reg})}, \quad S_3 = F_3^{(\mathrm{reg})}.$$

We also checked this coincidence for SnapPea census manifold **m009** under the same assumption.

4. Summary

Conclusions & Discussions:

- On hyperbolic 3-manifold side, we have shown the systematic computation of the WKB expansion of the Jones polynomial.
- On topological string side, we have computed the free energies on the basis of Eynard-Orantin's topological recursion.
- We compared $\mathbf{S_k}$ and $\mathbf{F_k}$ explicitly for figure eight knot case.
- For disk (NZ function) and annulus (Reidemeister torsion), we find the exact correspondence under our set-up.
- We expect that coincidence is found, if the regularizations for constant $\mathbf{G^n}$ is assumed.

Future Direcrtions

- The higher order terms in topological recursion. [Brini]
- Stokes phenomenon with higher order terms (Exact WKB)

  [Witten],[F-Manabe-Murakami-Terashima]

- Abelian branch:
  There is another expansion point with trivial holonomy representation at $\ell = 1$. $\Rightarrow$ Different expansion is found:

$$J_n(K;q) = \exp\left[S_1^{(\mathrm{abel})}(u) + \sum_{k=1}^{\infty} \hbar^k S_{k+1}^{(\mathrm{abel})}(u)\right], \quad S_1^{(\mathrm{abel})}(u) = \frac{1}{\Delta_K(m)}.$$

$\Delta_K(m)$: Alexander polynomial

- Investigations on the arithmeticity conjecture [DGLZ]

$$S_n^{(\mathrm{geom})}(0) \in \mathbb{K} = \mathbb{Q}(\mathrm{tr}\Gamma).$$

e.g. Fig.8 case $\Rightarrow \mathbb{K} = \mathbb{Q}(\sqrt{-3})$.

- Toward the free fermion realization of $SU(2)$ Chern-Simons gauge theory.

Back-ups

On Hyperbolic Geometry

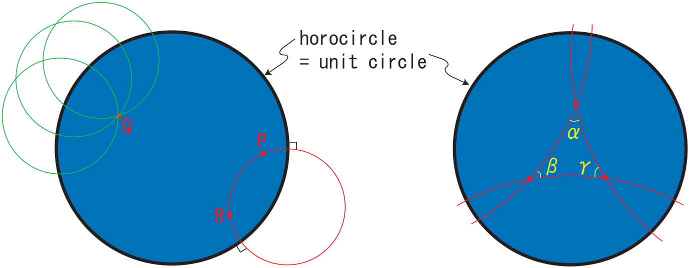
Non-Euclidean Geometry

Hyperbolic Geometry: One of non-Euclidean Geometry

Gauss, Boyai, and Lobachevsky found in 19th century.

$\Rightarrow$ The parallel postulate of Euclidean geometry is not imposed.

Poincaré's disk model



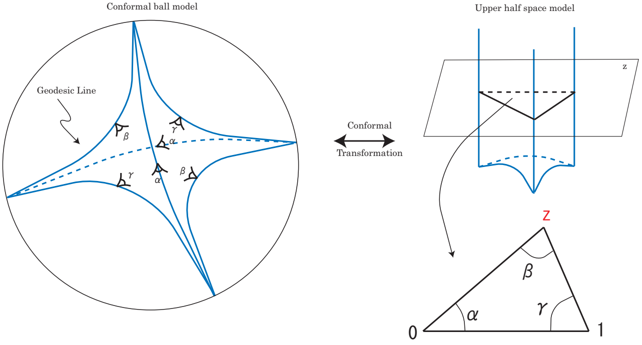
- The line is an arc of a circle orthogonal to the horocircle.
- If two lines are not intersecting, they are called parallel.
- The area **A** of triangle is determined by three inner angles.

$$\mathbf{A} = \pi - \alpha - \beta - \gamma.$$

Vertices are located at horocircle $\Rightarrow$ ideal triangle $(\mathbf{A} = \pi)$

Hyperbolic 3-manifold

- The hyperbolic 3-manifold admits a complete hyperbolic metric $\mathbf{R_{ij}} = -\mathbf{2g_{ij}}$.
- Volume w.r.t. the hyperbolic metric is finite.
- The hyperbolic 3-manifold is simplicially decomposed into the ideal tetrahedra .



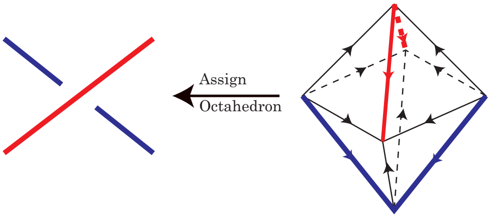
$\boxed{\alpha + \beta + \gamma = \pi}$

The ideal tetrahedron is specified by the dihedral angles α, β, γ. They are toggled into a shape parameter $\mathbf{z} \in \mathbb{C}$.

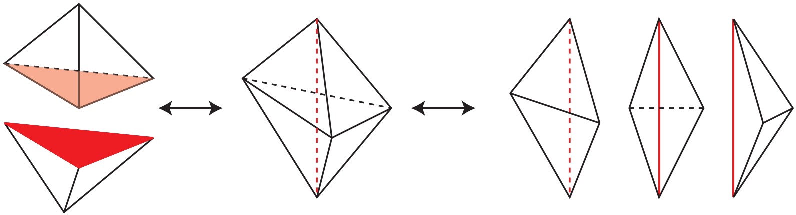
Simplicial Decomposition

Simplicial decomposition of the knot complement [Yokota]

1. Assign octahedron on each crossing of a knot

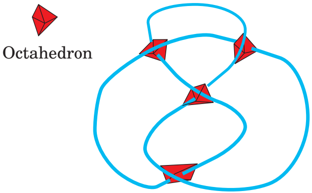


2. Reduce the number of ideal tetrahedra by Pachner 2-3 moves

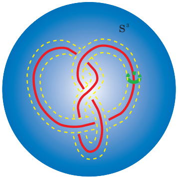


Example: Figure Eight Knot Complement:

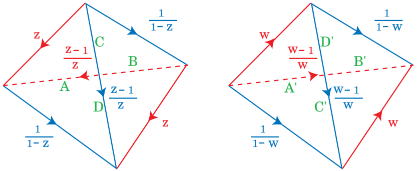
For figure eight knot one can assign 4 octahedron for each crossings.



Reducing the number of ideal tetrahedra, one finds that the complement is decomposed into 2 ideal tetrahedra.



$\cong$



Hyperbolic Volume

The volume of each ideal tetrahedra is determined w.r.t. hyperbolic metric on $\mathbb{H}^3$.

$$(x, y) \in \mathbb{R}^2, \quad z \in \mathbb{R}_+$$
$$ds^2_{\mathbb{H}^3} = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

After some elementary computations, one obtains [Milnor]

$$\mathrm{Vol}(T_{\alpha\beta\gamma}) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$$

$\Lambda(x)$: Lobachevsky's function

Dihedral angles for each ideal tetrahedra $\Rightarrow$ hyperbolic volume

$\boxed{\text{Mostow's rigidity theorem}}$

All topological informations are determined by $\pi_1(M)$

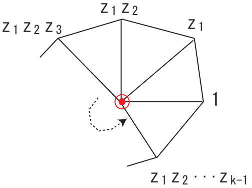
$\Rightarrow$ Dihedral angles are determined uniquely, if we solve gluing conditions .

$\Rightarrow$ Unique hyperbolic volume.

Gluing Conditions

There are two kinds of gluing conditions for ideal tetrahedra.

- Gluing conditions (bulk):



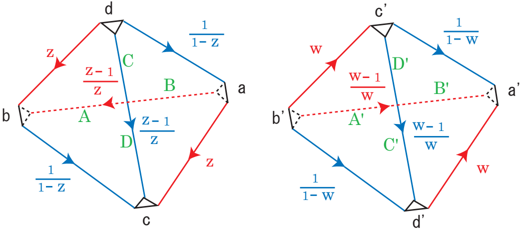
Gluing Condition

$$\prod_{i=1}^{k} z_i = 1$$

- Gluing conditions (boundary $\partial\mathbf{M} \simeq \mathbf{T}^2$):

Boundary is realized by chopping off small tetrahedra.

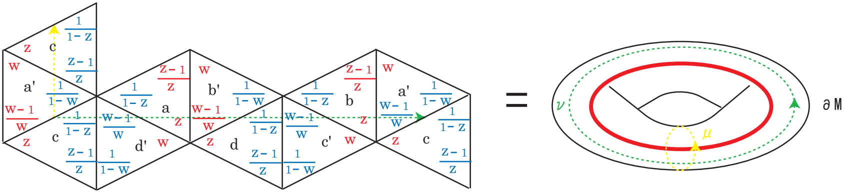
$\Rightarrow$ Each triangles are glued together completely.



