String and field theory realization of $T\bar{T}$ and related deformations

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Related works by O. Aharony, S. Datta, A. Hashimoto, N. Itzhaki, and Y. Jiang

References:
1905.00051 (Giveon, Kutasov, SC)
1806.09667 (Giveon, Kutasov, SC)
1805.06286 (Itzhaki,Giveon, Kutasov, SC)
1809.01915 (SC)
1907.07221 (Hashimoto, Kutasov)
1909.11118 (Hashimoto, Kutasov)
+work in progress

Introduction

In the past couple of years there has been some works on irrelevant deformations of CFT_2 that lead to solvable non-local theories. Some of the well studied examples are

1. $T\bar{T}$ **deformation** [(Smirnov,Zamolodchikov),(Cavagli,Negro,Szcsnyi,Tateo)]
2. $J\bar{T}$ **deformation** [(Giveon,Kutasov,SC),(Apolo,Song),(Guica)]
3. **General linear combination of** $T\bar{T}$, $J\bar{T}$ **and** $T\bar{J}$ [(Giveon,Kutasov,SC),(LeFloch,Mezei)]

All these deformations are irrelevant: involves "flowing up the RG flow"

Such processes are, in general, ambiguous: "irreversibility of RG flow"

It turns out that these theories are well defined and solvable. If the theory to start with is integrable, these deformations preserve integrability.

Motivation and research question

Recent progress in $T\bar{T}$ and related deformation raises a natural question: does holography shed any light on such theories?

The spectrum of $T\bar{T}$ deformed CFT has, for one sign of the coupling, a Hagedorn density of state. A natural question that arises at this point is: what is its interpretation from bulk point of view ?

1. In this talk I'll explain string worldsheet techniques to understand such deformations.
2. As we will see, this could be a very concrete way of realizing holography beyond AdS.

We restrict ourselves to CFTs that have a string theory dual in AdS_3.

The bulk dual of 2d CFT is string theory in AdS_3. The $T\bar{T}$ deformation is a double trace deformation of the theory. Unfortunately string theory doesn't give a very good understanding of such deformations.

It's nowhere close to being understood from the bulk point of view why such deformations are exactly solvable.

But that's NOT the end of the story!!!

String theory in AdS_3 contains operators that are closely related to $T\bar{T}, J\bar{T}$ & $T\bar{J}$. This was first realized by Itzhaki, Giveon, Kutasov in 2017.

1. **This approach to such deformations has been quite fruitful. The string theory techniques allows one to calculate many things (e.g. the deformed spectrum, modular properties of the deformed partition sum) that are usually hard to calculate from the field theory side.**

2. **It provides a concrete realization of holography beyond AdS (e.g. climbing out of the near horizon geometry of the F1 strings).**

I'll mostly discuss the most recent developments in these direction in relation to $T\bar{T}, J\bar{T}$ & $T\bar{J}$ deformations.

Aspects of String Theory in AdS_3

The worldsheet string theory in AdS_3 with the only NS-NS B-field turned on is given by the WZW model on the $SL(2,\mathbb{R})$ group manifold. The worldsheet action is invariant under $sl(2,\mathbb{R})_{L/R}$ current algebra at level k. The AdS_3 radius is related to the level as: $R_{AdS} = \sqrt{k}\ell_s$.

The $sl(2,\mathbb{R})_{L/R}$ symmetry algebra plays an important role in analyzing the symmetries, spectrum, and correlation functions of the spacetime theory.

Via the AdS/CFT correspondence, string theory on AdS_3 is dual to CFT_2 living on the boundary of AdS_3. For pure NS H-flux, the theory has the following properties.

1. **The spacetime theory has a normalizable $SL(2,\mathbb{C})$ invariant vacuum: (a) the NS vacuum that corresponds to global AdS_3 in the bulk, (b) the R vacuum that corresponds in the bulk to $M = J = 0$ BTZ blackhole.**

2. **The NS sector states contain a sequence of discrete states coming from the discrete series representation of $SL(2,\mathbb{R})$ followed by a continuum of long string states. The continuum starts at dimension $\sim \dfrac{k}{2}$.**
   (Maldacena,Ooguri 2001)

3. **The R sector states contain a continuum of long strings above a gap of order $\sim \dfrac{1}{k}$. Here the status of the discrete states is not very clear (to me at least).**

In this talk I'm going to stick to the long string states in the R sector.

