Deep into the Amplituhedron

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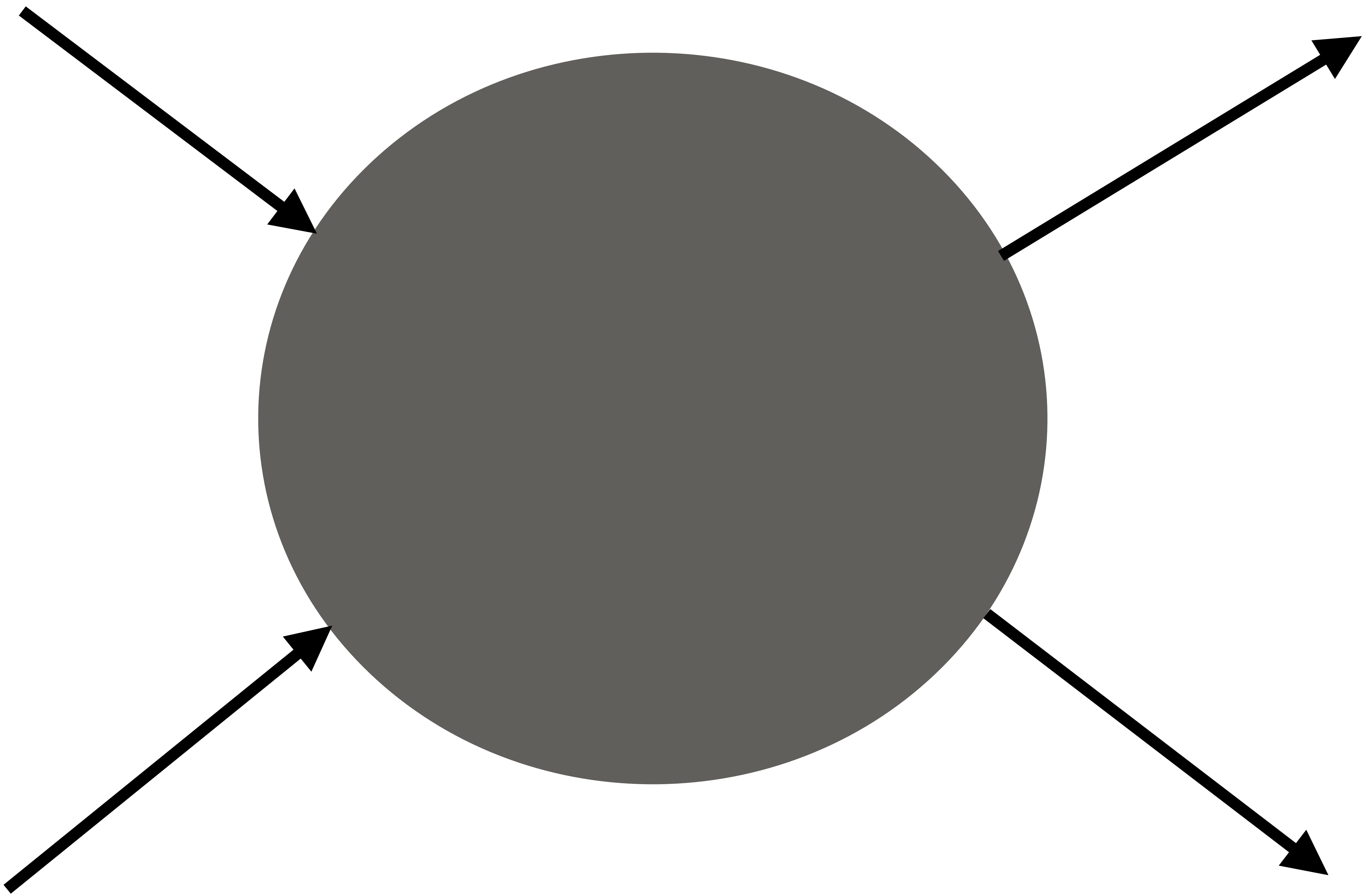
How to define/calculate
the perturbative S-matrix in QFT?



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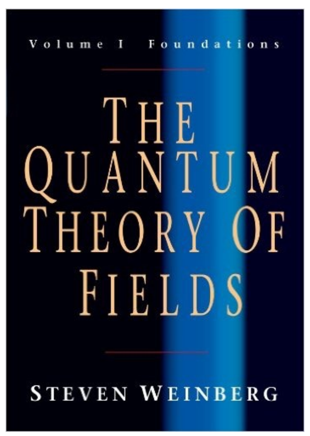
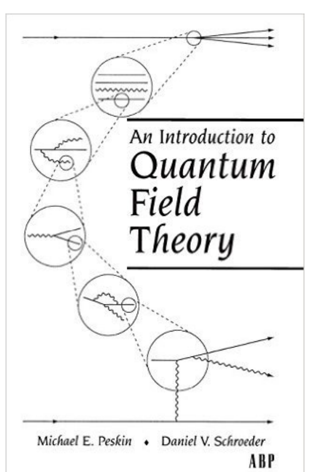
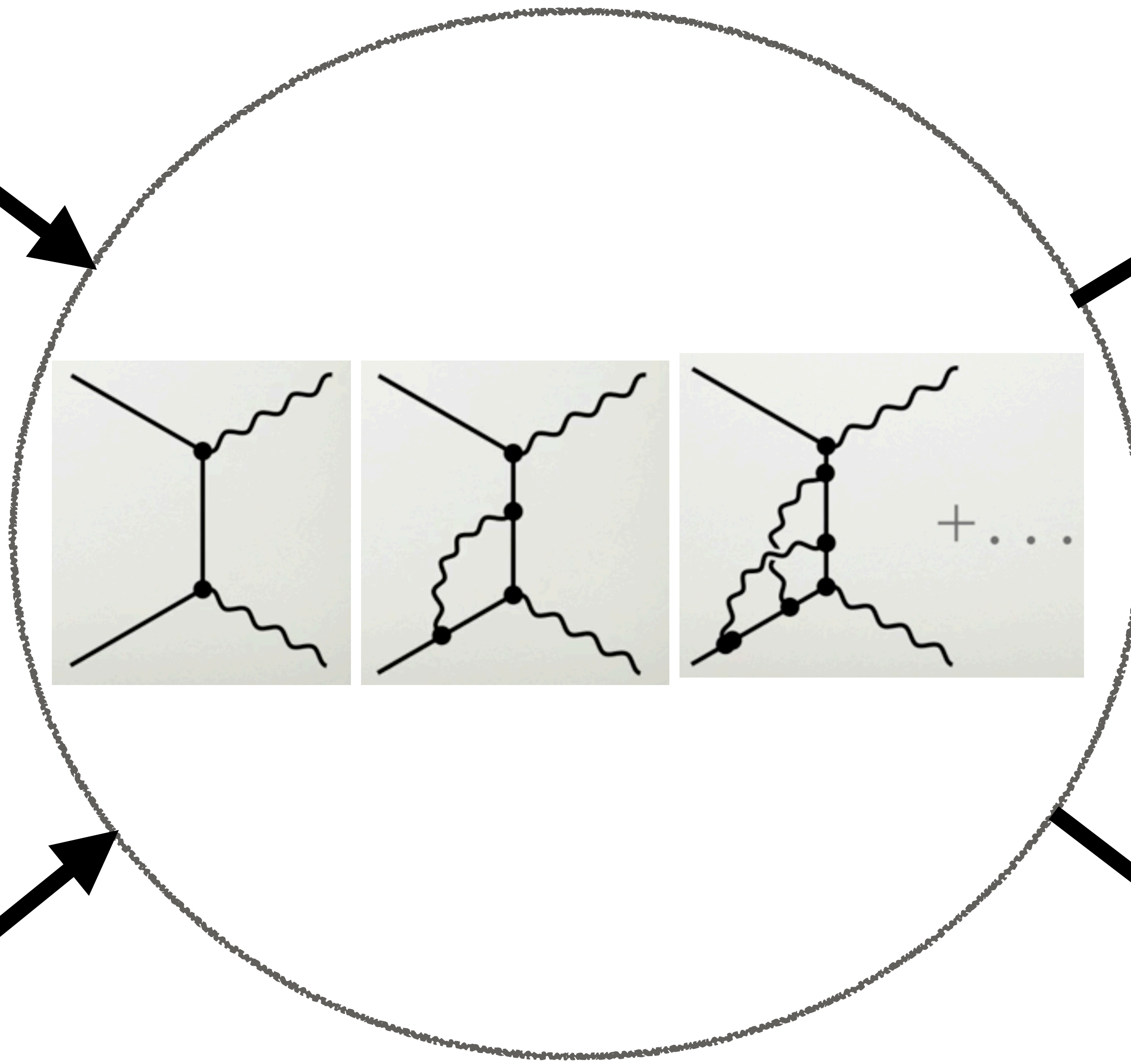
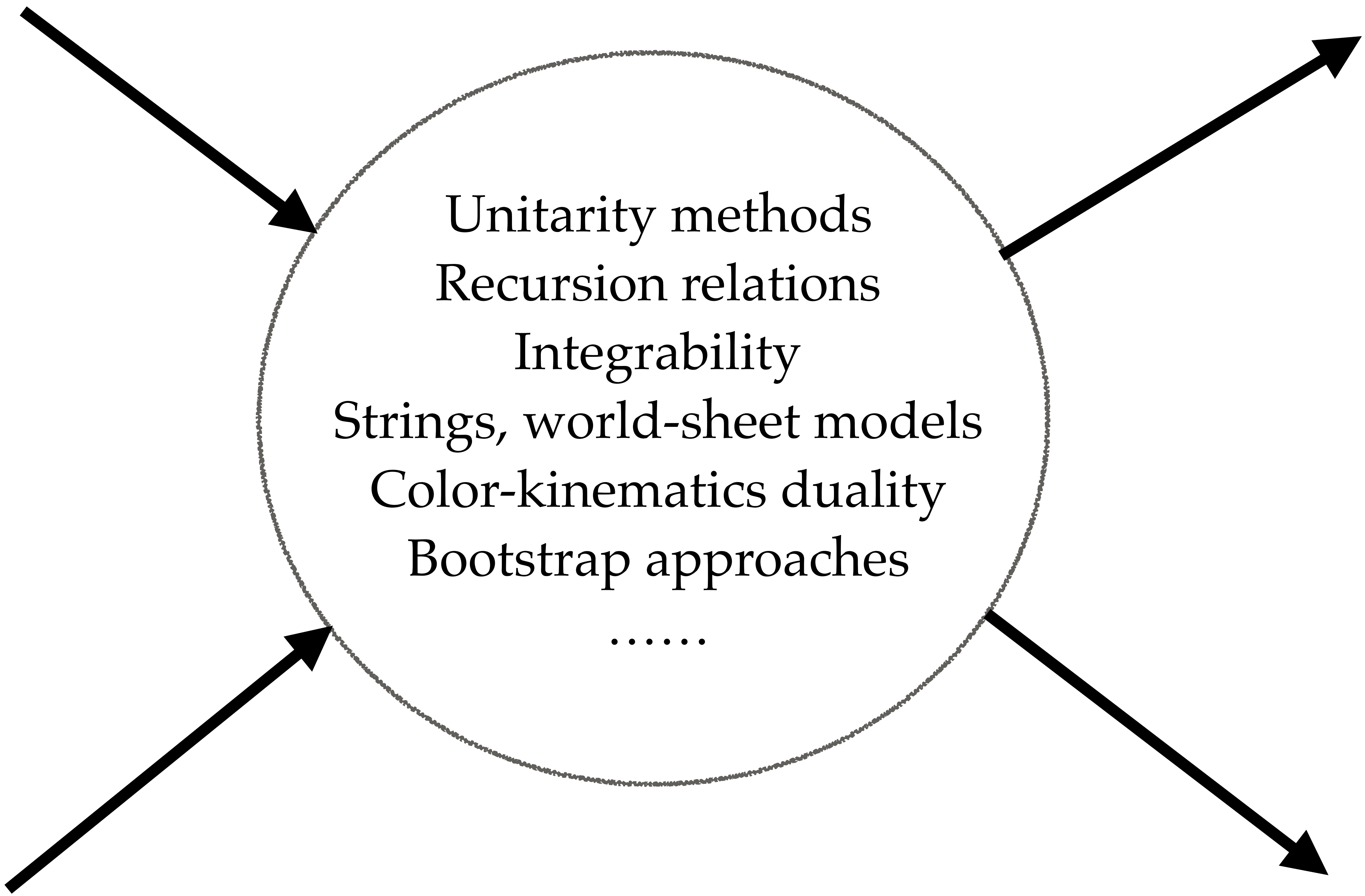