Cut around ideal points by horospheres

Volume of Fig.8 Knot Complement

Solving two conditions $\longrightarrow$ Hyperbolic volume



Gluing condition along edge

Red edge : $zw\frac{z-1}{z}\frac{w-1}{w}zw = 1$

Blue edge : $\frac{1}{1-z}\frac{1}{1-w}\frac{z-1}{z}\frac{w-1}{w}\frac{1}{1-z}\frac{1}{1-w} = 1$

$$\Rightarrow (z^2 - z)(w^2 - w) = 1.$$

Completeness condition

Meridian μ: $w(1 - z) = 1$

Longitude ν: $(z^2 - z)^2 = 1$

Solution:

$$\alpha_i = \beta_i = \gamma_i = \pi/3, \quad i = 1, 2,$$

$$\mathrm{Vol}(\mathbb{S}^3\backslash 4_1) = 6\Lambda(\pi/3) = 2,0298832\ldots$$

Incomplete Structure

Generalized

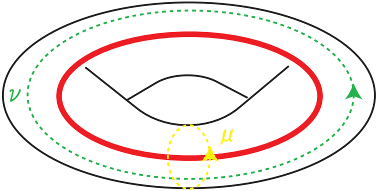
Neumann and Zagier discussed the deformation of the hyperbolic structure by changing the gluing condition of the boundary.

(Edge condition $\mathbf{z(z-1)w(w-1)=1}$ is not deformed.)

- Meridian μ: $\mathbf{w(1-z)=m^2}$
- Longitude ν: $\mathbf{(z/w)^2=\ell^2}$

Dehn surgery $\Rightarrow$ $\mathbf{M_u}$ has non-trivial $\mathbf{SL(2;\mathbb{C})}$ holonomy.

$$\rho(\mu)=\begin{pmatrix} \mathbf{m} & * \\ \mathbf{0} & \mathbf{m}^{-1} \end{pmatrix}, \quad \rho(\nu)=\begin{pmatrix} \ell & * \\ \mathbf{0} & \ell^{-1} \end{pmatrix}.$$



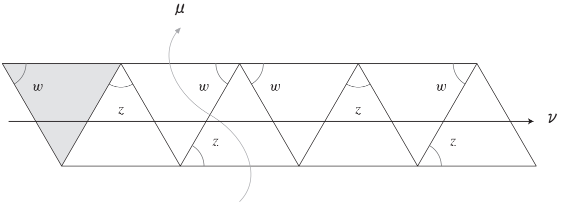
$$\mathbf{A_{4_1}(m,\ell)=m^4-m^2-2-m^{-2}+m^{-4}-\ell-\ell^{-1}=0.}$$

$\mathbf{A_K(m,\ell)}$: A-polynomial for knot K

$\{(\ell,\mathbf{m})\in\mathbb{C}^*\times\mathbb{C}^*|\mathbf{A_K}(\ell,\mathbf{m})=\mathbf{0}\}$: Character variety for knot **K**.

Complete Structure

Developing map of the boundary torus:



- Completeness condition:

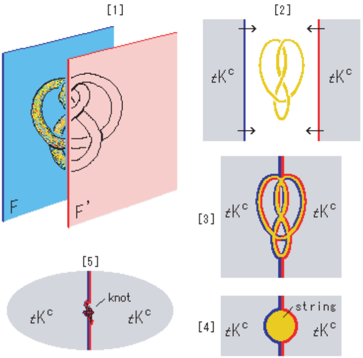
$$\sum_{i \in \mu} p_i = 0, \quad \sum_{i \in \nu} p_i = 0.$$

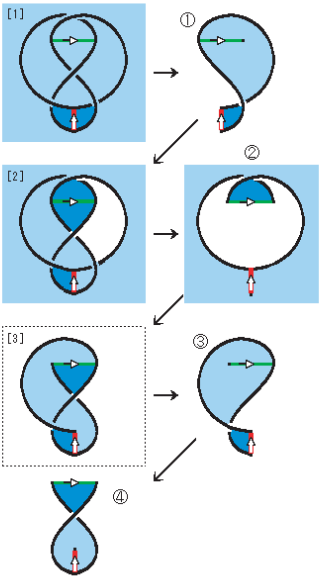
μ: Meridian cycle, ν: Longitude cycle

- Deformation of the completeness condition:

$$\sum_{i \in \mu} p_i = 2u, \quad \sum_{i \in \nu} p_i = v.$$

Explicit Gluing Processes





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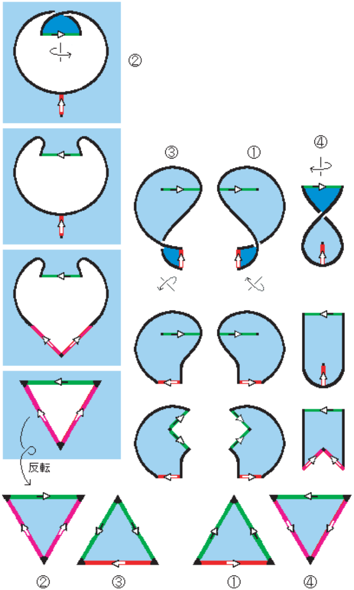
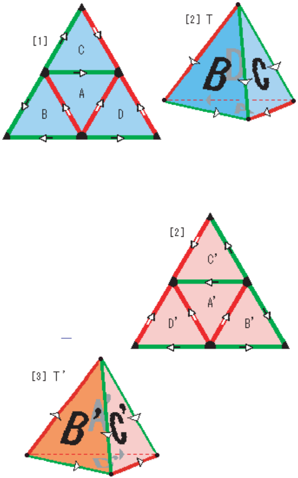




http://web.archive.org/web/20070713165857/http://www1.kcn.ne.jp/□iittoo,

Knot is localized at the tip of ideal tetrahedra.

Knot Invariants

Colored Jones Polynomial

Volume Conjecture

SU(2) Chern-Simons gauge theory:

$$S_{\rm CS}[A] = \frac{k}{4\pi}\int \mathrm{Tr}(AdA + \frac{2}{3}A\wedge A\wedge A).$$

The Wilson loop operator with spin j ($n = 2j+1$) representation:

$$W_n(K;\mathbb{S}^3) = \mathrm{tr}_n\left[P\exp\left(\oint_K A\right)\right].$$

The colored Jones polynomial $J_n(K;q)$ is related with the Wilson loop expectation value.

$$J_n(K;q=e^{4\pi i/(k+2)}) = \left\langle W_n(K;\mathbb{S}^3)\right\rangle / \left\langle W_n(U;\mathbb{S}^3)\right\rangle,$$

$$\left\langle W_n(U;\mathbb{S}^3)\right\rangle = \frac{q^n - q^{-n}}{q-q^{-1}}, \quad U: \ \text{unknot}.$$

Examples of Colored Jones Polynomial

Trefoil $\mathbf{3_1}$ and figure eight knot $\mathbf{4_1}$

$$J_n(3_1;q) = \sum_{k=0}^{n-1} \prod_{j=1}^{k} (-1)^k q^{k(k+3)/2} (q^{(n-j)/2} - q^{-(n-j)/2})(q^{(n+j)/2} - q^{-(n+j)/2}),$$

$$J_n(4_1;q) = \sum_{k=0}^{n-1} \prod_{j=1}^{k} (q^{(n-j)/2} - q^{-(n-j)/2})(q^{(n+j)/2} - q^{-(n+j)/2}).$$

Hyperbolic Knots : $\mathrm{Vol}(S^3\backslash K) \neq 0$

Non-hyperbolic Knots : $\mathrm{Vol}(S^3\backslash K)=0$

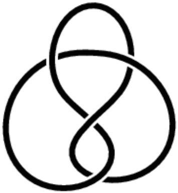
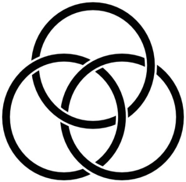
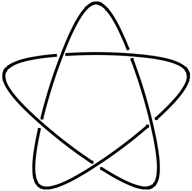


Figure eight knot



Borromean Ring

Trefoil (3,2)-torus knot

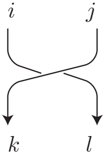


Solomon’ s Seal knot (5,2)-torus k

Volume Conjecture

Colored Jones Polynomial $J_n(K;q)$

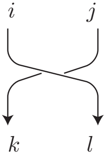
1. Assign $i_h = 0, \cdots, a-1$ for each segments in K.
2. Assign **R-matrix** for each crossings: $(a)_q := q^{a/2} - q^{-a/2}$



$$= R^{ij}_{kl}$$

$$i + j = k + l$$

$$l \geqq i\,,\ k \leqq j$$



$$= (R^{-1})^{ij}_{kl}$$

$$i + j = k + l$$

$$l \leqq i\,,\ k \geqq j$$

$$R^{ij}_{k\ell} = \sum_{m=0}^{\min(n-1-i,j)} \delta_{\ell,i+m}\delta_{k,j-m} \frac{(\ell)_q!(n-1-k)_q!}{(i)_q!(m)_q!(n-1-j)_q!} \times q^{(i-(n-1)/2)(j-(n-1)/2)-m(m+1)/4}.$$

$$(R^{-1})^{ij}_{k\ell} = \sum_{m=0}^{\min(n-1-i,j)} \delta_{\ell,i-m}\delta_{k,j+m} \frac{(k)_q!(n-1-\ell)_q!}{(j)_q!(m)_q!(n-1-i)_q!} \times (-1)^m q^{-(i-(n-1)/2)(j-(n-1)/2)-m(i-j)/2+m(m+1)/4}.$$

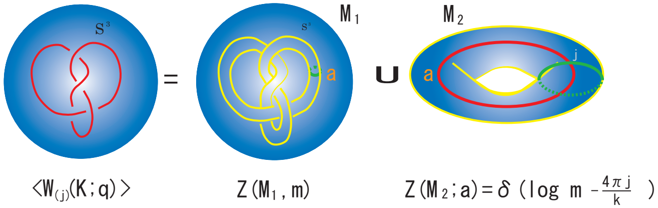
3. Sum all possible i_h's

Surgery and holonomy

In the topological field theory, the partition function is computed via the surgery procedure. [Atiyah]

$$\mathbf{Z_{CS}(M) = \int \mathcal{D}a\; Z(M_1; a) Z(M_2; a).}$$

a: Gauge field on the boundary $\partial M_1 = \partial M_2$.