1. The theory on a single long string was analyzed by Seiberg, Witten in 1999. For string theory on $AdS_3 \times \mathscr{N}$, the theory on a single long string is described by a sigma model on $\mathscr{M}_{6k}^{(L)} = \mathbb{R}_\phi \times \mathscr{N}$.

2. The theory on $\mathbb{R}_\phi$ has a linear dilaton with slope:
$$Q^{(L)} = (k-1)\sqrt{\frac{2}{k}}.$$

3. For example: String theory on $AdS_3 \times S_3 \times T^4$, which has $(4{,}4)$ superconformal symmetry, $\mathscr{M}_{6k}^{(L)} = \mathbb{R}_\phi \times SU(2)_k \times T^4$.

4. The effective coupling of the theory on the long string goes as $g_s \sim \exp(Q^{(L)}\phi)$. Thus the dynamics of the theory on the long strings become strongly coupled as they move towards the boundary.

A natural question that arises at this point is: what is the full boundary CFT for a given AdS_3 vacuum. The answer to that question is, in general, not known but there are reasons to believe that the theory on the long strings are described by the symmetric product CFT: $\dfrac{(\mathcal{M}_{6k}^{(L)})^p}{S_p}$.

There are strong evidences for this statement.

Please note

The full boundary theory is NOT $(\mathscr{M}^{(L)}_{6k})^p/S_p$!!!!

1. **The full boundary theory has an $SL(2,\mathbb{C})$ invariant normalizable vacuum but $(\mathscr{M}^{(L)}_{6k})^p/S_p$ doesn't have a normalizable $SL(2,\mathbb{C})$ invariant vacuum.**

2. **The entropy of the high energy states of the full boundary theory is given by:**

$$S \sim \sqrt{6pk}(\sqrt{h} + \sqrt{\bar{h}})\,.$$

This also agrees with the Bekenstein-Hawking entropy of BTZ blackhole. But, the entropy of the high energy states of $(\mathscr{M}^{(L)}_{6k})^p/S_p$ is given by:

$$S \sim \sqrt{6p\left(2-\frac{1}{k}\right)}(\sqrt{h} + \sqrt{\bar{h}})\,.$$

As claimed in one of the previous slides, the theory of the long string sector is given by the symmetric product CFT

$$\frac{(\mathscr{M}_{6k}^{(L)})^p}{S_p} \, .$$

There are strong evidences for this statement that I'll discuss next.

Evidence 1

- **Matrix string theory logic (Motl,DVV1997): If the theory on a string winding once around a circle is $\mathscr{M}$, then the symmetric product $\mathscr{M}^N/S_N$ provides the description of the Hilbert space of N free strings. The untwisted sector states describe N free strings each winding once around the circle; whereas the $\mathbb{Z}_w$ twisted states describe strings with winding w; general states of n strings with winding $(w_1, w_2, \cdots, w_n)$, where $\sum w_i = N$, are described in terms of the conjugacy classes of the permutation group S_N.**

Long strings in AdS_3 are weakly coupled in a wide range of positions in the radial direction, so the symmetric product description should be a good description of their dynamics in this regime.

Evidence 2

- **Spectrum of long strings: The spectrum of the long strings in the Ramond sector ($M = J = 0$ BTZ) is given by:**

$$E^w_{L/R} = \frac{1}{w}\left[-\frac{j(j+1)}{k} + N_{L/R} - \frac{1}{2}\right] = \frac{1}{w}E^1_{L/R}$$

$$E^w_{L/R} = \frac{R}{2}(E^w \pm P), \quad P \in \frac{1}{R}\mathbb{Z}$$

$$j = -\frac{1}{2} + is, \quad s \in \mathbb{R}$$

$R =$ **radius of the boundary circle,**

$N_{L/R} =$ **left and right moving excitation levels,**

$s \propto$ **radial momentum of the long string.**

To make contact with $(\mathscr{M}_{6k}^{(L)})^p/S_p$ one notes that in a symmetric product CFT, states in the $\mathbb{Z}_w$ twisted sector have energies

$$E_L^w = h_w - \frac{kw}{4}, \quad E_R^w = \bar{h}_w - \frac{kw}{4}.$$

$w=1$ **corresponds to the original CFT. For every state with dimension h_1 in that CFT there is a state in the $\mathbb{Z}_w$ twisted sector with dimension h_w given by**

$$h_w = \frac{h_1}{w} + \frac{k}{4}\left(w - \frac{1}{w}\right).$$

Thus $E_{L/R}^w = \frac{1}{w} E_{L/R}^1$.