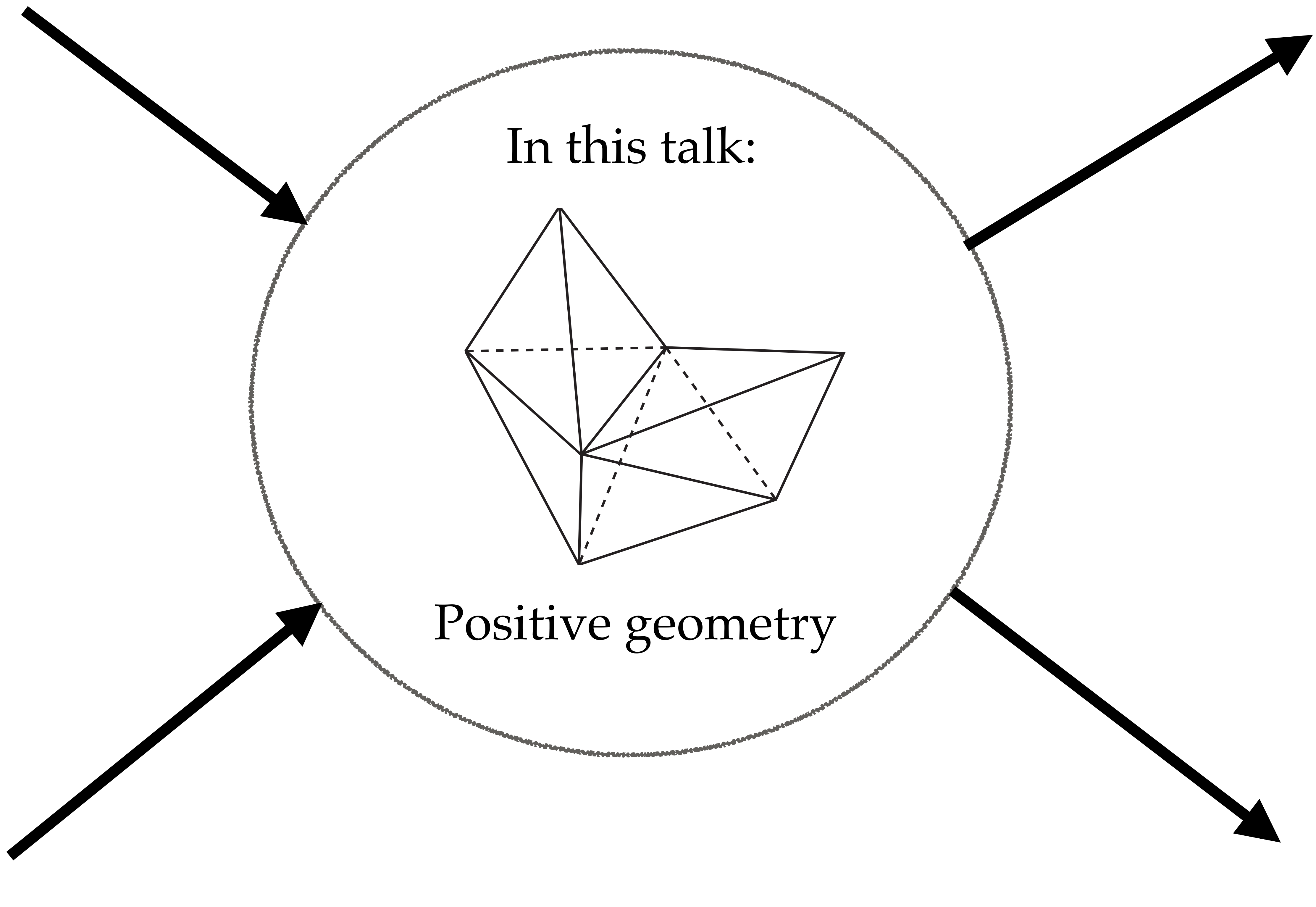




New picture?


Unitarity methods
Recursion relations
Integrability
Strings, world-sheet models
Color-kinematics duality
Bootstrap approaches
……

New picture?

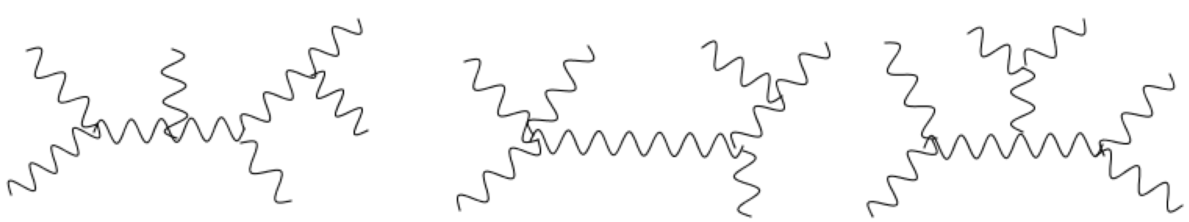


Motivation

- Practical approach: efficient computational method
- Theoretical motivation: understand all-loop order S-matrix, find a completely new framework
- Indirect way to attack bigger problems such as quantum gravity

Unexpected simplicity

- Need for new understanding: simplicity in scattering amplitudes invisible in Feynman diagrams
- Famous example: 2->4 gluon amplitudes in QCD



120 Feyman diagrams

$(k_1 \cdot k_4)(\epsilon_2 \cdot k_1)(\epsilon_1 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5)$

100 pages

Unexpected simplicity

- Need for new understanding: simplicity in scattering amplitudes invisible in Feynman diagrams

- Famous example: 2->4 gluon amplitudes in QCD

(Parke, Taylor 1985)

Helicity amplitude $M_6(1^-2^-3^+4^+5^+6^+)$

Color ordering $M_6 = \sum \mathrm{Tr}(T^{a_1}T^{a_2}\ldots T^{a_6})A_6(123456)$

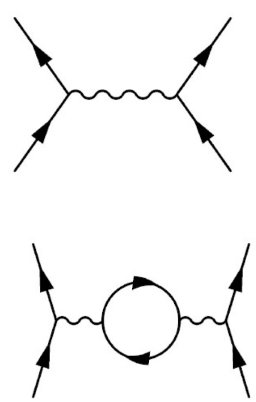
$$A_6 = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}$$

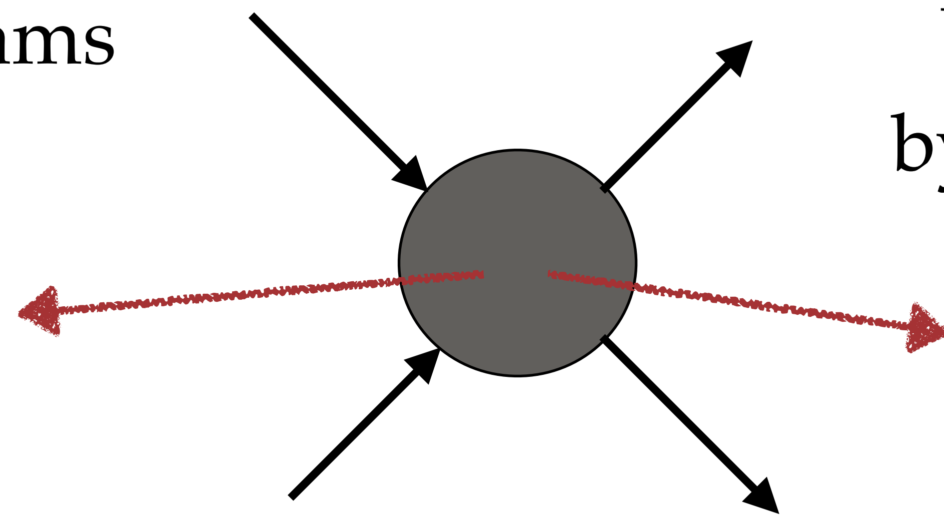
Maximal-helicity (**MHV**) violating amplitude

Change of strategy

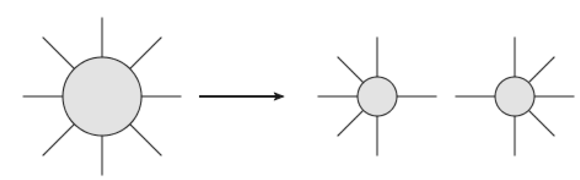
What is the scattering amplitude?

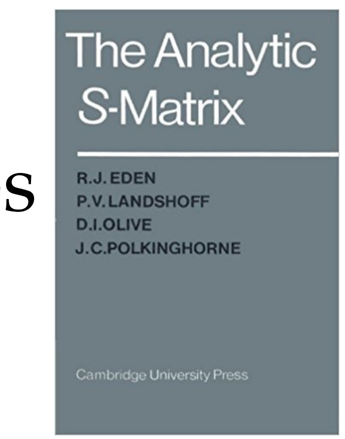
Feynman diagrams





Unique object fixed by physical properties





Modern methods use both:

- Calculate the amplitude directly
- Use perturbation theory

Lesson from Parke-Taylor:

- On-shell gauge invariant objects
- Helicity amplitudes $A_{n,k}$

Modern methods

- Very rich playground of ideas
  - Use of physical constraints: unitarity methods, recursion relations
  - Calculating loop integrals, study mathematical functions, symbols
  - Symmetries of N=4 SYM, UV of N=8 SUGRA, string amplitudes
- Connection between amplitudes and geometry
  - Canonical example is the geometry of worldsheet
  - CHY formula: write QFT amplitudes on worldsheet

(Cachazo, He, Yuan 2013)

$$A_n = \int \frac{dz_1 \dots dz_n}{\mathrm{Vol}[SL(2,C)]}\,\delta\left(\sum_{b\neq a}\frac{s_{ab}}{z_a - z_b}\right)\mathcal{I}_n$$

Positive geometry

- Geometric space defined using a set of inequalities

$$F_k(x_i) \geq 0$$

polynomials parametrize kinematics

- Define the differential form on this space $\Omega(x_i)$
  - Special form: logarithmic singularities on the boundaries

$$\Omega(x_i) \sim \frac{dx_i}{x_i} \quad \text{near boundary } x_i = 0$$

Simple examples

- Example: 1d interval

$$F(x) = x > 0$$

form: $\Omega = \frac{dx}{x} \equiv \mathrm{dlog}\, x$

$$F_1(x) = x - x_1 > 0$$
$$F_2(x) = x_2 - x > 0$$

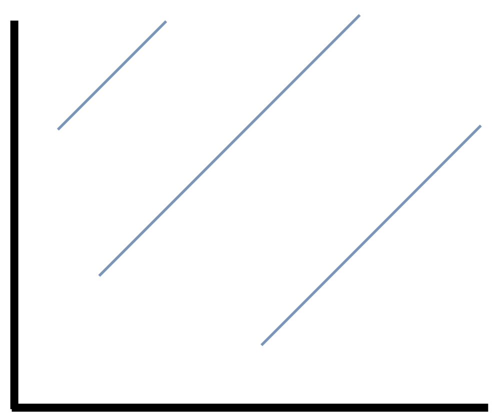
normalization: singularities are unit

$$\Omega = \frac{dx\,(x_1 - x_2)}{(x - x_1)(x - x_2)} = \mathrm{dlog}\left(\frac{x - x_1}{x - x_2}\right)$$