The holonomy $\rho(\mu)$ on ∂M is related with the $SL(2;\mathbb{C})$ holonomy around Wilson loop: $\mathbf{m_0 = \exp\left(\frac{4\pi j\sqrt{-1}}{k+2}\right)}$. [Murayama]

$$\left\langle W_n(K;S^3)\right\rangle = \int_{\mathcal{M}_{\partial M}} \mathcal{D}A\; Z_k(n; S^1 \times D^2)[A] \cdot Z_k(S^3 \backslash N(K))[A]$$

$$= \int du\; \delta\left(u - \frac{n-1}{k}\pi\sqrt{-1}\right) Z_k(M)[u] = Z_k(M)[u_0].$$

Computation of Volume Conjecture [Kashaev],[Murakami²]

$$\lim_{n\to\infty} \frac{\log |J_n(K, q = e^{\frac{2\pi\sqrt{-1}}{n}})|}{n} = \frac{1}{2\pi}\mathrm{Vol}(S^3\backslash N(K)).$$

$J_n(K; q)$: n-colored Jones Polynomial

Example: Figure 8 knot

$$J_n(4_1, q) = \sum_{k=0}^{n-1}\prod_{j=1}^{k}(q^{(n+j)/2} - q^{-(n+j)/2})(q^{(n-j)/2} - q^{-(n-j)/2}).$$

Specialize to $q = e^{2\pi\sqrt{-1}/n}$

$$J_n(4_1, e^{2\pi\sqrt{-1}/n}) = \sum_{k=0}^{n-1}|(q)_k|^2, \quad (q)_k := \frac{L(q^{k+1/2}; q)}{L(q^{1/2}; q)} \to \frac{S_{\frac{\pi}{n}}(\frac{\pi}{n} - \pi)}{S_{\frac{\pi}{n}}(\pi - 2\pi k/n)}$$

$S_\gamma(p) := \exp\left[\frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{px}}{\sinh(\pi x)\sinh(\gamma x)}\right]$: Faddeev integral of q-dilog.

Asymptotic behavior $\mathbf{n} \longrightarrow \infty$ $(\gamma = \frac{\pi}{\mathbf{n}} \longrightarrow \mathbf{0})$

$$\mathbf{S}_\gamma(\mathbf{p}) \sim \exp\left[\frac{1}{2\sqrt{-1}\gamma}\mathrm{Li}_2(-e^{\sqrt{-1}p})\right],$$

$$\rightarrow \quad \mathbf{J}_\mathbf{n}(4_1; e^{2\pi\sqrt{-1}}) \sim \int dz \exp\left[\frac{\sqrt{-1}n}{2\pi}\left(\mathrm{Li}_2(z) - \mathrm{Li}_2(z^{-1})\right)\right]$$

$$z := q^k$$

The saddle point of $\log|\mathbf{J}_\mathbf{n}(4_1, e^{2\pi\sqrt{-1}}/n)| \Rightarrow z_0 = e^{\pi\sqrt{-1}/3}$

Asymptotic value of Jones polynomial

$$2\pi \lim_{n\to\infty} \frac{\log|\mathbf{J}_\mathbf{n}(4_1; q = e^{2\pi\sqrt{-1}/n})|}{n}$$
$$= \quad 2\mathrm{Im}[\mathrm{Li}_2(z_0)] = 2,02988\cdots = \mathrm{Vol}(S^3\backslash N(K))$$

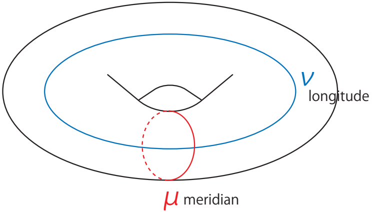
Fundamental Group and A-polynomial

Generalized Wirtinger

A-polynomial is determined by the fundamental group $\pi_1(\mathbb{S}^3 \backslash K)$.

$$\pi_1(\mathbb{S}^3 \backslash K) = \left\{ x, y | x\omega = \omega y \right\},$$

$$\omega_{4_1} := xy^{-1}x^{-1}y, \quad \omega_{3_1} := xy.$$



The meridian and longitude holonomies are identified as

$$\mu = x,$$
$$\nu_{4_1} = xy^{-1}xyx^{-2}yx^{-2}yxy^{-1}x^{-1}, \quad \nu_{3_1} = yx^2yx^{-4},$$

Holonomy rep. of hyperbolic mfd. $\rho \in PSL(2;\mathbb{C})/\Gamma$, Γ: discrete subgp.

$$\rho(\mu) = \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(\nu) = \begin{pmatrix} \ell & * \\ 0 & \ell^{-1} \end{pmatrix}.$$

Examples of A-polynomial

Applying these holonomy representations, one finds the constraint eqnation on $(\ell, \mathbf{m})$.

$$\mathbf{A}_{4_1}(\ell, \mathbf{m}) = \ell + \ell^{-1} + (\mathbf{m}^4 - \mathbf{m}^2 - 2 - \mathbf{m}^{-2} + \mathbf{m}^{-4}) = 0,$$
$$\mathbf{A}_{3_1}(\ell, \mathbf{m}) = \ell + \mathbf{m}^6 = 0.$$

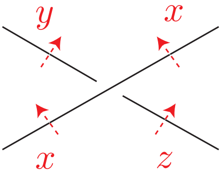
Generalized Volume Conjecture

Wirtinger Presentation of Knot Group

Generalized A-polynomial

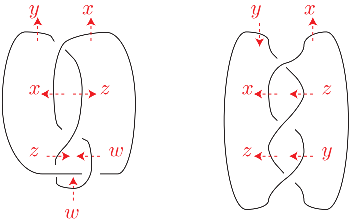
The fundamental group for the knot complement is computed via Wirtinger presentation. The algorithm is briefly summarized as follows:

1. For each intervals, non-commuting operators are assigned.



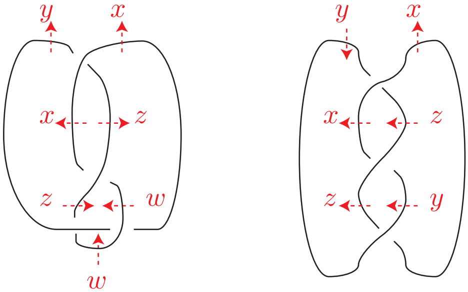
$$xy = zx$$

2. Assign non-commuting operators **x**, **y**, **z**, **w**,... for each line segments.



3. Eliminate extra operators except for **x** and **y** by crossing rule.

Meridian and Longitude in Wirtinger Algorithm Generalized

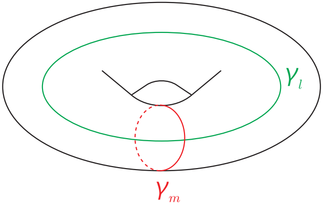


- The meridian is identified with the operator at the base point on the knot.
$$\mu = \mathbf{x}.$$
- The longitude ν is identified with the ordered product of $\mathbf{x_i}$'s which are assigned for the transversal interval at each crossings.
$$\nu = \prod_{\text{i:under crossings}} \mathbf{x_i^{\epsilon_i}},$$

$$\nu_{4_1} = \mathbf{w x^{-1} y z^{-1}}, \quad \nu_{3_1} = \mathbf{y x z x^{-3}}.$$

Properties of A-polynomial [CCGLS]

- Reciprocal $A_K(m,\ell) = \pm A_K(1/m, 1/\ell)$
- Under the change of $\pi_1(\partial M)$ basis (γ_m, γ_ℓ)



$$\begin{pmatrix} \gamma_\ell \\ \gamma_m \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma_\ell \\ \gamma_m \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;C)$$

$$\Rightarrow A_K(m,\ell) \to A_K(m^a \ell^{-c}, m^{-b}\ell^d)$$

- Tempered
  Face of Newton polygon define cyclotomic polynomial in 1-variable

Logarithmic Mahler Measure

The logarithmic Mahler measure for the polynomial $\mathbf{P(z_1, \cdots, z_n)}$ are defined as follows:

$$\mathbf{m(P)} = \frac{1}{(2\pi i)^n} \int_{|z_1|=1} \frac{dz_1}{z_1} \cdots \int_{|z_n|=1} \frac{dz_n}{z_n} \log |P(z_1, \cdots, z_n)|.$$

Jensen's formula

Let $\mathbf{P(z)}$ be a 1-parameter polynomial with complex coefficients.

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z} \log |P(z)| = \log |a_0| + \log^+ |a_i|,$$

$$P(z) = a_0 \prod_{i=1}^{d} (z - a_i),$$

where

$$\log^+ x = \begin{cases} \log x & \text{for} \quad |x| > 1, \\ 0 & \text{for} \quad |x| < 1. \end{cases}$$

Applying the Jensen's formula for each variable $\mathbf{z_i}$, one can evaluate the logarithmic Mahler measure.