The string theory spectrum has exactly the same form.

Evidence 3

The third piece of evidence comes from the study of irrelevant deformations $(T\bar{T}, J\bar{T}\ \&\ T\bar{J})$ **that we discuss next.**

Solvable irrelevant deformations of string theory in AdS_3

1. String theory on AdS_3 contains an operator $D(x, \bar{x})$ (Kutasov,Seiberg1999) that shares many properties in common with the operator $T\bar{T}$ e.g. $D(x, \bar{x})$ is a $(2,2)$ quasi-primary operator of the spacetime Virasoro and has the same OPE with the stress tensor as the $T\bar{T}$ operator.

2. However $D(x, \bar{x})$ is not equal to the $T\bar{T}$ operator; $T\bar{T}$ is double trace but $D(x, \bar{x})$ is single trace.

3. But $D(x, \bar{x}) = \sum_i^p D_i(x, \bar{x}) = \sum_i^p T_i\bar{T}_i$ where $D_i = T_i\bar{T}_i$ is the $T\bar{T}$ operator of the theory on a single long string $\mathscr{M}_{6k}^{(L)}$. Thus $D(x, \bar{x})$ is an operator of the symmetric product $(\mathscr{M}_{6k}^{(L)})^p/S_p$.

We will consider deformation of the long string symmetric product theory by $D(x,\bar{x})$. This corresponds to deforming the i^{th} block $\mathscr{M}^{(L)}_{6k}$ by the operator $D_i(x,\bar{x}) = T_i\bar{T}_i$ and then symmetrize.

The deformation $D(x,\bar{x})$ of the spacetime theory induces on the worldsheet a truly marginal deformation:

$$\int_{\partial AdS} d^2x D(x,\bar{x}) \sim \int_{\text{ws}} d^2z J^-_{SL}\bar{J}^-_{SL}$$

where J^-_{SL} & $\bar{J}^-_{SL}$ are the left and right moving null $SL(2,\mathbb{R})$ currents.

These are current-current deformation of the worldsheet theory and hence exactly solvable.

The above discussion has a natural generalization that involves Kac-Moody currents. This time we start with $AdS_3 \times \mathcal{N}$ such that $\mathcal{N}$ contains a left moving $U(1)$ current $J(x)$.

Similar to the $T\bar{T}$ case, one can construct the operator $A(x, \bar{x})$ in the long string sector, that corresponds to $J\bar{T}$ deformation of the individual block $\mathcal{M}_{6k}^{(L)}$ and then symmetrized. As in the case of $T\bar{T}$, the deformation of the spacetime CFT by $A(x, \bar{x})$ induces on the worldsheet theory on $AdS_3 \times S^1$ a truly marginal deformation:

$$\int_{\partial AdS} d^2x A(x, \bar{x}) \sim \int_{\rm ws} d^2z K \bar{J}^-_{SL}$$

where K the left moving $U(1)$ current on the worldsheet.

We thus study the deformation of the sigma model on $AdS_3 \times S^1$.

One can add to the boundary Lagrangian an arbitrary combination of single trace $T\bar{T}, J\bar{T}$ & $T\bar{T}$ coupling to get a solvable theory. The induced deformation on the worldsheet is given by

$$\delta \mathscr{L}_{ws} = \lambda J^-_{SL} \bar{J}^-_{SL} + \epsilon_+ K \bar{J}^-_{SL} + \epsilon_- J^-_{SL} \bar{K} \, .$$

The worldsheet deformations are truly marginal and of the form current-current deformation. Such worldsheet deformations are exactly solvable.

We thus study deformation the sigma model on $AdS_3 \times S^1$.

Deformation of the worldsheet theory

The worldsheet action on $AdS_3 \times S^1$ is given by

$$S = \frac{k}{2\pi}\int d^2z \left(\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}\bar{\partial}\gamma + \frac{1}{k}\partial y\bar{\partial}y \right).$$

We consider deformation of the form

$$\delta\mathscr{L}_{ws} = \lambda J^-_{SL}\bar{J}^-_{SL} + \epsilon_+ K\bar{J}^-_{SL} + \epsilon_- J^-_{SL}\bar{K}.$$