Simple examples

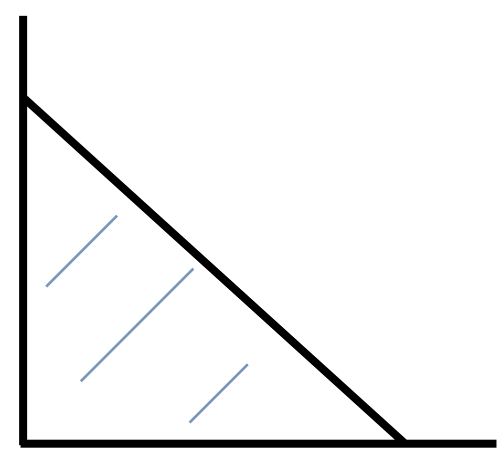
- Example: 2d region

$$x > 0 \qquad y > 0$$



$$x = (0, \infty) \qquad y = (0, \infty)$$

$$\Omega = \frac{dx}{x}\frac{dy}{y}$$



$$x > 0 \qquad y > 0 \qquad 1 - x - y > 0$$

$$x = (0, 1 - y) \qquad y = (0, \infty)$$

$$\Omega = \frac{(y-1)\,dx}{x(x+y-1)} \wedge \frac{dy}{y} = \frac{(y-1)\,dx\,dy}{xy(x+y-1)}$$

- General positive geometry: more than just boundaries

Positive Grassmannian

- Consider space of (2x4) matrices modulo GL(2)

  Real Grassmannian $G(2,4)$ $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$

- Positive Grassmannian $G_+(2,4)$

  All (2x2) minors $(ij) > 0$ $\longrightarrow$ not all of them are boundaries

Positive Grassmannian

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  Shouten identity $(13)(24) = (12)(34) + (14)(23)$

  ↓ product positive; $(12)(34) + (14)(23)$: all positive

  $(13),(24) > 0$ ↙ ↘ $(13),(24) < 0$

  can not set (13)=0 unless set several others to zero

Positive Grassmannian

- Positive Grassmannian $G_+(2,4)$

  Fix GL(2): choose parametrization $\begin{pmatrix} 1 & x & 0 & -y \\ 0 & w & 1 & z \end{pmatrix}$ $\quad x, y, z, w > 0$

- Boundaries: $(12), (23), (34), (14) = 0$

- Logarithmic form:

$$\Omega = \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\frac{dw}{w} = \frac{d^{2\times 4}C}{\text{vol}[GL(2)]}\frac{1}{(12)(23)(34)(14)}$$

Positive geometry for amplitudes

- Amplituhedron: planar N=4 SYM (Arkani-Hamed, JT) (Arkani-Hamed, Thomas, JT)
  - Tree-level and all-loop integrand
- Associahedron: biadjoint scalar at tree-level (Arkani-Hamed, Bai, He, Yan)
  - Connection to CHY, recent work on 1-loop
- More: cosmological polytopes, CFT, EFT (Arkani-Hamed, Benincasa, Huang, Shao)
- Gravituhedron: tree-level GR??? (JT, in progress)

Note: at the moment, no work on the final (integrated) loop amplitudes space of functions is too complicated, we can not play this game

Amplituhedron

(Arkani-Hamed, JT 2013)

(Arkani-Hamed, Thomas, JT 2017)

Amplitudes in planar N=4 SYM

- Large N limit: only planar diagrams, cyclic ordering
- $\mathcal{N}=4$ superfield: $\Phi = G_+ + \widetilde{\eta}_A \Gamma_A + \cdots + \epsilon_{ABCD}\widetilde{\eta}^A\widetilde{\eta}^B\widetilde{\eta}^C\widetilde{\eta}^D G_-$
- Superamplitudes: $\mathcal{A}_n = \sum_{k=2}^{n-2} \mathcal{A}_{n,k}$ → Component amplitudes with power $\tilde{\eta}^{4k}$

Contains $A_n(\underbrace{- - \cdots -}_{k} + + \cdots +)$

N^{k-2}MHV amplitude

- Tree-level + loop integrand
  - conformal invariant
  - dual conformal invariant

} Yangian PSU(2,2|4)

broken after integration due to IR divergencies

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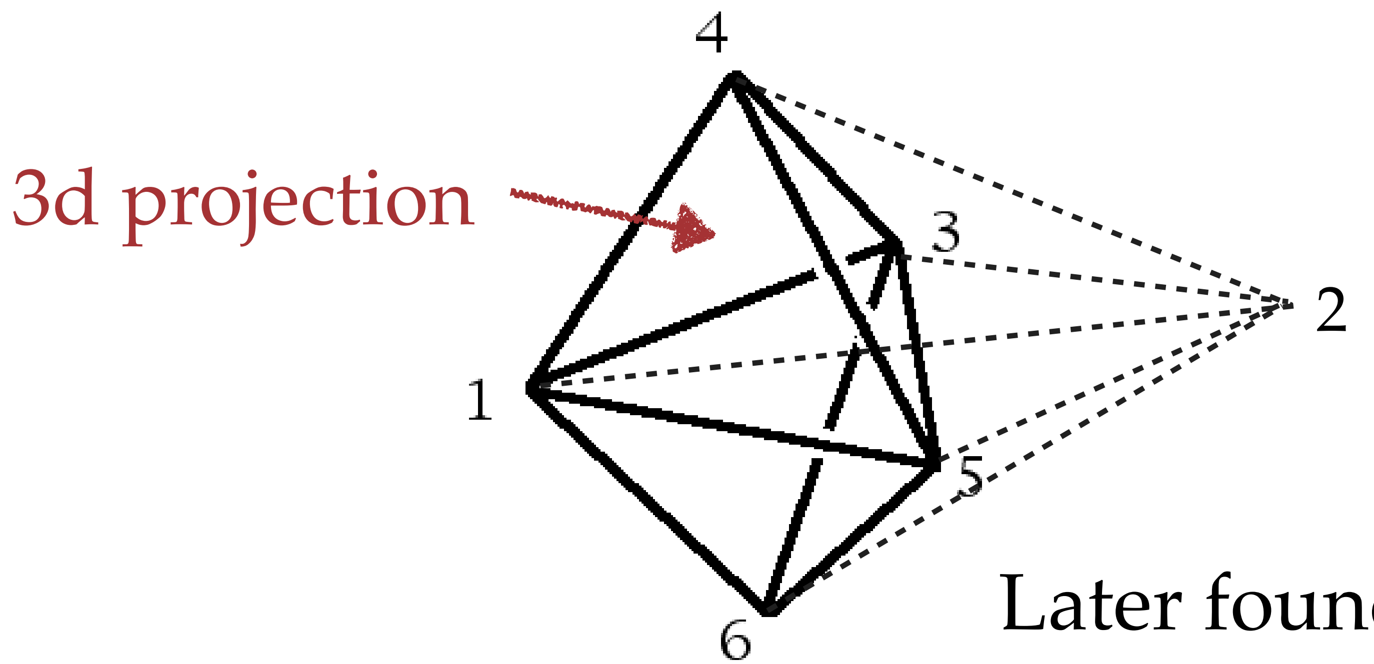
N^{k-2}MHV amplitude

denote $k-2 \to k$

MHV amplitude: $k=0$

Amplitudes as volumes of polytopes

- The simplest example is the 6pt NMHV amplitude pioneered by Andrew Hodges in 2009



$$A_6 = \int_P dV$$

Volume in dual momentum twistor space

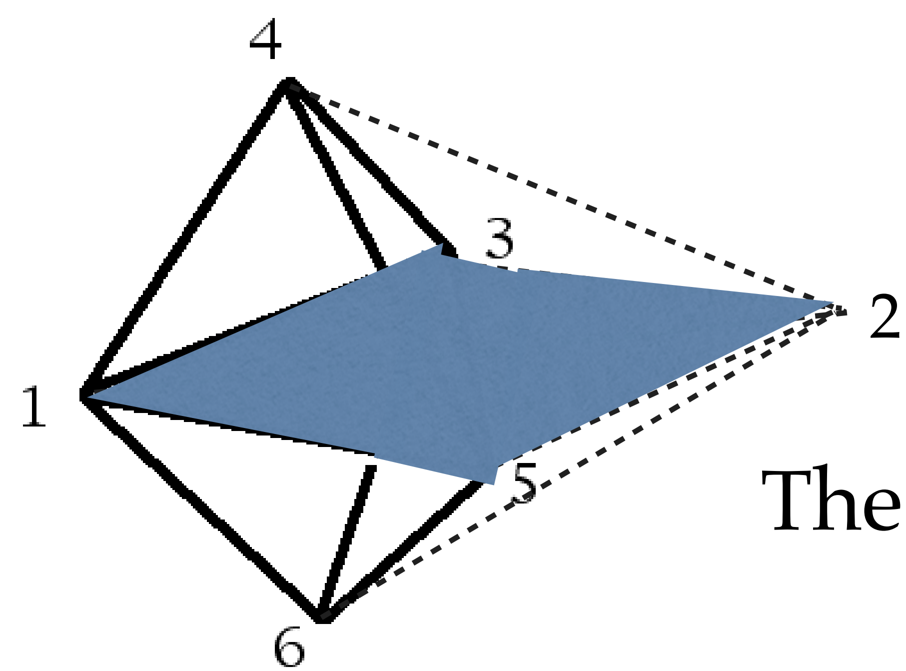
Later found this is equal to logarithmic form on the "cyclic polytope" in P^4

(Arkani-Hamed, Bourjaily, Cachazo, Hodges, JT 2010)

Triangulation

(Hodges 2009)

- Calculation: triangulation in terms of elementary building blocks



Divide into two simplices by cutting the polyhedron with (1235) plane

The first only depends on (12345) and second on (12356)

$$A_6 = [12345] + [12356]$$

each simplex is associated with "R-invariant"
this correctly reproduces amplitude

From kinematics to geometry

(Arkani-Hamed, Thomas, JT 2017)

- Change of kinematics: $p_i, \epsilon_j \to Z_k \in P^3 \quad k = 1, 2, \ldots, n$

points in projective space

$\ell_j \to (Z_A Z_B)_j \in P^3$

lines in projective space

$\mathcal{A}_{n,k,\ell}$

positive geometry in P^{3+k}

projection

kinematical space P^3

Definition of the **Amplituhedron**

Form with logarithmic singularities on the boundaries = **amplitude**

Back to 6pt NMHV amplitude

- Definition of the space:

(5x5) determinants

$$\langle Z_1 Z_2 Z_3 Z_4 Z_5 \rangle, \langle Z_1 Z_2 Z_3 Z_4 Z_6 \rangle, \cdots > 0$$ convex hall of points

positive geometry in convex P^4 Z_j

projection

kinematical space P^3 z_j

projection: $Y : P^4 \to P^3$

such that $\langle z_i z_{i+1} z_j z_{j+1} \rangle > 0$

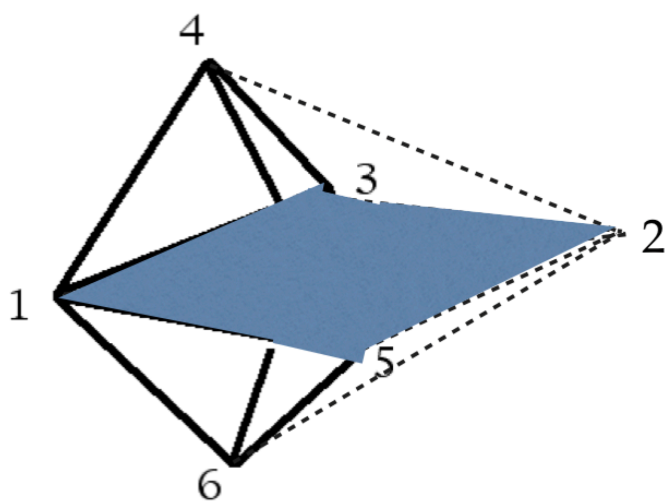
these are boundaries: poles in S-matrix

$$\sim (p_{i+1} + p_{i+2} + \ldots p_j)^2$$

In this case (boundaries)>0 completely specifies the projection, hence the space in P^3