Logarithmic Mahler Measure for A-polynomial

A-polynomial is a reciprocal polynomial with 2-variable $\mathbf{A}(\ell^{-1}, \mathbf{m}^{-1}) = \mathbf{m}^{a}\ell^{b}\mathbf{A}(\ell, \mathbf{m})$. This property simplifies the logarithmic Mahler measure $\mathbf{m}(\mathbf{A})$

$$\pi \mathbf{m}(\mathbf{A}) = \sum_{i=1}^{d} \int_0^{\pi} \log^{+} |\ell_k(e^{2\pi\sqrt{-1}u})| du,$$

$$\mathbf{A}(\ell, \mathbf{m}) = \ell^{p}\mathbf{m}^{q} \prod_{i=1}^{d} (\ell - \ell_k(\mathbf{m})).$$

Examples: Logarithmic Mahler measure [Boyd]

$$\pi \mathbf{m}(\mathbf{A}_{4_1}) = 2\pi d_3, \quad \pi \mathbf{m}(\mathbf{A}_{m009}) = \frac{1}{2}\pi d_7.$$

where

$$d_f = L'(\chi_{-f}, -1), \quad L(\chi_{-f}, s) = \sum_{n=1}^{\infty} \chi_{-f}(n) \frac{1}{n^s}.$$

χ_{-f}: real odd primitive character for the discriminant $-f$.

Bianchi manifold $\mathbf{M_f}$: $\mathbf{M_f} = \mathbb{H}^3/\Gamma$, $\Gamma = \mathbf{PSL}(2; \mathcal{O}_{\mathbb{Q}(\sqrt{-f})})$

$$\mathrm{Vol}\mathbf{M_f} = \frac{f\sqrt{f}}{24} L(\chi_{-f}, 2).$$

Once Punctured Torus Bundle over $\mathbb{S}^1$

Once punctured torus bundle over $\mathbb{S}^1$ is classified by the holonomy group.

$$\mathbf{M}(\varphi) = (\mathbb{T}^2 \backslash \{0\})/(x, 0) \sim (\varphi(x), 1).$$

The holonomy φ has two distinct eigenvalue $\Rightarrow$ $\mathbf{M}(\varphi)$ admit hyperbolic structure.

$$\varphi = \mathbf{L}^{s_1}\mathbf{R}^{t_1}\mathbf{L}^{s_2}\mathbf{R}^{t_2}\cdots\mathbf{L}^{s_n}\mathbf{R}^{t_n}, \quad s_i,\ t_i \in \mathbb{N}$$
$$\mathbf{L} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

- $\varphi = \mathbf{LR} \Rightarrow \mathbf{M}(\mathbf{LR}) = \mathbb{S}^3 \backslash 4_1$.
- $\varphi = \mathbf{L}^2\mathbf{R} \Rightarrow \mathbf{M}(\mathbf{L}^2\mathbf{R}) =$SnapPea census manifold **m009**.

3D Gravity

Physical meaning of the volume conjecture

Einstein-Hilbert action for 3D Euclidean gravity

$$I_{\mathrm{EH}}[g_{ij}] = -\frac{1}{4\pi}\int_M d^3x\sqrt{g}(R-2\Lambda).$$

Normalizing the cosmological constant to $\Lambda = 1$, the Einstein equation yields to

$$R_{ij} = -2g_{ij}. \quad \Rightarrow \quad (\mathrm{Hyperbolic}\ \ 3-\mathrm{manifold})$$

The same equation is also derived from the equation of motion of the following action. (1st order formulation)

$$I_{\mathrm{grav}}[e,\omega] = \frac{1}{2\pi}\int_M \mathrm{Tr}\left(e\wedge R(\omega) - \frac{1}{3}e\wedge e\wedge e\right),$$

e_i^a:dreibein, ω_i^a:spin connection $(a, i = 1, \cdots, 3)$

$$g_{ij} := \sum_{a=1}^{3} e_i^a e_j^a, \quad R(\omega) := d\omega + \omega\wedge\omega, \quad e = \sum_{a,i=1}^{3} e_i^a T^a_{SU(2)\mathrm{adj}} dx^i.$$

There is a topological term which gives rise to the same equation of motion.

$$\mathbf{I}_{\mathrm{CS}}[\mathbf{e}, \omega] = \frac{1}{4\pi} \int_{\mathbf{M}} \mathrm{Tr}\Big(\omega \wedge \mathbf{d}\omega - \mathbf{e} \wedge \mathbf{de} + \frac{2}{3} \omega \wedge \omega \wedge \omega - 2\omega \wedge \mathbf{e} \wedge \mathbf{e} \Big).$$

In general, the 1st order action yields to

$$\mathbf{I}_{\mathrm{gCS}} = \mathbf{k} \mathbf{I}_{\mathrm{CS}} + \sqrt{-1} \sigma \mathbf{I}_{\mathrm{grav}}.$$

Let $\mathbf{A}$, $\bar{\mathbf{A}}$ and $\mathbf{t}$, $\bar{\mathbf{t}}$ be the linear combinations

$$\mathbf{A} := \omega + \sqrt{-1}\mathbf{e}, \quad \bar{\mathbf{A}} := \omega - \sqrt{-1}\mathbf{e}, \quad \mathbf{t} := \mathbf{k} + \sigma, \quad \bar{\mathbf{t}} := \mathbf{k} - \sigma.$$

3D grav.w/ neg. c.c. $\Leftrightarrow$ $\mathbf{SL(2;\mathbb{C})}$ Chern-Simons gauge theory

$$\mathbf{I}_{\mathrm{gCS}} = \frac{\mathbf{t}}{8\pi} \int_{\mathbf{M}} \mathrm{Tr}\left[\mathbf{A} \wedge \mathbf{dA} + \frac{2}{3} \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \right] + \frac{\bar{\mathbf{t}}}{8\pi} \int_{\mathbf{M}} \mathrm{Tr}\left[\bar{\mathbf{A}} \wedge \mathbf{d}\bar{\mathbf{A}} + \frac{2}{3} \bar{\mathbf{A}} \wedge \bar{\mathbf{A}} \wedge \bar{\mathbf{A}} \right].$$

Under on-shell condition, the value of the action yields to

$$\mathsf{I}_{\mathrm{grav}}[\mathbf{e},\omega] \sim \int_{\mathsf{M}} \mathrm{Tr}\,\mathbf{e}\wedge\mathbf{e}\wedge\mathbf{e} = \mathrm{Vol}(\mathsf{M}).$$
$$\mathsf{I}_{\mathrm{CS}}[\mathbf{e},\omega] \sim \mathrm{CS}(\mathsf{M}).$$

$\Rightarrow$ Leading terms of $\log \mathbf{Z}_{\mathrm{CS\ grav.}}$ in the WKB expansion gives rise to the volume and Chern-Simons invariants .

Classical solution of $\mathbf{SL}(2;\mathbb{C})$ Chern-Simons gauge theory

The classical solution $\mathbf{F}=\mathbf{0}=\bar{\mathbf{F}}$ is given by the holonomy representation ρ.

$$\begin{array}{ccc} \rho : \pi_1(\mathsf{M}) & \longrightarrow & \mathbf{SL}(2;\mathbb{C}) \\ \Uplus & & \Uplus \\ \mathcal{C} & \mapsto & \rho = \mathrm{P}\exp\left[\oint_{\mathsf{C}} \mathsf{A}\right]. \end{array}$$

Moduli space $\mathbf{L}$ of the solution for $\mathbf{F}=\bar{\mathbf{F}}=\mathbf{0}$ on $\mathbf{M}=\mathbb{S}^3\backslash\mathbf{N}(\mathbf{K})$:

$$\begin{aligned} \mathsf{L} &= \mathrm{Hom}_{\mathbb{C}}\big(\pi_1(\mathbb{S}^3\backslash\mathsf{N}(\mathsf{K}));\mathsf{SL}(2;\mathbb{C})\big)/\mathrm{Gauge\ equiv.} \\ &= \left\{(\mathsf{m},\ell)\in\left(\mathbb{C}^{\times}\right)^2 \middle| \mathsf{A}_{\mathsf{K}}(\mathsf{m},\ell)=\mathbf{0}\right\}. \end{aligned}$$

The partition function for $\mathbf{SL(2;\mathbb{C})}$ Chern-Simons gauge theory on $\mathbf{M}$ is expanded perturbatively as

$$\mathbf{Z_{gCS}(M;m) = \exp(\sqrt{-1}S)\sqrt{T_K(M;m)} + \mathcal{O}(1/k, 1/\sigma).}$$

- Geometric quantization on $\mathbf{L}$ $\Rightarrow$ Leading term $\mathbf{S}$ [Gukov]

$$\begin{aligned}\mathbf{S} &= \sqrt{-1}\frac{\sigma}{\pi}\int_\gamma (-\log|\ell|d(\arg m) + \log|m|d(\arg\ell)),\\ &\quad + \frac{k}{\pi}\int_\gamma (\log|m|d(\log|m|) + \arg\ell d(\arg m)),\end{aligned}$$

γ: 1-dimensional cycle in L

In the case of $\mathbf{k = \sigma}$, the leading term simplifies.

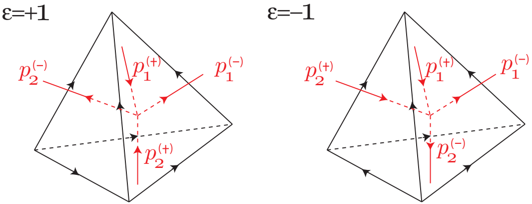
$$\mathbf{S} = \frac{\mathbf{k}}{\pi}\int_\gamma \log\ell(m)d(\log m)$$

- One loop term $\mathbf{T_K(M;m)}$ is the Reidemeister torsion of the hyperbolic manifold.