The full deformed action is given by

$$S(\lambda, \epsilon_+, \epsilon_-) = \frac{k}{2\pi}\int d^2z\Big(\partial\phi\bar{\partial}\phi + h\partial\bar{\gamma}\bar{\partial}\gamma + \frac{2\epsilon_+ h}{\sqrt{k}}\partial y\bar{\partial}\gamma + \frac{2\epsilon_- h}{\sqrt{k}}\partial\bar{\gamma}\bar{\partial}y + \frac{f^{-1}h}{k}\partial y\bar{\partial}y \Big)$$

where $h^{-1} = \lambda - 4\epsilon_+\epsilon_- + e^{-2\phi}, \quad f^{-1} = \lambda + e^{-2\phi}$**.**

Residual symmetries:

$$
\begin{array}{c}
SL(2,\mathbb{R})_{L,null} \times SL(2,\mathbb{R})_{R,null} \times U(1)_L \times U(1)_R \\
\Big\downarrow \\
U(1)_{L,null} \times U(1)_{R,null} \times U(1)_L \times U(1)_R
\end{array}
$$

The deformed background (after KK reduction):

$$ds^2 = k(d\phi^2 + hd\gamma d\bar{\gamma} - fh(\epsilon_+ d\gamma + \epsilon_- d\bar{\gamma})^2),$$

$$e^{2\Phi} = g_s^2 e^{-2\phi} h,$$

$$B_{\gamma\bar{\gamma}} = \frac{kf}{2}\,.$$

In addition there are also KK scalar and fields.

Recover known theories

1. $\lambda \neq 0, \epsilon_{\pm} = 0$: setup for $T\bar{T}$ deformed CFT
    - For $\lambda > 0$, background interpolates between AdS_3 in IR and flat spacetime with a linear dilaton in the UV.
    - For $\lambda > 0$, the boundary field theory interpolates between CFT_2 in the IR and "Little String Theory " in the UV.
    - For $\lambda < 0$, the bulk geometry has naked singularity and closed time like curves.

2. $\lambda = 0, \epsilon_{+} \neq 0, \epsilon_{-} = 0$: setup for $J\bar{T}$ deformed CFT
    - Background interpolates between AdS_3 in IR and null-$WAdS_3$ spacetime in UV for both signs of ϵ_+.
    - The boundary field theory interpolates between CFT_2 in the IR to something very similar to Null-Warpped CFT in the UV.
    - The dual spacetime has closed timelike curves.
    - This is possibly related to $WAdS_3/WCFT_2$ but so far the connection has not yet been well understood.

Calculate Spectrum

We follow the standard textbook techniques to calculate the spectrum. Since we are interested in long strings propagating at large ϕ it is convenient to express the deformed Hamiltonian in the free Wakimoto variables.

$$\mathscr{L} = -\partial\phi_+\bar{\partial}\phi_- - \partial\phi_-\bar{\partial}\phi_+ + \hat{\lambda}\partial\phi_+\bar{\partial}\phi_+ + 2\hat{\epsilon}_+\partial y\bar{\partial}\phi_+ + 2\hat{\epsilon}_-\partial\phi_+\bar{\partial}y + \partial y\bar{\partial}y + \mathscr{L}_\phi$$

$$\mathscr{L}_\phi = \partial\phi\bar{\partial}\phi - \sqrt{\frac{2}{k}}\hat{R}\phi, \quad (\lambda, \epsilon_\pm^2) = \frac{R^2}{2\alpha'}(\hat{\lambda}, \hat{\epsilon}_\pm^2)$$

$$\gamma = i\phi_-, \qquad \bar{\gamma} = i\bar{\phi}_-, \qquad \phi_\pm = \frac{1}{\sqrt{2}}(\phi_0 \pm \phi_1)$$

$$\phi_\mu(z)\phi_\nu(w) \sim -\eta_{\mu\nu}\ln(z-w); \quad \eta_{\mu\nu} = \mathrm{diag}(-1,1)$$

At large ϕ this is a free theory.