Back to 6pt NMHV amplitude

- Triangulation -> differential form -> amplitude



$$\Omega_6 = [12345] + [12356]$$ two simplicies

$\longrightarrow$ project to P^3

change variables: $x_i \to z_k$

Logarithmic form: $\Omega = \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\frac{dx_4}{x_4}$

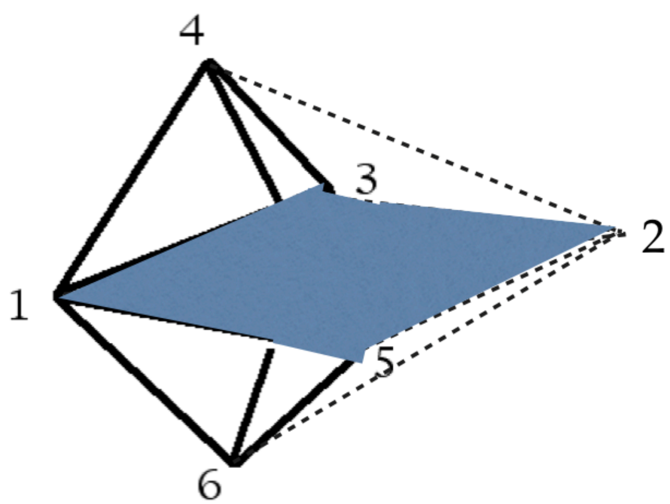
$$[12345] = \frac{(\langle 1234\rangle dz_5 + \langle 2345\rangle dz_1 + \langle 3451\rangle dz_2 + \langle 4512\rangle dz_3 + \langle 5123\rangle dz_4)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$$

$$[12356] = \frac{(\langle 1235\rangle dz_6 + \langle 2356\rangle dz_1 + \langle 3561\rangle dz_2 + \langle 5612\rangle dz_3 + \langle 6123\rangle dz_5)^4}{\langle 1235\rangle\langle 2356\rangle\langle 3561\rangle\langle 5612\rangle\langle 6123\rangle}$$

where $\langle 1234\rangle \equiv \langle z_1 z_2 z_3 z_4\rangle$

Back to 6pt NMHV amplitude

- Triangulation -> differential form -> amplitude



$$A_6 = [12345] + [12356]$$ two simplicies

$\longrightarrow$ project to P^3

Differential form

$\downarrow$

Superfunction

Replace: $dz_j \to \eta_j$

$$[12345] = \frac{(\langle 1234\rangle\eta_5 + \langle 2345\rangle\eta_1 + \langle 3451\rangle\eta_2 + \langle 4512\rangle\eta_3 + \langle 5123\rangle\eta_4)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$$

$$[12356] = \frac{(\langle 1235\rangle\eta_6 + \langle 2356\rangle\eta_1 + \langle 3561\rangle\eta_2 + \langle 5612\rangle\eta_3 + \langle 6123\rangle\eta_5)^4}{\langle 1235\rangle\langle 2356\rangle\langle 3561\rangle\langle 5612\rangle\langle 6123\rangle}$$

compare to amplitude ✓

where $\langle 1234\rangle \equiv \langle z_1 z_2 z_3 z_4\rangle$

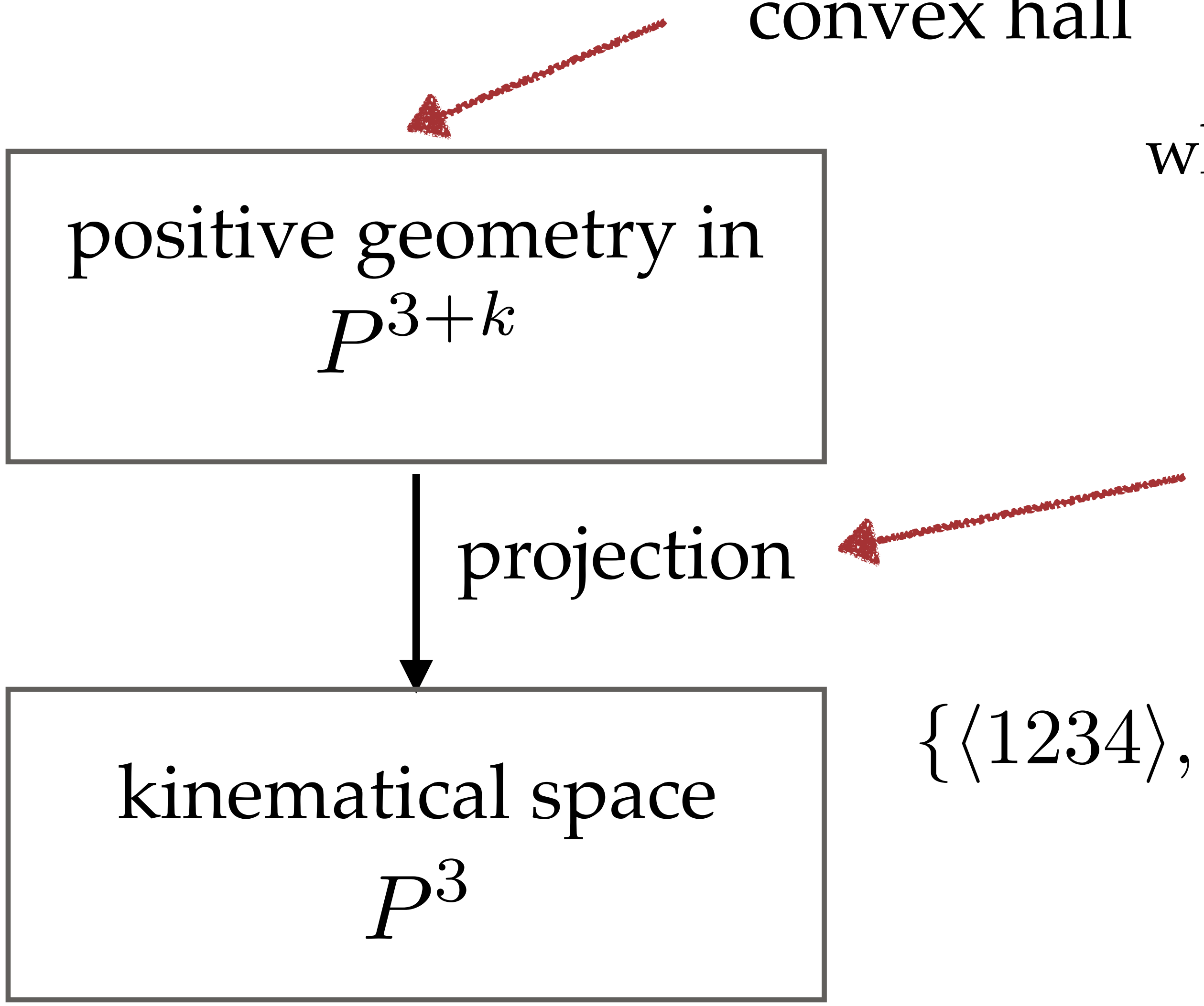
Definition of Amplituhedron

(Arkani-Hamed, Thomas, JT 2017)

- Constraints on positive geometry and the projection

convex hall $\langle Z_{a_1} Z_{a_2} Z_{a_3} \ldots Z_{a_{k+3}} \rangle > 0$

where $\langle \ldots \rangle$ is (k+3)x(k+3) determinant



$$\begin{array}{ccc} Z_a & \xrightarrow{Y} & z_a \\ \cap & & \cap \\ P^{k+3} & & P^3 \end{array}$$

$\langle z_i z_{i+1} z_j z_{j+1} \rangle > 0$

$\{\langle 1234 \rangle, \langle 1235 \rangle, \ldots, \langle 123n \rangle\}$ has k sign flips

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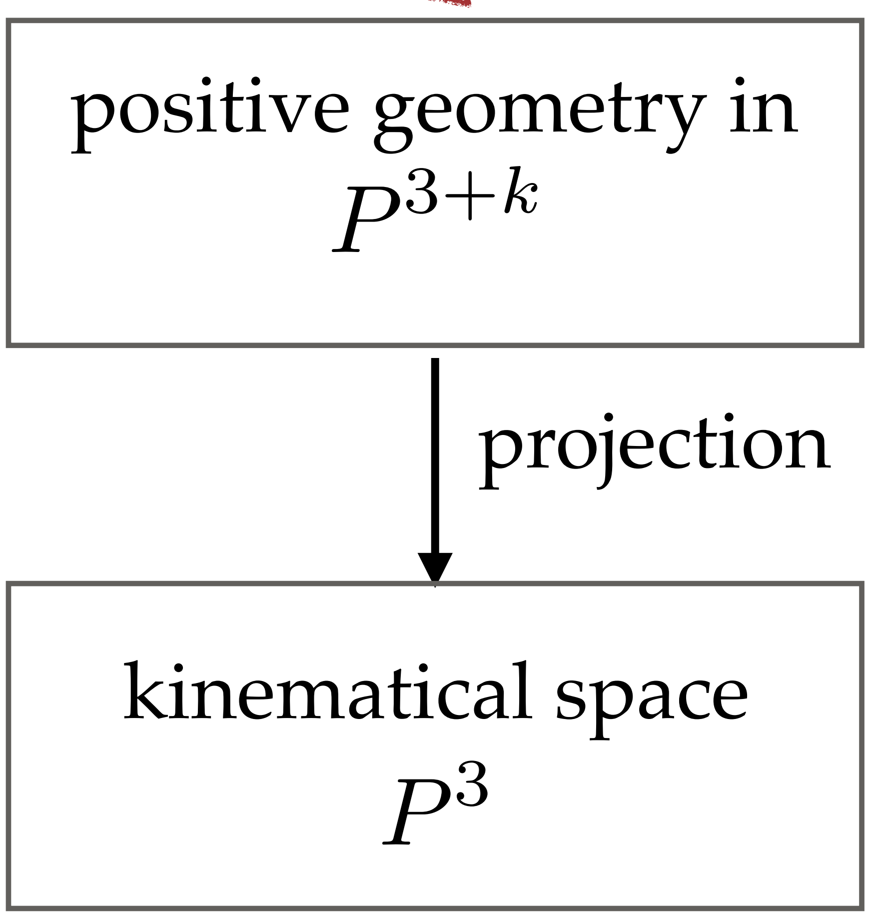

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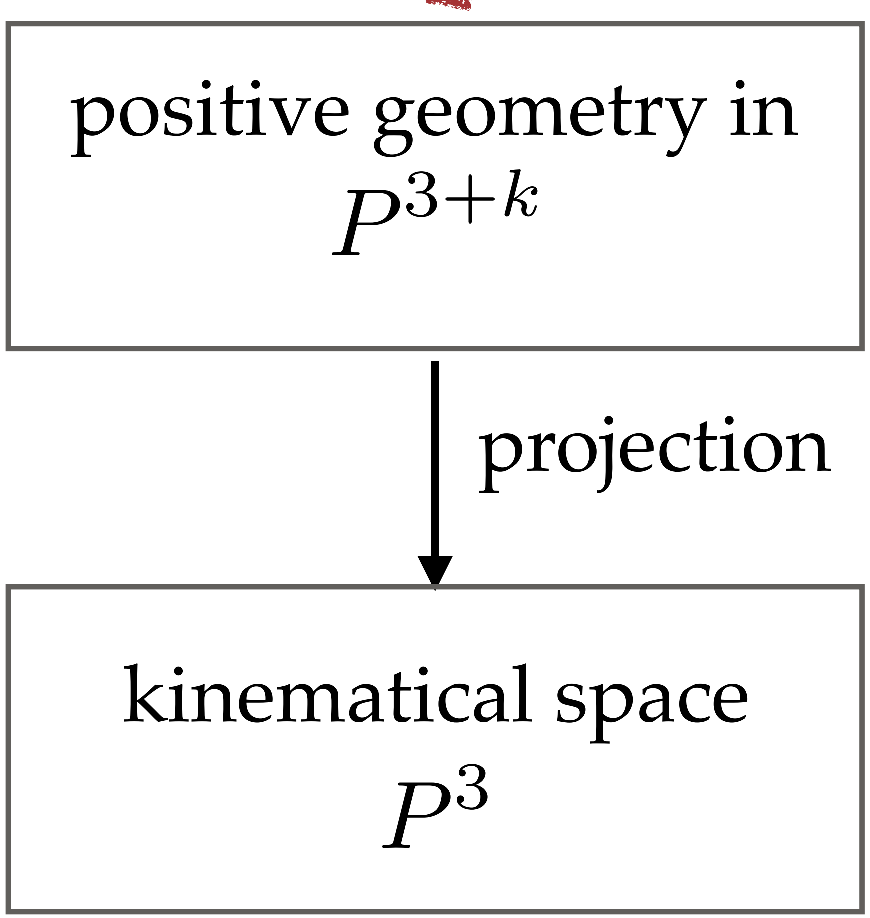
example: $\{\langle 1234 \rangle, \langle 1235 \rangle, \langle 1236 \rangle, \langle 1237 \rangle\}$

n=7	+	+	-	+
k=2	+	-	+	+

Definition of Amplituhedron

(Arkani-Hamed, Thomas, JT 2017)