On Hikami's Invariant

State Integral Model

Hikami proposed a state integral model which gives a topological invariant for hyperbolic 3-manifold. This model can be seen as the $\mathbf{SL(2;\mathbb{C})}$ analogue of Turaev-Viro model.



For each ideal tetrahedra the following factors are assigned.

$$\langle \mathbf{p}_1^{(-)}, \mathbf{p}_2^{(-)} | \mathbf{S} | \mathbf{p}_1^{(+)}, \mathbf{p}_2^{(+)} \rangle = \frac{\delta\left(\mathbf{p}_1^{(-)} + \mathbf{p}_2^{(-)} - \mathbf{p}_1^{(+)}\right)}{\sqrt{4\pi\hbar/\mathbf{i}}} \Phi_\hbar(\mathbf{p}_2^{(+)} - \mathbf{p}_2^{(-)} + \mathbf{i}\pi + \hbar)$$

$$\times \mathbf{e}^{\frac{1}{2\hbar}\left[\mathbf{p}_1^{(-)}(\mathbf{p}_2^{(+)} - \mathbf{p}_2^{(-)}) + \frac{\mathbf{i}\pi\hbar}{2} - \frac{\pi^2 - \hbar^2}{6}\right]}, \quad \mathbf{z} = \mathbf{e}^{\mathbf{p}_2^{(+)} - \mathbf{p}_2^{(-)}}.$$

$$\langle \mathbf{p}_1^{(-)}, \mathbf{p}_2^{(-)} | \mathbf{S}^{-1} | \mathbf{p}_1^{(+)}, \mathbf{p}_2^{(+)} \rangle = \frac{\delta\left(\mathbf{p}_1^{(-)} - \mathbf{p}_1^{(+)} - \mathbf{p}_2^{(+)}\right)}{\sqrt{4\pi\hbar/\mathbf{i}}} \frac{1}{\Phi_\hbar(\mathbf{p}_2^{(-)} - \mathbf{p}_2^{(+)} - \mathbf{i}\pi - \hbar)}$$

$$\times \mathbf{e}^{\frac{1}{2\hbar}\left[-\mathbf{p}_1'(\mathbf{p}_2^{(-)} - \mathbf{p}_2^{(+)}) - \frac{\mathbf{i}\pi\hbar}{2} + \frac{\pi^2 - \hbar^2}{6}\right]}, \quad \mathbf{z} = \mathbf{e}^{\mathbf{p}_2^{(-)} - \mathbf{p}_2^{(+)}}.$$

Quantum Dilogarithm

The function $\Phi_\hbar(p)$ is called quantum dilogarithm

$$\Phi_\hbar(p) = \exp\left[\frac{1}{4}\int_{\mathbb{R}_+} \frac{e^{xz/(\pi i)}}{\sinh x \sinh \hbar x/(\pi i)} \frac{dx}{x}\right].$$

This function satisfies the pentagon relation

$$\Phi_\hbar(\hat{p})\Phi_\hbar(\hat{q}) = \Phi_\hbar(\hat{q})\Phi_\hbar(\hat{p}+\hat{q})\Phi_\hbar(\hat{p}), \quad [\hat{q},\hat{p}] = 2\hbar.$$

The perturbative expansion:

$$\Phi_\hbar(p_0+p) = \exp\left[\sum_{n=0}^{\infty} B_n\left(\frac{1}{2}+\frac{p}{2\hbar}\right)\mathrm{Li}_{2-n}(-e^{p_0})\frac{(2\hbar)^{n-1}}{n!}\right].$$

$$\mathrm{Li}_k(z) := \sum_{n=0}^{\infty} \frac{z^n}{n^k},\ B_n(x) = \sum_{k=0}^{n} {}_nC_k b_k x^{n-k}.$$

Hikami's Invariant

The partition function for the simplicially decomposed hyperbolic 3-manifold is defined by

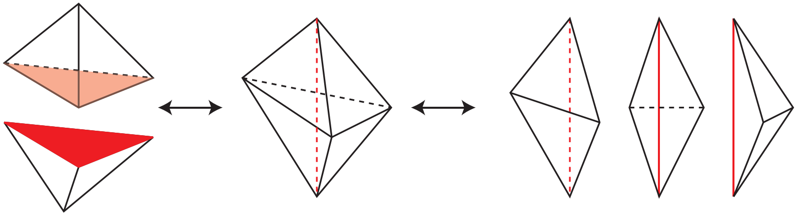
$$\mathbf{Z}_{\hbar}(\mathbf{M};\mathbf{u}) = \sqrt{2}\int \mathbf{dp}\,\delta_{\mathbf{C}}(\mathbf{p};\mathbf{u})\,\delta_{\mathbf{G}}(\mathbf{p})\prod_{\mathbf{i}=1}^{\mathbf{N}}\langle \mathbf{p}_{2\mathbf{i}-1}^{(-)},\mathbf{p}_{2\mathbf{i}}^{(-)}|\mathbf{S}^{\epsilon_{\mathbf{i}}}|\mathbf{p}_{2\mathbf{i}-1}^{(+)},\mathbf{p}_{2\mathbf{i}}^{(+)}\rangle,$$

$\delta_{\mathbf{G}}(\mathbf{p})$ Gluing condition along edges. $(\mathbf{z}_{\mathbf{i}} = \mathbf{e}^{\epsilon_{\mathbf{i}}(\mathbf{p}_{2\mathbf{i}}^{(+)}-\mathbf{p}_{2\mathbf{i}}^{(-)})})$

$\delta_{\mathbf{C}}(\mathbf{p};\mathbf{u})$: Gluing condition around meridian and longitude.

$$\sum_{\mathbf{i}\in\mu}\mathbf{p}_{\mathbf{i}} = \mathbf{2u}, \quad \mathbf{u} = \mathbf{0} \Rightarrow \text{Complete}$$

This partition function is invariant under 2-3 Pachner moves by pentagon relation.

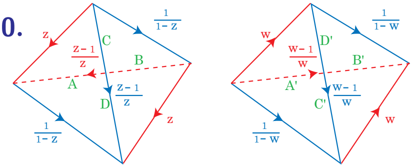


Saddle Point of Hikami's Invariant

Now we discuss $\hbar \to 0$ limit of the partition function of the state integral model. The leading term $\mathcal{O}(1/\hbar)$ is found by the steepest descent method.

$$Z_\hbar(M;u) \sim \int \prod_i dp_i e^{\frac{V(p_i)}{2\hbar}}, \quad \Phi_\hbar(p) \sim \exp\left[\frac{1}{\hbar}\mathrm{Li}_2(-e^p)\right],$$

$$\frac{\partial V(p_i)}{\partial p_i} = 0.$$



Example: Figure eight knot complement

$$\begin{aligned} Z_\hbar(\mathbb{S}^3\backslash 4_1;u) &= \frac{e^{u+2\pi i u/\hbar}}{\sqrt{2\pi\hbar}} \int dp \frac{\Phi_\hbar(p+i\pi+\hbar)}{\Phi_\hbar(-p-2u-\pi i-\hbar)} \\ &\sim \frac{e^{u+2\pi i u/\hbar}}{\sqrt{2\pi\hbar}} \int dp\, e^{-\frac{1}{2\hbar}V(p)}, \end{aligned}$$

$$V(p) = \left[\mathrm{Li}_2(e^p) - \mathrm{Li}_2(e^{-p-2u}) - 4u(u+p)\right].$$

Saddle Point Value of Figure Eight Knot Complement

The solution of the saddle point $\partial V(p;u)/\partial p = 0$ is

$$p_0(u) = \log\left[\frac{1-m^2-m^4+\sqrt{1-2m^2-m^4-2m^6+m^8}}{2m^3}\right], \quad m := e^u.$$

Complete case:

For $u = 0$, the saddle point value yields to $p_0 = e^{\pi i/3}$. Plugging this value into the above $V(p)$, one finds

$$V(p_0) = \mathrm{Li}_2(e^{\pi i/3}) - \mathrm{Li}_2(e^{-\pi i/3}) = 2,02988.. = \mathrm{Vol}(\mathbb{S}^3\backslash 4_1).$$

Incomplete case:

The saddle point value of the potential $V(p_0, u)$ satisfies the Neumann-Zagier's relation.

$$v := \frac{\partial V(p_0(u))}{\partial u}, \quad \ell = e^v,$$
$$A_{4_1}(\ell, m) = \ell + \ell^{-1} + m^4 + m^2 + 1 + m^{-2} + m^{-4} = 0,$$

Perturbative Expansion of Hikami's Invariant [Dimofte et.al.]