Low lying vertex operators

Low lying vertex operators with oscillations in the transverse space:

$M = J = 0$ **BTZ**

$$V_{\Delta,\bar{\Delta}} = e^{\sqrt{\frac{2}{k}} j(\phi+\bar{\phi})} V^{w}_{E_{L,R}}$$

$$V^{w}_{E_{L,R}} = e^{iw\phi_{+}+iE_{L}\phi_{-}} e^{iw\bar{\phi}_{+}+iE_{R}\bar{\phi}_{-}}$$

$J^{-}_{\mathrm{SL}} \bar{J}^{-}_{\mathrm{SL}} + K \bar{J}^{-}_{\mathrm{SL}} + J^{-}_{\mathrm{SL}} \bar{K}$ **deformed massless BTZ** $\times\, S^1$

$$V_{\Delta,\bar{\Delta}} = e^{\sqrt{\frac{2}{k}} j(\phi+\bar{\phi})} V^{w}_{E_{L,R};q_{L,R}}$$

$$V^{w}_{E_{L,R};q_{L,R}} = e^{iw\phi_{+}+iE_{L}\phi_{-}+iq_{L}y} \times e^{iw\bar{\phi}_{+}+iE_{R}\bar{\phi}_{-}+iq_{R}\bar{y}}$$

Physical vertex operators

Consider physical vertex operators in the (-1,-1) picture:

$$V_{\text{phys}} = e^{-\varphi} e^{-\bar{\varphi}} V_{\Delta,\bar{\Delta}} V^{N,\bar{N}}_{\text{internal}} \,.$$

On-shell conditions:

$$\Delta + N - \frac{1}{2} = 0, \quad \bar{\Delta} + \bar{N} - \frac{1}{2} = 0.$$

Dimensions:

$$\Delta = \frac{P_L P_L^t}{2} - \frac{j(j+1)}{k}, \quad \bar{\Delta} = \frac{P_R P_R^t}{2} - \frac{j(j+1)}{k} \,.$$

$$P_{L/R} = (n^t + m^t(B \mp G))e^*, \quad e^*(e^*)^t = \frac{G^{-1}}{2} \,.$$

Dimensions Δ, $\bar{\Delta}$

Massless BTZ:

$$\Delta = -wE_L - \frac{j(j+1)}{k}, \quad j \in -\frac{1}{2} + i\mathbb{R}$$

$$\bar{\Delta} = -wE_R - \frac{j(j+1)}{k}$$

$$\bar{\Delta} - \Delta = nw$$

Deformed massless BTZ $\times S^1$**:**

$$\Delta = \frac{q_L^2}{2} - E_L\left(w + \frac{\hat{\lambda}}{2}E_R\right) + \hat{\epsilon}_+ q_L E_R + \hat{\epsilon}_- q_R E_L + \frac{1}{2}(\hat{\epsilon}_+ E_R + \hat{\epsilon}_- E_L)^2 - \frac{j(j+1)}{k}$$

$$\bar{\Delta} = \frac{q_R^2}{2} - E_R\left(w + \frac{\hat{\lambda}}{2}E_L\right) + \hat{\epsilon}_+ q_L E_R + \hat{\epsilon}_- q_R E_L + \frac{1}{2}(\hat{\epsilon}_+ E_R + \hat{\epsilon}_- E_L)^2 - \frac{j(j+1)}{k}$$

$$\bar{\Delta} - \Delta = \frac{1}{2}(q_R^2 - q_L^2) + wn$$

Spectrum

Using on-shell condition and $E_L(0) = h_w - \frac{kw}{4}$ for undeformed ($\mathbb{Z}_w$ twisted) symmetric product CFT $(\mathscr{M}_{6k}^{(L)})^N / S_N$ with $c_{\mathscr{M}_{6k}^{(L)}} = 6k$, one can read off the deformed spectrum as:

$$
\begin{aligned}
E_L(0) &= h_w - \frac{kw}{4} \\
&= E_L\left(w + \frac{\hat{\lambda}}{2} E_R\right) - \frac{1}{\omega}\left(\hat{\epsilon}_+ q_L E_R + \hat{\epsilon}\_q_R E_L + \frac{1}{2}(\hat{\epsilon}_+ E_R + \hat{\epsilon}\_E_L)^2\right) \\
E_R(0) &= \bar{h}_w - \frac{kw}{4} \\
&= E_R\left(w + \frac{\hat{\lambda}}{2} E_L\right) - \frac{1}{\omega}\left(\hat{\epsilon}_+ q_L E_R + \hat{\epsilon}\_q_R E_L + \frac{1}{2}(\hat{\epsilon}_+ E_R + \hat{\epsilon}\_E_L)^2\right)
\end{aligned}
$$

1. **For $w = 1$ the spectrum is that of the deformed block $\mathscr{M}_{\text{def}}^{(L)}$.**
2. **For $w > 1$ the spectrum is that of the $\mathbb{Z}_w$ twisted sector of the symmetric product $(\mathscr{M}_{\text{def}}^{(L)})^N / S_N$.**