- Constraints on positive geometry and the projection



convex hall $\langle Z_{a_1} Z_{a_2} Z_{a_3} \dots Z_{a_{k+3}} \rangle > 0$

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$$\langle z_i z_{i+1} z_j z_{j+1} \rangle > 0$$

$\{\langle 1234 \rangle, \langle 1235 \rangle, \dots, \langle 123n \rangle\}$ has k sign flips

analogue of (13)>0 for $G_+(2,4)$

Definition of Amplituhedron

(Arkani-Hamed, Thomas, JT 2017)

- Tree-level $\mathcal{A}_{n,k,\ell=0}$

$$\langle Z_{a_1} Z_{a_2} Z_{a_3} \dots Z_{a_{k+3}} \rangle > 0$$

$$\langle z_i z_{i+1} z_j z_{j+1} \rangle > 0$$

$$\{\langle 1234 \rangle, \langle 1235 \rangle, \dots, \langle 123n \rangle\}$$ has k sign flips

$4k$ form → $\Omega_{n,k}(z_j) \to A_{n,k}(z_j, \widetilde{\eta}_j)$ ← **tree-level amplitude**

- Loop integrand $\mathcal{A}_{n,k,\ell}$

for each line (loop momentum): $\langle (AB)_j z_i z_{i+1} \rangle > 0$

$$\{\langle (AB)_j 12 \rangle, \langle (AB)_j 13 \rangle, \dots \langle (AB)_j 1n \rangle\}$$ has (k+2) sign flips

for each pair of lines: $\langle (AB)_j (AB)_k \rangle > 0$

$4k + 4\ell$ form → $\Omega_{n,k,\ell}(z_j, (AB)_k) \to \mathcal{I}_{n,k,\ell}(z_j, \widetilde{\eta}_j, (AB)_k)$ ← **loop integrand**

From geometry to amplitudes

- Amplituhedron: space of points and lines in projective space
  - triangulate the space into "simplices" = elementary regions for which the form is trivial dlog form and sum them

$$\Omega = \frac{dx_1}{x_1}\frac{dx_2}{x_2}\dots\frac{dx_{4k+4\ell}}{x_{4k+4\ell}} \qquad \text{where} \quad x_k = f(z_i,(AB)_j)$$

- Turn the physics problem of calculating scattering amplitudes to a math problem of **triangulations**

- Same S-matrix: physics properties are consequences of positivity geometry of Amplituhedron

Exploring Amplituhedron

(Arkani-Hamed, JT 2013) (Arkani-Hamed, Thomas, JT 2017)

(Arkani-Hamed, Langer, Yelleshpur Srikant, JT 2018) (Rao 2017, 2018) (Kojima 2018)

(Langer, Kojima to appear) (Herrmann, Langer, Zheng, JT, to appear)

Triangulations

- Systematic approach to triangulating the Amplituhedron is still missing

- A number of non-trivial explicit calculations
  - higher k and higher ℓ complicated
  - known results up to 2-loop (for MHV configuration) for any n

(Arkani-Hamed, Rao, Kojima, Langer, JT)

- All-loop order data for boundaries (cuts of integrand)

(Arkani-Hamed, Yellshpur, Langer, JT)

- Recent work on the rigorous understanding of Amplituhedron for simple cases

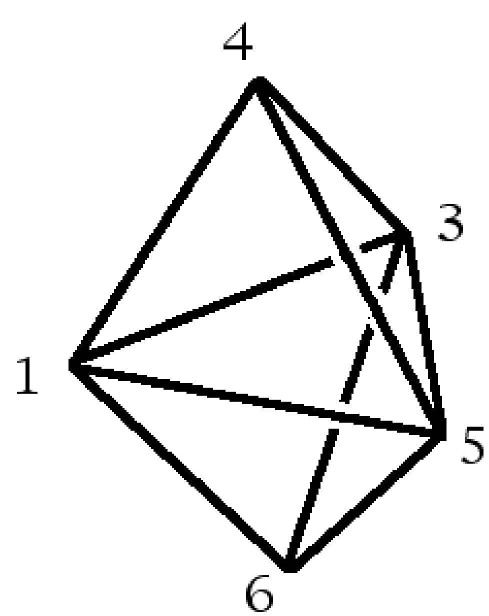
(Williams, Lam, Postnikov, Karp, Galashin)

Dual Amplituhedron

- Original idea: amplitude = volume -> much desired

One case understood: NMHV tree

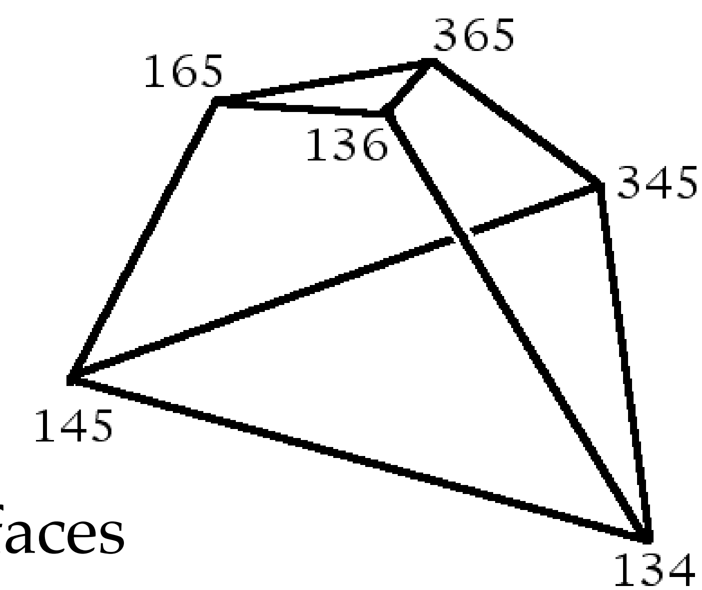
(Hodges)



A = dlog form

dual

vertices <-> faces
edge <-> edges
faces <-> vertices

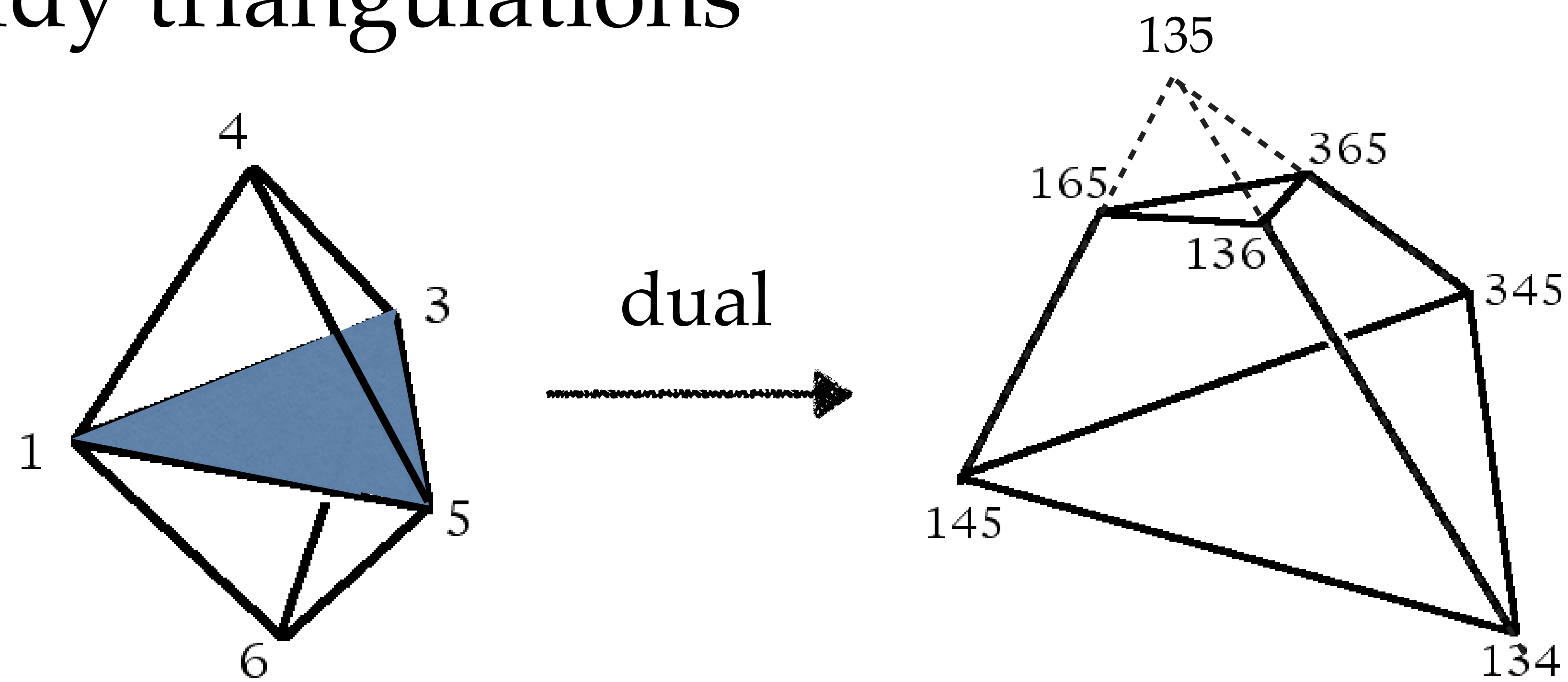


A = volume

- We do not know how to dualize Amplituhedron

Dual Amplituhedron

- Idea: study triangulations



- Internal triangulations of Amplituhedron = external triangulations of dual Amplituhedron
  - Explicit triangulations -> deduce the dual Amplituhedron

(Langer, Zhen, JT, in progress)

Four point problem

- For MHV amplitudes (k=0) there is no projection

$$Z_j = z_j$$

positive geometry
= kinematical space P^3

$$\langle z_{a_1} z_{a_2} z_{a_3} z_{a_4} \rangle > 0$$

4pt only: $\langle 1234 \rangle > 0$

- For 4pt (2->2 scattering) the all-loop problem can be phrased in a simple way: geometry of ℓ lines in P^3

for each: $\langle (AB)_j 12 \rangle, \langle (AB)_j 23 \rangle, \langle (AB)_j 34 \rangle, \langle (AB)_j 14 \rangle > 0$ boundaries

$\{\langle (AB)_j 12 \rangle, \langle (AB)_j 13 \rangle, \langle (AB)_j 14 \rangle\}$
+ - +