Utilizing the expansion of the quantum dilogarithm function, one can expand the partition function $\mathbf{Z}_\hbar(\mathbf{M};\mathbf{u})$ w.r.t. $\hbar$.

$$\mathbf{Z}_\hbar(\mathbf{M};\mathbf{u}) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{2\pi i u}{\hbar}+u} \int dp\; e^{\Upsilon(\hbar,p;u)}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} e^{u+V(p_0)/\hbar} \int dp\; e^{-\frac{b^2}{2\hbar}p^2} \exp\left[\frac{1}{\hbar}\sum_{j=3}^{\infty}\Upsilon_{j,-1}p^j + \sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\Upsilon_{i,k}p^j\hbar^k\right]$$

$$\Upsilon(\hbar, p_0+p; u) = \sum_{j=0}^{\infty}\sum_{k=-1}^{\infty}\Upsilon_{j,k}(p_0,u)p^j\hbar^k, \quad b(p,u) := -\frac{\partial^2}{\partial p^2}V(p;u).$$

Neglecting $\mathcal{O}(e^{-\mathrm{const}/\hbar}) \Rightarrow$ Gaussian integral $\int dp\; p^n e^{-\frac{b}{2\hbar}p^2}$.

$$\mathbf{Z}_\hbar(\mathbf{M};\mathbf{u}) = \exp\left[\frac{1}{\hbar}V(p_0) - \frac{1}{2}\log b + u + \sum_{k=1}^{\infty}\hbar^k S_{k+1}\right].$$

Computational Results

$$\ell(m) = \frac{1 - 2m^2 - 2m^4 - m^6 + m^8 + (1 - m^4)\sqrt{1 - 2m^2 + m^4 - 2m^6 + m^8}}{2m^4},$$

$$S_0'(u) = V'(p_0(u)) = \log \ell(m), \quad \sigma_0(m) := m^{-4} - 2m^{-2} + 1 - 2m^2 + m^4,$$

$$S_1(u) = -\frac{1}{2}\log b(p,u) = -\frac{1}{2}\log\left[\frac{\sqrt{\sigma_0(m)}}{2}\right], \quad u = \log m$$

$$S_2(u) = \frac{-1}{12\sigma_0(m)^{3/2}m^6}(1 - m^2 - 2m^4 + 15m^6 - 2m^8 - m^{10} + m^{12}).$$

$$S_3(u) = \frac{2}{\sigma_0(m)^3 m^6}(1 - m^2 - 2m^4 + 5m^6 - 2m^8 - m^{10} + m^{12}).$$

Topological Recursion for 2-cut

Bergmann Kernel for 2-cut Curve

For the curve $\mathbf{y^2 = \sigma(x)}$ with 2-cuts, the Bergmann kernel is given explicitly. [Akemann][BKMP][Manabe]

$$\frac{B(x_1, x_2)}{dx_1\, dx_2} = \frac{dx_1\, dx_2}{\sqrt{\sigma(x_1)\sigma(x_2)}} \left(\frac{\sqrt{\sigma(x_1)\sigma(x_2)} + f(x_1, x_2)}{2(x_1 - x_2)^2} + \frac{G}{4} \right),$$

$$f(p, q) := p^2q^2 - pq(p+q) - \frac{1}{6}(p^2 + 4pq + q^2) - (p+q) + 1.$$

G: Constant that makes $\mathbf{B(x_1, x_2)}$ zero A-period.

$$G = \frac{e_3}{3} - \frac{E(k)}{K(k)}(q_1 - q_2)(q_3 - q_4),$$

$$e_3 = \frac{(q_1 + q_2)(q_3 + q_4) - 2(q_1q_2 + q_3q_4)}{12}, \quad k = \frac{(q_1 - q_2)(q_3 - q_4)}{(q_1 - q_3)(q_2 - q_4)}.$$

Regularization

From curve on $\mathbb{C}$ to $\mathbb{C}^*$

One has to change variables from $\mathbb{C}$ to $\mathbb{C}^*$ to discuss the mirror curve. [Marino]

$$y(x) = \frac{a(x) \pm \sqrt{\sigma(x)}}{c(p)}, \quad \sigma(x) := \prod_{i=1}^{2n}(x - q_i).$$

- Change of variables:

$$y \to v := \log y = \log\left[\frac{a(x) + \sqrt{\sigma(x)}}{c(x)}\right].$$

The branching structure of **v** is captured by the following identity:

$$\log\left[\frac{a + \sqrt{\sigma}}{c}\right] = \frac{1}{2}\log\frac{a^2 - \sigma}{c^2} + \tanh^{-1}\left(\frac{\sqrt{\sigma}}{a}\right).$$

The effective curve is given by

$$y(x) = \frac{1}{x}\tanh^{-1}\left[\frac{\sqrt{\sigma(x)}}{a(x)}\right] =: M(x)\sqrt{\sigma(x)},$$

$$M(x) = \frac{1}{x\sqrt{\sigma(x)}}\tanh^{-1}\left[\frac{\sqrt{\sigma(x)}}{a(x)}\right] : \text{Moment fn.}$$

Results for **m009**

Once Punctured Torus Bundle over $\mathbb{S}^1$ [Jorgensen]

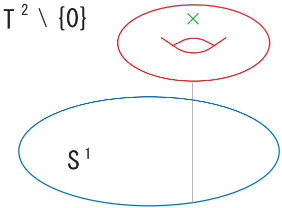
Once punctured torus bundle over $\mathbb{S}^1$ is classified by the holonomy group.

$$\mathbf{M}(\varphi) = (\mathbb{T}^2 \backslash \{0\})/(\mathbf{x}, 0) \sim (\varphi(\mathbf{x}), 1).$$

The holonomy φ has two distinct eigenvalue
$\Rightarrow$ $\mathbf{M}(\varphi)$ admit hyperbolic structure.

$$\varphi = \mathbf{L}^{s_1}\mathbf{R}^{t_1}\mathbf{L}^{s_2}\mathbf{R}^{t_2}\cdots\mathbf{L}^{s_n}\mathbf{R}^{t_n}, \quad s_i,\ t_i \in \mathbb{N}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$



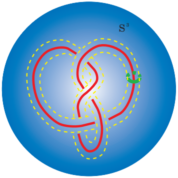
Examples:

- $\varphi = \mathbf{LR} \Rightarrow \mathbf{M}(\mathbf{LR}) = \mathbb{S}^3 \backslash 4_1$.
- $\varphi = \mathbf{L}^2\mathbf{R} \Rightarrow \mathbf{M}(\mathbf{L}^2\mathbf{R}) =$ SnapPea census manifold **m009**.

Examples of Simplicial Decomposition

The simplicial decomposition of the once punctured torus bundle over circle is performed explicitly.

- $\varphi = \mathbf{LR}$ case:



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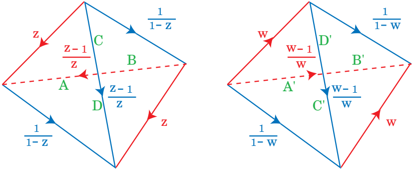
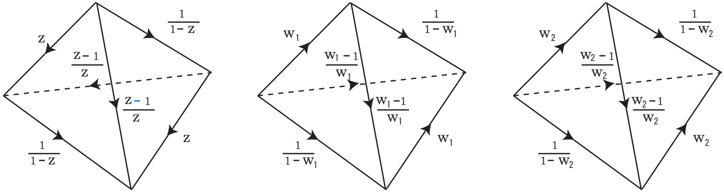
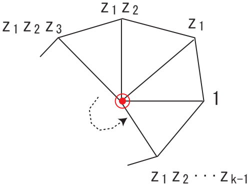


- $\varphi = \mathbf{L^2R}$ case:



Gluing Conditions for **m009**

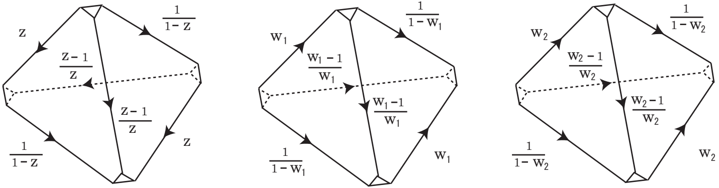
- Gluing condition for edges



Gluing Condition

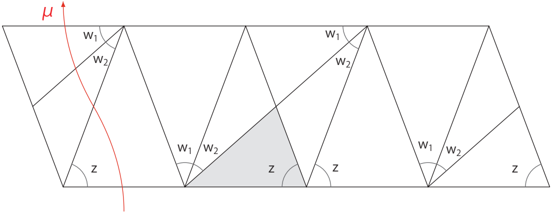
$$\prod_{i=1}^{k} z_i = 1$$

- Gluing conditions (boundary $\partial \mathbf{M} \simeq \mathbf{T}^2$):
  Boundary is realized by chopping off small tetrahedra.
  $\Rightarrow$ Each triangles are glued together completely.