Spectrum for $w = 1$

For $w = 1$ the spectrum is that of the deformed block $\mathscr{M}^{(L)}_{\text{def}}$.

$$ER = E_L + E_R = n + \frac{1}{2A}\left(-B - \sqrt{B^2 - 4AC}\right)$$

$$A = \frac{1}{4}\left((\hat{\epsilon}_+ + \hat{\epsilon}_-)^2 - \hat{\lambda}\right)$$

$$B = -1 + \hat{\epsilon}_+ q_L + \hat{\epsilon}_- q_R + n\hat{\epsilon}_-(\hat{\epsilon}_+ + \hat{\epsilon}_-) - \frac{\hat{\lambda} n}{2}$$

$$C = 2\left(\bar{h}_1 - \frac{c}{24} - \frac{q_R^2}{2}\right) + (q_R + n\hat{\epsilon}_-)^2$$

If the boundary field theory deformation is given by $-tT\bar{T} + \mu_+ J\bar{T} + \mu_- T\bar{J}$, then the worldsheet couplings are related to the spacetime couplings as

$$t = \frac{\pi R^2 \hat{\lambda}}{2}, \quad \mu_\pm = 2R\hat{\epsilon}_\pm .$$

1. $A > 0$: the theory is sick, high energy states become complex

2. $A < 0$: the spectrum is real

- For a given negative A the high energy spectrum has Hagedorn entropy, $S = \beta_H E$ where

$$\beta_H = 4\pi R\sqrt{\frac{C|A|}{12}}.$$

In particular, $\beta_H \to 0$ as $|A| \to 0$. The theory has multiple scales.

- The theory in the limit $|A| \to 0$ is interesting. The inverse Hagedorn temperature vanishes. The behavior of the energies depend on the sign of B. States with $B > 0$ decouples. The states for which $B < 0$ survive and their energies are given by $E = n + \frac{C}{|B|}$. The entropy goes as

$$S \sim \sqrt{|B|E}.$$

This is intermediate between Cardy and Hagedorn.

Spectrum vs Geometry

1. **In some region in the coupling space, the spectrum of the boundary theory is healthy (real energies). In the complementary region it is sick.**

2. **In same region in the coupling space, the geometries are healthy (no singularities, CTC's, etc). In the complementary region it is sick.**

Thus there is an "one to one correspondence" between COMPLEX ENERGIES of the boundary theory and CLOSED TIMELIKE CURVES in the dual geometry.

Conclusion

1. Studied a class of solvable deformations of the worldsheet theory that correspond to irrelevant deformations of the boundary theory.

2. Computed the spectrum of the deformed field theory from the string theory analysis.

3. “Proposed a holographic dual.”

4. Our technique provide a systematic way of realizing holography in non-AdS backgrounds.

5. String theory analysis provides further evidence in support of symmetric product description of the long string sector of the spacetime theory.

Thank you!

Backup slides

$$J_{SL}(x;z) = 2xJ^3_{SL}(z) - J^+_{SL}(z) - x^2 J^-_{SL}(z)$$

$$\Phi_h(x;z) = \frac{1}{\pi}\left(\frac{1}{|\gamma - x|^2 e^{\phi} + e^{-\phi}}\right)^{2h}$$

$$T(x) = \frac{1}{2k}\int d^2z \left(\partial_x J_{SL}\partial_x \Phi_1 + 2\partial_x^2 J_{SL}\Phi_1\right)\bar{J}_{SL}(\bar{x};\bar{z})$$

$$D(x,\bar{x}) = \int d^2z \left(\partial_x J_{SL}\partial_x + 2\partial_x^2 J_{SL}\right)\left(\partial_{\bar{x}}\bar{J}_{SL}\partial_{\bar{x}} + 2\partial_{\bar{x}}^2\bar{J}_{SL}\right)\Phi_1(\bar{x};\bar{z}),$$

$$J(x) = -\frac{1}{k}\int d^2z K(z)\bar{J}_{SL}(\bar{x};\bar{z})\Phi_1(x,\bar{x};z,\bar{z})$$

$$A(x,\bar{x}) = \int d^2z K(z)\left(\partial_{\bar{x}}\bar{J}_{SL}\partial_{\bar{x}}\Phi_1 + 2\partial_{\bar{x}}^2\bar{J}_{SL}\Phi_1\right)$$