→ $\langle (AB)_j 13 \rangle < 0$

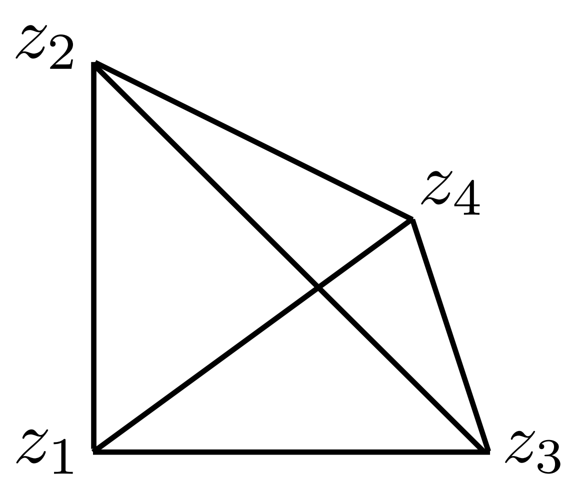
$\langle (AB)_j 24 \rangle < 0$

not boundaries
inequalities needed

pair of lines: $\langle (AB)_j (AB)_k \rangle > 0$ boundaries

Four point problem

- Geometry of ℓ lines in P^3



$\langle 1234 \rangle > 0 \longrightarrow$ fix

$$z = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

draw in 3-d space

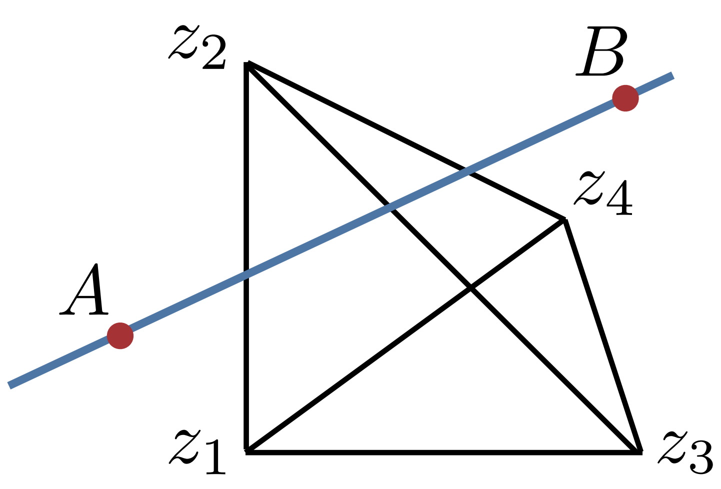
$$v = \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$$

space of z is completely fixed

Four point problem

- Geometry of ℓ lines in P^3 $\quad \langle 1234 \rangle > 0 \longrightarrow$ fix

One line



$$z = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

draw in 3-d space

$$v = \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$$

space of z is completely fixed

Line in this space

$$A = z_1 + x z_2 + -y z_3$$
$$B = z_3 + w z_2 + z z_4$$

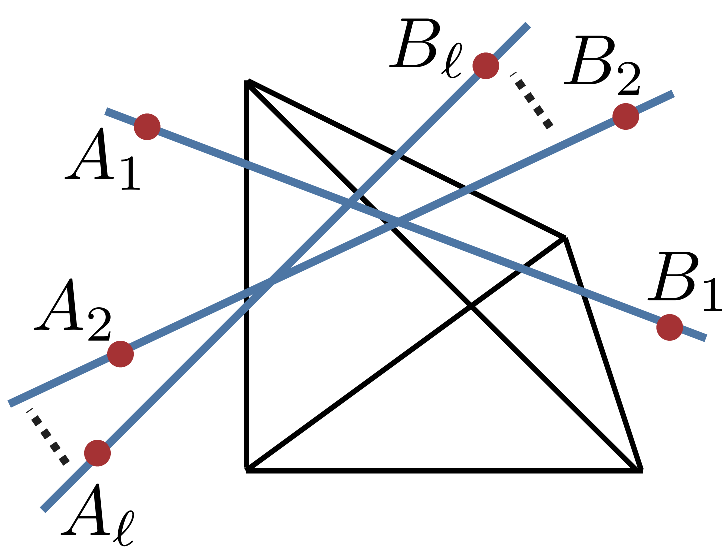
matrix of coefficients $\quad D = \begin{pmatrix} 1 & x & 0 & -y \\ 0 & w & 1 & z \end{pmatrix}$

positive constraints: x,y,z,w>0

Four point problem

- Geometry of ℓ lines in P^3

Many lines



each line $D_j = \begin{pmatrix} 1 & x_j & 0 & -y_j \\ 0 & w_j & 1 & z_j \end{pmatrix}$

$$x_j, y_j, w_j, z_j > 0$$

mutual positivities

$$(x_j - x_k)(z_j - z_k) + (w_j - w_k)(y_j - y_k) < 0$$

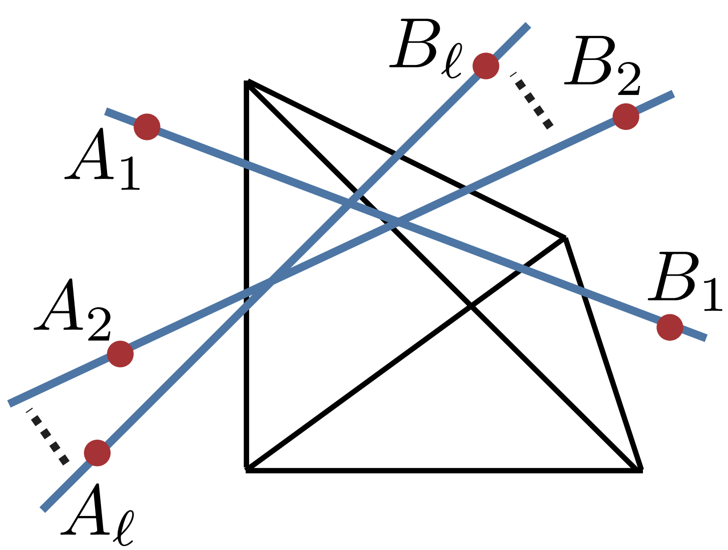
- Triangulation: break the space into elementary regions

$$x \in (x_{min}, x_{max}) \quad \rightarrow \quad \Omega = \frac{(x_{max} - x_{min})\, dx}{(x - x_{min})(x - x_{max})}$$ for each parameter

Four point problem

- Geometry of ℓ lines in P^3

Many lines



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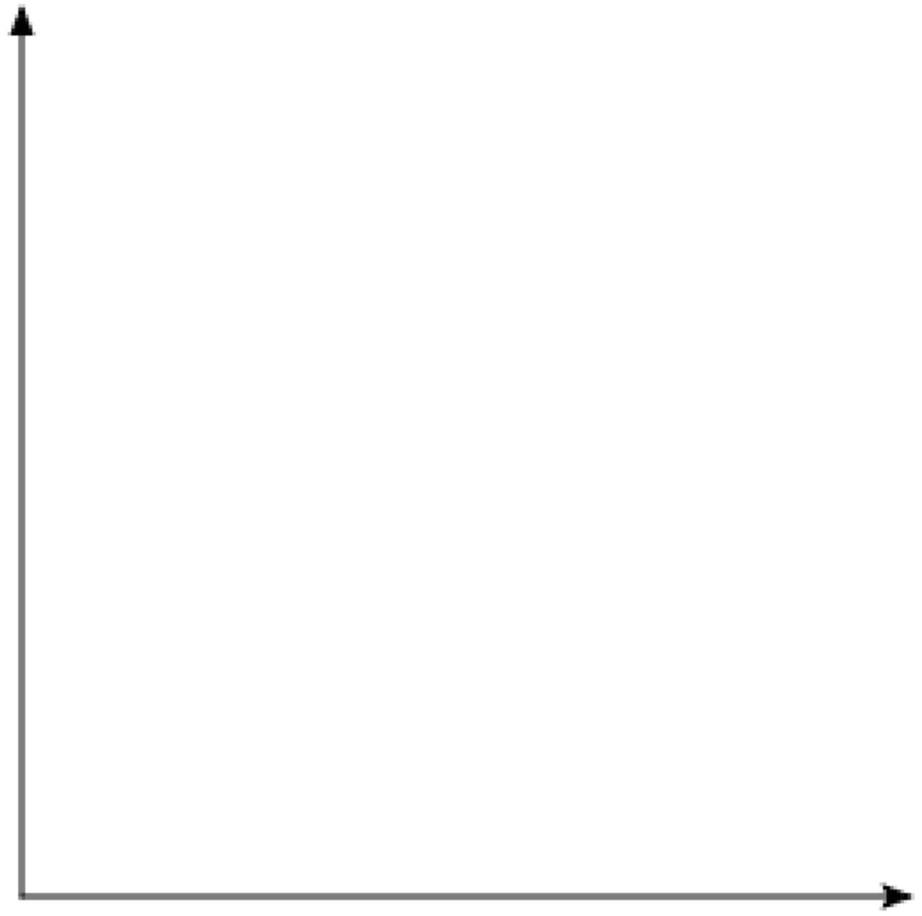
$$x \in (x_{min}, x_{max}) \quad \to \quad \Omega = \frac{(x_{max} - x_{min})\, dx}{(x - x_{min})(x - x_{max})}$$ for each parameter