Complete Structure

Developing map of the boundary torus:



- Completeness condition:

$$\sum_{\mathbf{i}\in\boldsymbol{\mu}} \mathbf{p_i} = \mathbf{0}, \quad \sum_{\mathbf{i}\in\boldsymbol{\nu}} \mathbf{p_i} = \mathbf{0}.$$

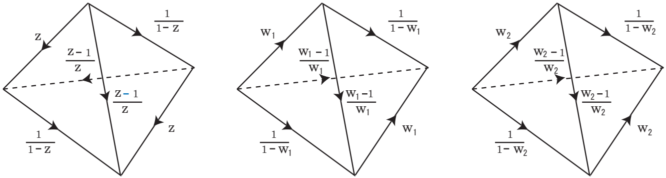
$\boldsymbol{\mu}$: Meridian cycle, $\boldsymbol{\nu}$: Longitude cycle

- Deformation of the completeness condition:

$$\sum_{\mathbf{i}\in\boldsymbol{\mu}} \mathbf{p_i} = \mathbf{2u}, \quad \sum_{\mathbf{i}\in\boldsymbol{\nu}} \mathbf{p_i} = \mathbf{v}.$$

Example: SnapPea Census Manifold m009

$$\mathbf{Z}_\hbar(\mathrm{m009};\mathrm{u})$$
$$= \int \prod_{i=1}^{6} \mathrm{d}p_i \delta_C(p;u) \langle p_1, p_5|S^{-1}|p_6, p_3\rangle \langle p_6, p_4|S^{-1}|p_2, p_5\rangle \langle p_3, p_2|S|p_4, p_1\rangle$$
$$= \frac{1}{\sqrt{2\pi\hbar}} \int \mathrm{d}p_1 \mathrm{d}p_2 e^{-\frac{2}{\hbar}\left[u(u+p_1+p_2)+\frac{1}{2}(p_1+p_2/2)^2-\frac{\pi^2}{12}-\frac{\hbar^2}{12}-\frac{\pi}{4}\hbar\right]-u}$$
$$\times \frac{\Phi_\hbar(-p_1-2u+i\pi+\hbar)}{\Phi_\hbar(-p_1-p_2-2u-\pi i-\hbar)\Phi_\hbar(2p_1+p_2+2u-\pi i-\hbar)}$$



Shape parameters & Meridian holonomy:

$$z_1 = e^{p_1-p_2}, \quad w_1 = e^{p_3-p_5}, \quad w_2 = e^{p_5-p_4},$$
$$p_3 - p_4 - p_1 + p_2 = 2u.$$

Saddle Point of Hikami's Invariant

Now we discuss $\hbar \to 0$ limit of the partition function of the state integral model. The leading term $\mathcal{O}(1/\hbar)$ is found by the steepest descent method.

$$Z_\hbar(M;u) \sim \int \prod_i dp_i e^{\frac{V(p_i)}{2\hbar}}, \quad \Phi_\hbar(p) \sim \exp\left[\frac{1}{\hbar}\mathrm{Li}_2(-e^p)\right],$$

Example: SnapPea census manifold **m009**

$$Z_\hbar(m009;u) \sim \int dp_1 dp_2\; e^{-\frac{1}{2\hbar}V(p_1,p_2;u)},$$

$$V(p_1,p_2) = \mathrm{Li}_2(e^{-p_1-2u}) - \mathrm{Li}_2(e^{-p_1-2p_2-2u}) - \mathrm{Li}_2(e^{2p_1+2p_2+2u}) \\ -4u(u+p_1+2p_2) - 2(p_1+p_2)^2 + \frac{\pi^2}{6}.$$

A solution of the saddle point $\partial V(p_j;u)/\partial p_i = 0$ is

$$p_1^{(0)}(u) = \log\left[\frac{-1+m^2+m^4+\sqrt{1-2m^2-5m^4-2m^6+m^8}}{2m^3}\right],$$

$$p_2^{(0)}(u) = \frac{1}{2}\log\frac{1+m^2 e^{p_1^{(0)}}}{m^2(1+m^2)e^{2p_1^{(0)}}}, \quad m := e^u.$$

Saddle Point Value of m009

Complete case:

For $\mathbf{u} = \mathbf{0}$ the saddle point is $(e^{p_1^{(0)}}, e^{2p_2^{(0)}}) = (\frac{7+i\sqrt{7}}{4}, \frac{-1-i\sqrt{7}}{2})$.

Plugging these values into $\mathbf{V}(\mathbf{p}_1^{(0)}, \mathbf{p}_2^{(0)})$, one finds

$$\begin{aligned} \mathbf{V}(\mathbf{p}_1^{(0)}, \mathbf{p}_2^{(0)}) &= i[2,66674\ldots - i2\pi^2 \cdot 0,02083\ldots] \\ &= i[\mathrm{Vol}(m009) + 2\pi^2 i\mathrm{CS}(m009)]. \end{aligned}$$

Incomplete case:

The saddle point value of the potential $\mathbf{V}(\mathbf{p}_0, \mathbf{u})$ satisfies the Neumann-Zagier's relation.

$$\mathbf{v} := \frac{\partial \mathbf{V}(\mathbf{p}_0(\mathbf{u}), \mathbf{p}_1(\mathbf{u}))}{\partial \mathbf{u}}, \quad \ell = e^{\mathbf{v}},$$

$$A_{m009}(\ell, m) = m^2\ell^{-1} + m^4\ell - 1 + 2m^2 + 2m^4 - m^6 = 0.$$

Remark [Boyd-Rodriguez-Villegas]

The volume is also given by the logarithmic Mahler measure

$\mathrm{Vol}(m009) = \pi m(A_{m009}) = d_7/2$.

Perturbative Expansion of Hikami's Invariant

Utilizing the expansion of the quantum dilogarithm function, one can expand the partition function $\mathbf{Z}_\hbar(\mathbf{M};\mathbf{u})$ w.r.t. $\hbar$. q-dilog

$$
\begin{aligned}
Z_\hbar(m009;u) = \frac{e^{u+\frac{1}{\hbar}V(p_1^{(0)},p_2^{(0)})}}{2\sqrt{2}\pi\hbar}\int dp_1 dp_2\; e^{-\frac{b_{11}p_1^2+b_{22}p_2^2+2b_{12}p_1p_2}{2\hbar}} \\
\times \exp\left[\frac{1}{\hbar}\sum_{i+j=3}^{\infty}\Upsilon_{i,j,-1}p_1^i p_2^j + \sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty} p_1^i p_2^j \hbar^k\right],
\end{aligned}
$$

$$
b_{ij}(p_1,p_2) := -\frac{\partial^2}{\partial p_i \partial p_j}V(p_1,p_2).
$$

Neglecting $\mathcal{O}(e^{-\mathrm{const}/\hbar})$

$\Rightarrow$ Gaussian integrals $\int dp_1 dp_2\; p_1^a p_2^b e^{-\frac{b_{ij}p_ip_j}{2\hbar}}$.

$$
Z_\hbar(M;u) = \exp\left[\frac{1}{\hbar}V(p_i^{(0)}(u)) - \frac{1}{2}\log\det b + \sum_{k=1}^{\infty}\hbar^k S_{k+1}(u)\right].
$$

Computational Results

$$\ell(m) = \frac{-1 + m^2 + m^4 + \sqrt{1 - 2m^2 - 5m^4 - 2m^6 + m^8}}{2m^3}$$

$$S_0'(u) = V'(p_1^{(0)}(u), p_2^{(0)}(u)) = \log \ell(m), \quad \sigma_0(m) := m^{-4} - 2m^{-2} - 5 - 2m^2 + m^4,$$

$$S_1(u) = -\frac{1}{2} \log \det b(p,u) = -\frac{1}{2} \log \left[\frac{\sqrt{\sigma_0(m)}}{2} \right], \quad u = \log m$$

$$S_2(u) = \frac{-1}{48\sigma_0(m)^{3/2} m^6} (5 - 11m^2 + 22m^4 + 105m^6 + 22m^8 - 11m^{10} + 5m^{12}).$$

$$S_3(u) = \frac{2}{\sigma_0(m)^3 m^{12}} m^4 (1 - m^2 + m^4)(1 + 9m^2 + 4m^4 - 9m^6 + 4m^8 + 9m^{10} + m^{12}).$$

$S_1(u)$ coincides with the Reidemeister torsion. [Porti]

Computational Results for **m009** in top. string

- 2nd order term:
  The spectral invariants $\bar{\mathcal{F}}^{(0,3)}$ and $\bar{\mathcal{F}}^{(1,1)}$ are

- $\frac{1}{3!}\bar{\mathcal{F}}^{(0,3)}(x) = -\frac{8w^2+36w^2+6w+19}{48\sigma(x)^{3/2}}, \quad w := \frac{x+x^{-1}}{2},$
- $\bar{\mathcal{F}}^{(1,1)}(x) = -\frac{(40-72G)w^3+(-12+156G)w^2+(-210+42G)w-217-147G}{336\sigma(x)^{3/2}}.$

G: Constant in the Bergmann kernel on 2-cut curve

$$G = \frac{(q_1+q_2)(q_3+q_4)-2(q_1q_2+q_3q_4)}{12} - \frac{E(k)}{K(k)}(q_1-q_2)(q_3-q_4).$$

The function $\mathbf{F_2}$ yields to

$$F_2 = -\frac{(16+72G)w^3+(264-156G)w^2+(252-42G)w+350+147G}{336\sigma(x)^{3/2}}.$$

- 3rd order term:
  The spectral invariants $\bar{\mathcal{F}}^{(0,4)}$ and $\bar{\mathcal{F}}^{(1,2)}$ are

$$\frac{1}{4!}\bar{\mathcal{F}}^{(0,4)}(x) = \frac{64w^6 + 832w^5 - 144w^4 + 3168w^3 + 1532w^2 - 2060w + 1257}{768\sigma(x)^3},$$

$$\frac{1}{2!}\bar{\mathcal{F}}^{(1,2)}(x) = \frac{7862w^6 - 116544w^5 + 341968w^4 + 841120w^3 - 443884w^2 - 350644w + 556003}{112896\sigma(x)^3}$$

$$+G\frac{144w^4 - 816w^3 + 952w^2 + 1988w - 931}{4704\sigma(x)^2} + G^2\frac{(6w-7)^2}{12544\sigma(x)}.$$

Summing these contributions, we find $\mathbf{F_3}$.