Quadratic conditions: hard to solve

High school problem $gg \to gg$

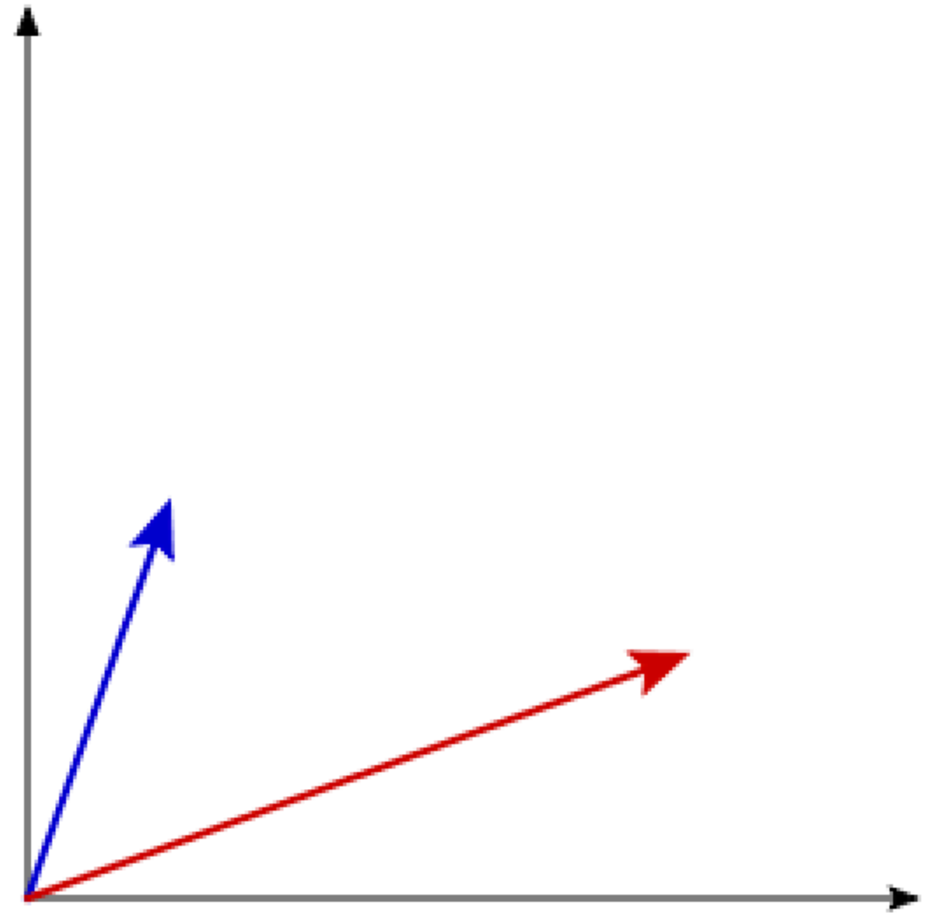
- Positive quadrant



High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

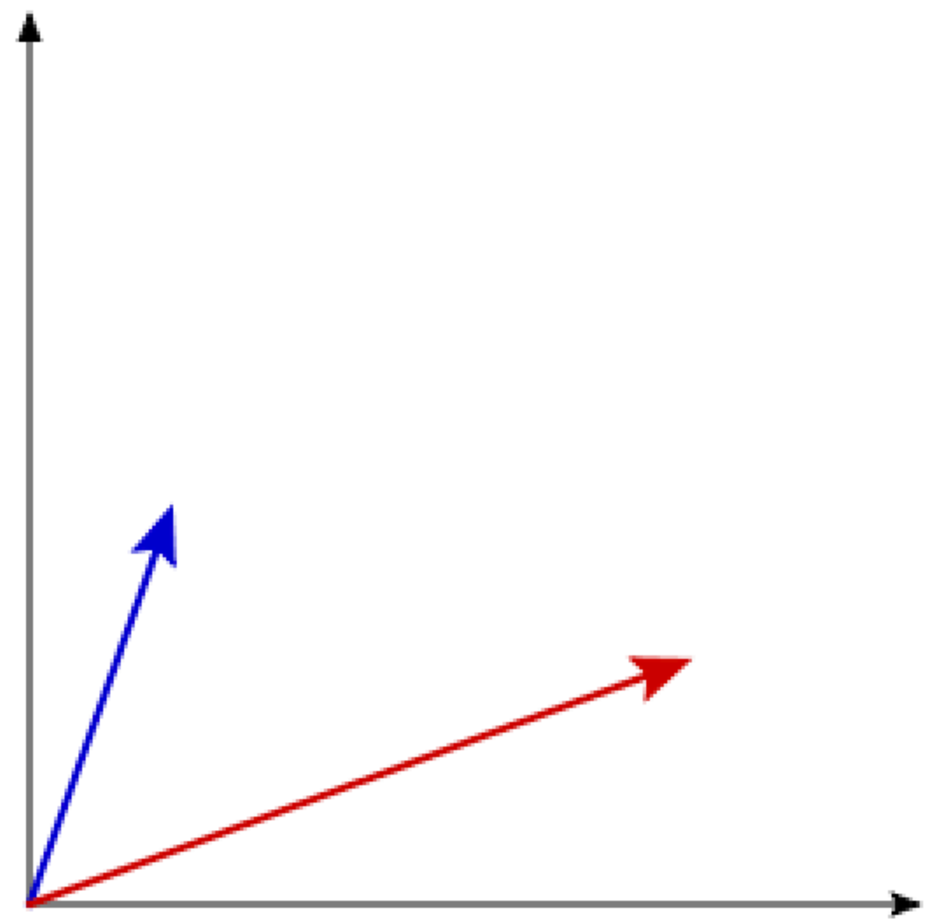


$$\mathrm{Vol}\,(1) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}$$

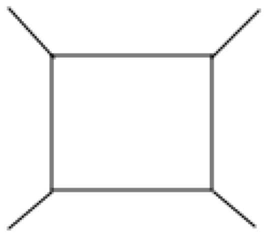
High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$



$$\mathrm{Vol}\,(1) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1} =$$

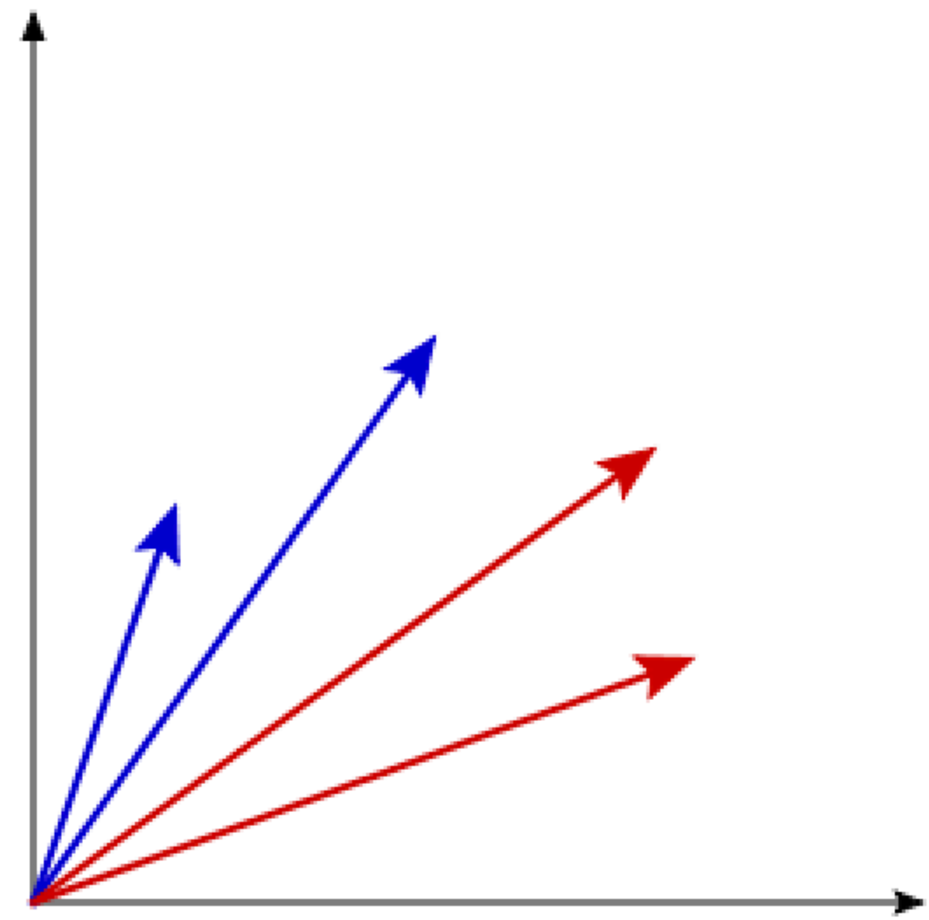


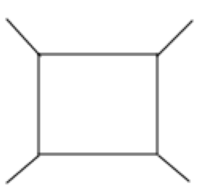
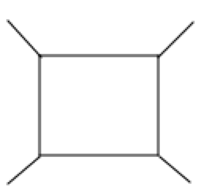
High school problem $\quad gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



$$[\mathrm{Vol}\,(1)]^2 = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}\frac{dz_2}{z_2}\frac{dw_2}{w_2} =$$  $\times$ 

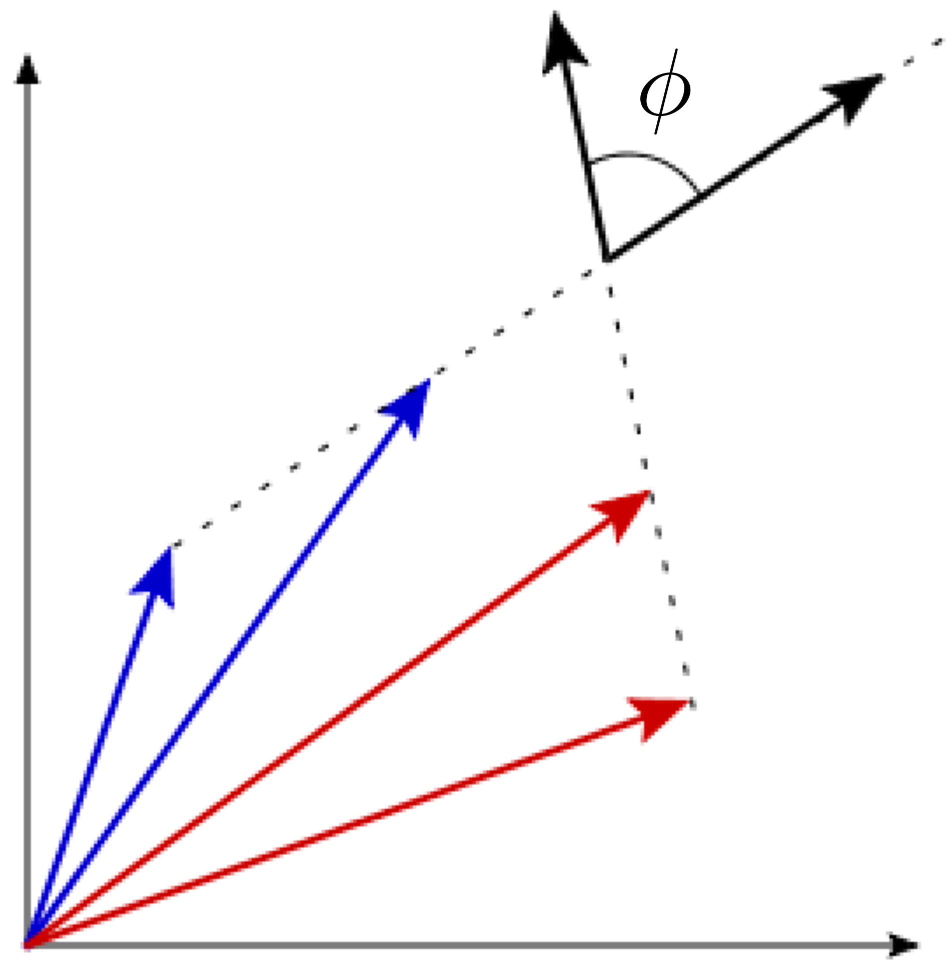
High school problem $gg \to gg$

- Positive quadrant

- Vectors

$$\color{red}{\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}} \quad \color{blue}{\vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}}$$

$$\color{red}{\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}} \quad \color{blue}{\vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}}$$



- Impose: $\color{red}{(\vec{a}_2 - \vec{a}_1)} \cdot \color{blue}{(\vec{b}_2 - \vec{b}_1)} \leq 0 \qquad \phi > 90^o$