$$F_3 = \frac{370391 - 326732w - 109340w^2 + 653408w^3 + 160400w^4 + 2880w^5 + 8635w^6}{56448\sigma(x)^3}$$

$$+G\frac{144w^4 - 816w^3 + 952w^2 + 1988w - 931}{4704\sigma(x)^2} + G^2\frac{(6w-7)^2}{12544\sigma(x)}.$$

Comparing Results

$$y(x) = \frac{1 - 2x - 2x^2 - x^3 + (1-x)\sqrt{1 - 2x - 5x^2 - 2x^3 + x^4}}{2x^2}$$

$$F_0 = \int d\log x \ \log y(x),$$

$$F_1 = \frac{1}{2}\log\frac{1}{\sqrt{-7 - 4w + 4w^2}}, \quad w = \frac{x + x^{-1}}{2},$$

$$F_2^{(\mathrm{reg})} = -\frac{(16 + 72G_{\mathrm{reg}}^{(1)})w^3 + (264 - 156G_{\mathrm{reg}}^{(1)})w^2 + (252 - 42G_{\mathrm{reg}}^{(1)})w + 350 + 147G_{\mathrm{reg}}^{(1)}}{336\sigma(x)^{3/2}}$$

$$F_3^{(\mathrm{reg})} = \frac{370391 - 326732w - 109340w^2 + 653408w^3 + 160400w^4 + 2880w^5 + 8635w^6}{56448\sigma(x)^3}$$

$$+G_{\mathrm{reg}}^{(1)}\frac{144w^4 - 816w^3 + 952w^2 + 1988w - 931}{4704\sigma(x)^2} + G_{\mathrm{reg}}^{(2)}\frac{(6w-7)^2}{12544\sigma(x)}.$$

We find

$$S_0 = F_0 + \mathrm{linear}, \quad S_1 = F_1, \quad S_2 = F_2^{(\mathrm{reg})}, \quad S_3 = F_3^{(\mathrm{reg})}.$$

We also checked this coincidence for fig.8 knot complement under the same assumption.

Level-Rank Large n Duality

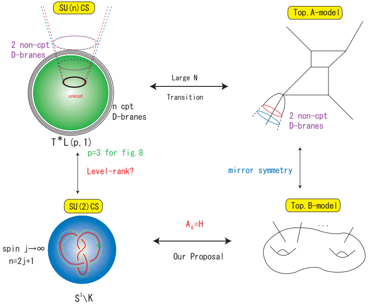
Topological Vertex/CS computation

$$Z_D(x_1,\cdots,x_n)=\sum_R Z_R \mathrm{Tr}_R V,\quad V=\mathrm{diag}(x_1,\cdots,x_p).$$

$$\log Z_D=\sum_{g=0}^{\infty}\sum_{h=1}^{\infty}\sum_{w_1,\cdots,w_h}\frac{1}{h!}g_s^{2g-2+h}F^{(g)}_{w_1,\cdots,w_h}\mathrm{Tr}V^{w_1}\cdots\mathrm{Tr}V^{w_h}.$$

We have identified

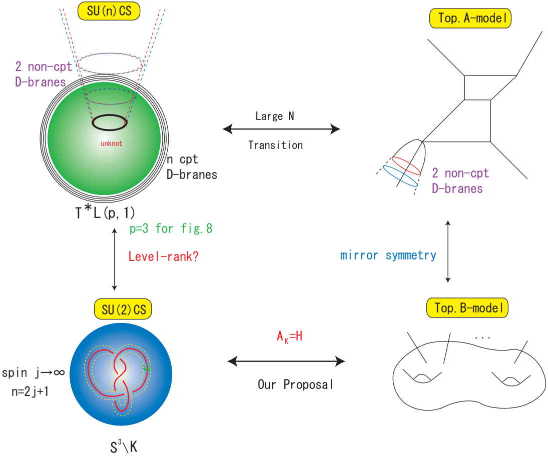
$$V=\mathrm{diag}(x_1,x_2)\leftrightarrow\rho(\mu)=\mathrm{diag}(m,m^{-1})\in SL(2;\mathbb{C}),\quad m=e^u.$$



Motivation of Our Research

- Realization of 3D quantum gravity in top. string
- Non-perturbative completion (e.g. Witten's ECFT)
- Integrability ($\mathcal{D}$-module structure) of knot invariants
- Large **n** duality not for rank but for level

  $\Rightarrow$ Novel class of duality

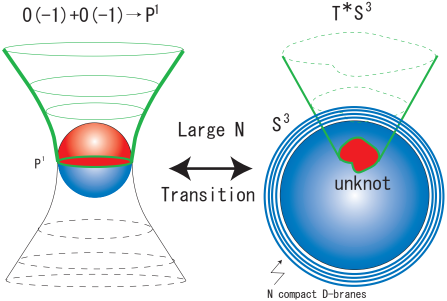


$\mathcal{D}$-module structure in top. string

Large N transition

[Gopakumar-Vafa],[Ooguri-Vafa]

One of the famous open/closed duality in topological string is large **N** transition .



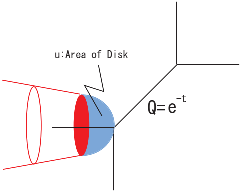
On **N** D-branes, **U(N)** Chern-Simons gauge theory is realized.

$$Z_{\text{open}}\left(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1;u\right)=\sum_R e^{-uR}\left\langle W_R(\bigcirc;q)\right\rangle^{U(N)}.$$

$e^{-g_s}=q=e^{\frac{2\pi\sqrt{-1}}{k+N}}$: string coupling,

$t=g_sN$: volume of $\mathbb{P}^1$, $\quad$ u: Area of disk.

Example: $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbf{P}^1$



$$\mathsf{Z}(t,u) = \exp[\sum_{g,h} g_s^{2g-2+h} \mathcal{F}^A_{g,h}(t,u)] = \mathsf{Z}_{\mathrm{closed}}(e^{-t};q) \cdot \mathsf{Z}_{\mathrm{open}}(e^{-u};q)$$

$$= \mathsf{M}(Q;q) \cdot \frac{\mathsf{L}(e^{-u};q)}{\mathsf{L}(Qe^{-u};q)}. \quad q := e^{-g_s}, \quad g_s : \text{string coupling}$$

$$\mathsf{M}(x;q) := \prod_{n=1}^{\infty} \frac{1}{(1-xq^n)^n} \quad : \text{McMahon function}$$

$$\mathsf{L}(x;q) := \prod_{n=1}^{\infty} (1-xq^{n-1/2}) \quad : \text{Quantum Dilogarithm}$$

Mirror Symmetry [Candelas et.al.],[Hosono et.al.]

$$\text{A-model on } \mathbf{X} \simeq \text{B-model on } \mathbf{X}^\vee$$

$$\mathbf{H}^{1,1}(\mathbf{X}) \simeq \mathbf{H}^{2,1}(\mathbf{X}^\vee), \quad \mathbf{H}^{2,1}(\mathbf{X}) \simeq \mathbf{H}^{1,1}(\mathbf{X}^\vee).$$

The bi-rational map between A-model and B-model is so-called mirror map .

$$\langle \mathcal{O}_1^{\mathbf{B}} \cdots \mathcal{O}_{\mathbf{n}}^{\mathbf{B}} \rangle^{\mathbf{classical}}_{\mathbf{B-model}} \quad \longrightarrow \quad \langle \mathcal{O}_1^{\mathbf{A}} \cdots \mathcal{O}_{\mathbf{n}}^{\mathbf{A}} \rangle^{\mathbf{quantum}}_{\mathbf{A-model}}$$

Mirror CY of conifold

$$\mathbf{X}^\vee = \{(\mathbf{z}, \mathbf{w}, \mathbf{x}, \mathbf{y}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^2 | \mathbf{zw} = \mathbf{H}(\mathbf{x}, \mathbf{y})\},$$
$$\mathbf{H}(\mathbf{x}, \mathbf{y}) = \mathbf{1} - \mathbf{Qx} - \mathbf{y} + \mathbf{xy},$$
$$\boldsymbol{\Sigma} := \{(\mathbf{x}, \mathbf{y}) \in \mathbf{C}^\times \times \mathbf{C}^\times \mid \mathbf{H}(\mathbf{x}, \mathbf{y}) = \mathbf{0}\}$$

$\mathbf{Z_{open}}$ is given by a one-point function (BA-function) of a free fremion on $\boldsymbol{\Sigma}$ inside mirror CY. [ADKMV]

$$\mathbf{Z_{open}(X;u) = \langle \psi(e^{-u}) \rangle_{\Sigma}}.$$

$\boxed{\text{Schrödinger equation}}$ (conjecture):

$$\mathbf{\hat{H}(e^{-\hat{x}}, e^{\hat{p}}) Z_{open}(X;u) = 0},$$

$$\mathbf{\hat{x} := u - g_s/2, \quad \hat{p} := -g_s \partial_u, \quad [\hat{x}, \hat{p}] = g_s, \quad e^{-\hat{x}} e^{\hat{p}} = q e^{\hat{p}} e^{-\hat{x}}}.$$

Actually the open string partition function for conifold satisfies

$$\mathbf{\left[1 - e^{-g_s \partial_u} - Q e^{-u} q^{1/2} + (e^{-u} q^{1/2})(e^{-g_s \partial_u})\right] Z_{open}(u) = 0}.$$