Subset of configurations allowed: triangulate

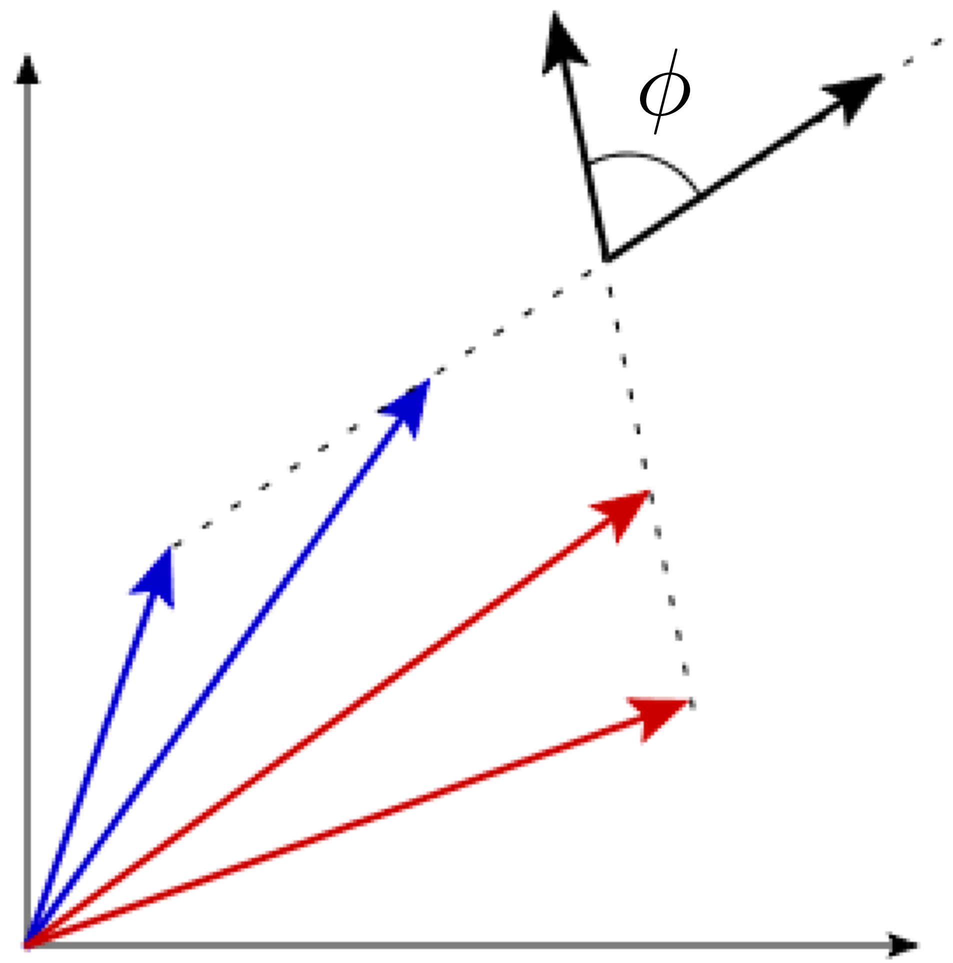
High school problem $gg \to gg$

- Positive quadrant

- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



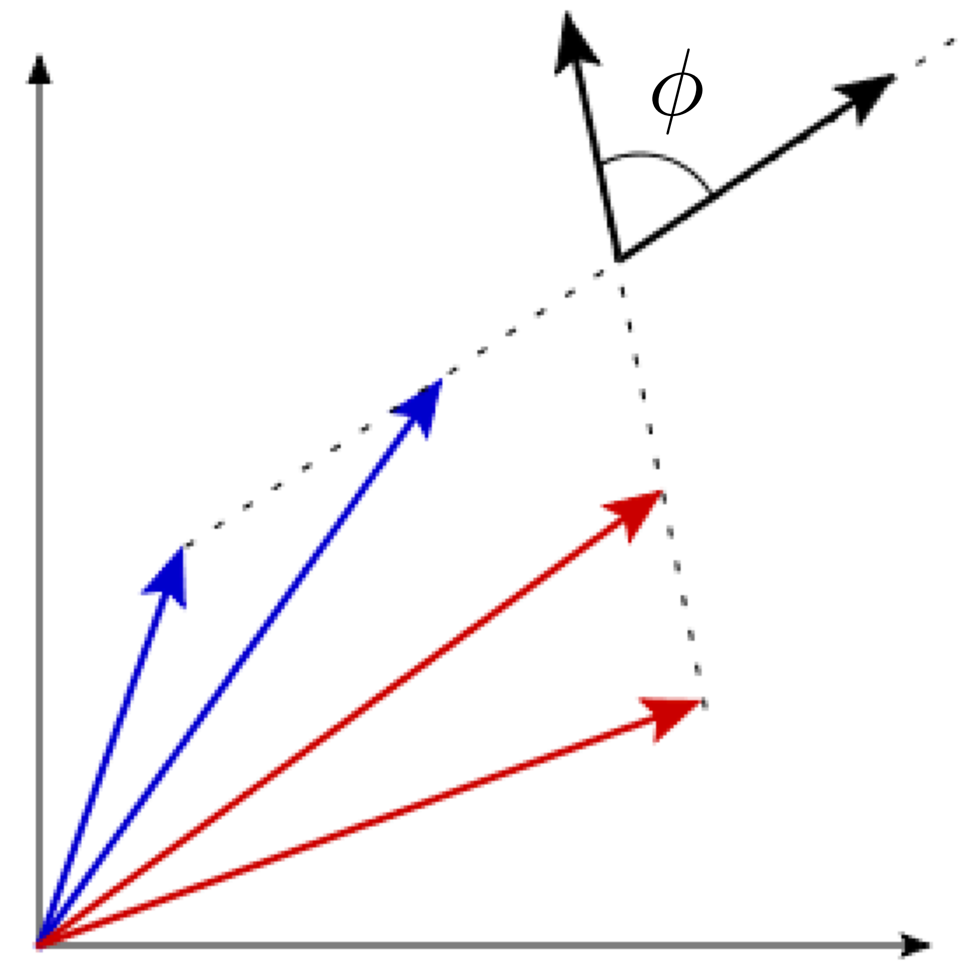
$$\mathrm{Vol}\,(2) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}\frac{dz_2}{z_2}\frac{dw_2}{w_2}\left[\frac{\vec{a}_1\cdot\vec{b}_2+\vec{a}_2\cdot\vec{b}_1}{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_2-\vec{b}_1)}\right]$$

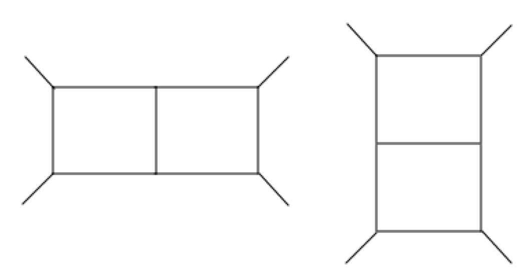
High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



$\mathrm{Vol}\,(2) =$ 

High school problem $gg \to gg$

- Positive quadrant
- Vectors

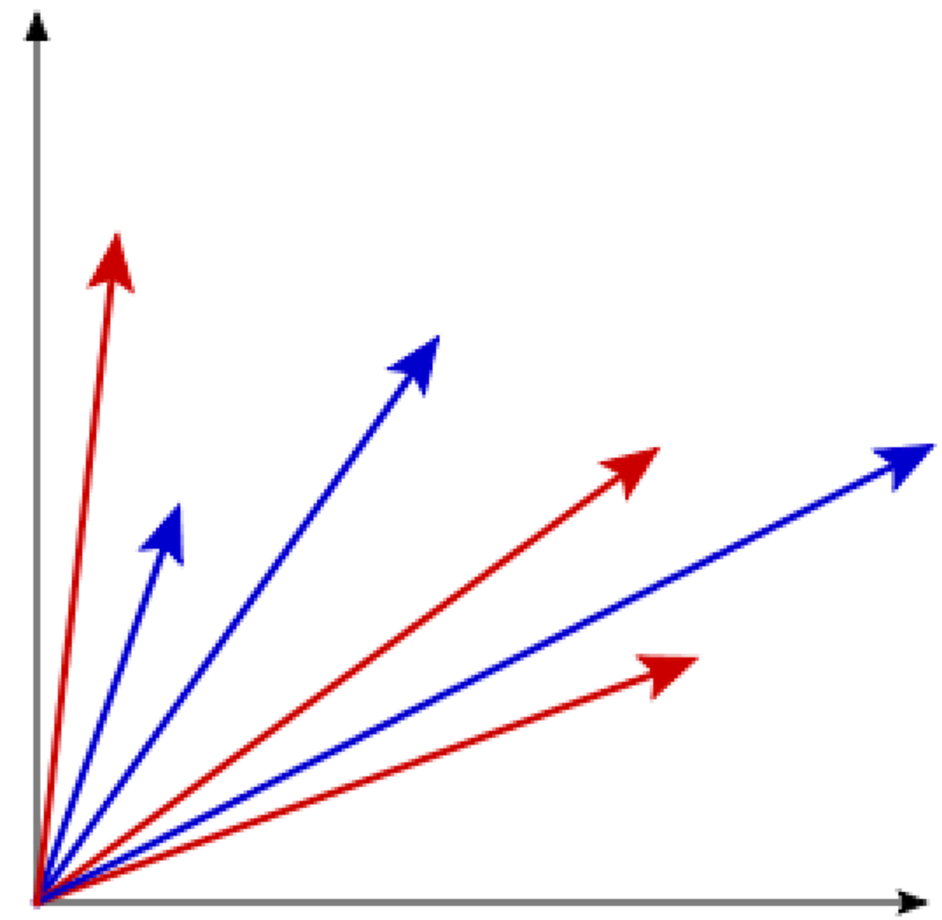
  $\vec{a}_1, \vec{a}_2, \vec{a}_3 \qquad \vec{b}_1, \vec{b}_2, \vec{b}_3$

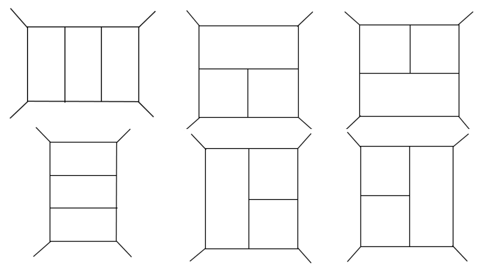
- Conditions

$$(\vec{a}_1 - \vec{a}_2) \cdot (\vec{b}_1 - \vec{b}_2) \leq 0$$

$$(\vec{a}_1 - \vec{a}_3) \cdot (\vec{b}_1 - \vec{b}_3) \leq 0$$

$$(\vec{a}_2 - \vec{a}_3) \cdot (\vec{b}_2 - \vec{b}_3) \leq 0$$



$\mathrm{Vol}\,(3) =$ 

High school problem $gg \to gg$

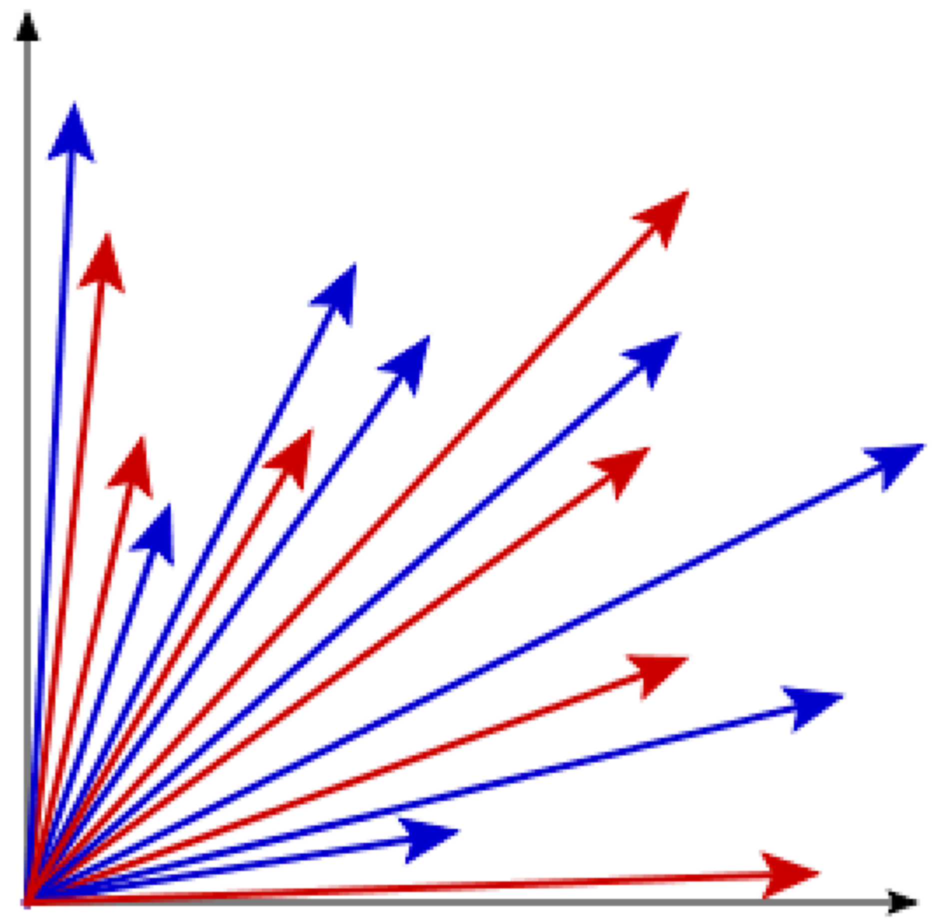
- Positive quadrant
- Vectors

$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_\ell \quad \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_\ell$

- Conditions

$$(\vec{a}_i - \vec{a}_j) \cdot (\vec{b}_i - \vec{b}_j) \leq 0$$

for all pairs i, j



$\mathrm{Vol}\,(\ell) = \ldots\ldots$

Amplituhedron recap

- Calculating perturbative amplitudes (tree-level, integrand) in this theory is reduced to the math problem
    - Define geometry, kinematical data are input
    - Triangulations and calculating differential forms
- Can not derive Amplituhedron from QFT
    - We can prove that the volume function satisfies all properties of scattering amplitudes: factorization etc.
- For planar N=4 SYM this is a new definition for the S-matrix
- General: no dictionary between Lagrangian and geometry

Step 1.1.1. in the program

- Maybe this is very special and no reformulation exists in general, maybe it exists but it is something else
- Right/wrong: analyze "theoretical data", look for new structures, make proposals and check them
- Step-by-step process, all steps require new ideas
  - Lower supersymmetry, other theories, spins, masses
  - Final (integrated) amplitudes
  - **UV physics, renormalization**
  - …..

Thank you!