Effective theories for a nonrelativistic field in an expanding universe

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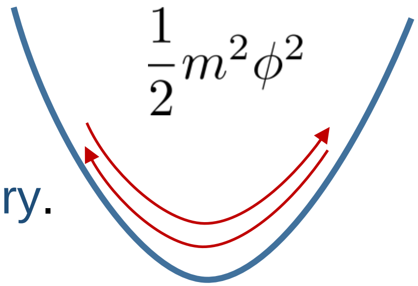
Based on 2005.05388 with M.H. Namjoo and D.I. Kaiser

Introduction

Nature of cold collisionless dark matter (CDM) is one of the puzzles of theoretical cosmology.

Ultra-light particles are one of the well-motivated candidates: QCD-axion, string theory-axion.



They form a BEC with large number density of particles

so that the dynamics can be modeled with a classical field theory.

While fundamental models are relativistic field theories, It seems plausible to have an effective description for classical nonrelativistic field with gravity for low energy phenomena.

Sikivie [2006]
Ferreira [2020]
Hertzberg [2016]
Guth, Hertzberg and Prescod-Weinstein [2014]
Brambilla, Pineda, Soto and Vairo [2004]

Introduction

The common practice is to write

main time dependence

$$\phi(t,\mathbf{x}) = \frac{1}{\sqrt{2m}}\left(e^{-imt}\psi(t,\mathbf{x}) + e^{imt}\psi^*(t,\mathbf{x})\right)$$

low energy field

1

The equation of motion reads

$$i\dot{\psi} + i\frac{3}{2}H\left(\psi - e^{2imt}\psi^*\right) - \frac{3}{2}\frac{H}{m}\dot{\psi} - \frac{1}{2m}\ddot{\psi} + \cdots = 0$$

2

Background and fluctuations are described by

$$\rho = m|\psi|^2 \propto a^{-3} \qquad \ddot{\delta} + 2H\dot{\delta} - c_s^2\frac{\nabla^2\delta}{a^2} = \frac{\rho}{2{M_{\rm Pl}}^2}\delta$$

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Outline

Suitable field redefinition 1

Nonrelativistic variables 2

Effective Imperfect fluid 3

Namjoo, Guth and Kaiser [2017]
Mukaida, Takimoto and Yamada [2016]
Braaten, Mohapatra and Zhang [2018]

A suitable field redefinition

$$\phi(t,\mathbf{x}) = \frac{1}{\sqrt{2m}}\left(e^{-imt}\psi(t,\mathbf{x}) + e^{imt}\psi^*(t,\mathbf{x})\right)$$

Number of d.o.f does not match.

One approach is to consider it as a canonical transformation in phase space

$$\phi = \frac{1}{\sqrt{2m}}\mathcal{P}^{-1/2}\left(e^{-imt}\psi + e^{imt}\psi^*\right)$$

$$\pi = -i\sqrt{\frac{m}{2}}\mathcal{P}^{1/2}\left(e^{-imt}\psi - e^{imt}\psi^*\right)$$

$$\pi = \dot{\phi}$$

nonlocal operator

$$\mathcal{P} = \sqrt{1 - \frac{\nabla^2}{m^2}}$$

Namjoo, Guth and Kaiser [2017]

For a general curved metric, it is not straightforward to define such an operator.

A suitable field redefinition

$$\phi(t,\mathbf{x}) = \frac{1}{\sqrt{2m}}\left(e^{-imt}\psi(t,\mathbf{x}) + e^{imt}\psi^*(t,\mathbf{x})\right)$$

It does not uniquely determine ψ but has a gauge freedom

$$\psi(t,\mathbf{x}) \to \psi(t,\mathbf{x}) + ie^{imt}\eta(t,\mathbf{x})$$

We can fix the gauge such that

$$\dot{\phi} = -i\sqrt{\frac{m}{2}}\left(e^{-imt}\psi - e^{imt}\psi^*\right)$$

Note that all these can be done for a general metric

A suitable field redefinition

For a general metric we can obtain a Schrodinger-like equation

nonzero for interacting case

$$ig^{00}\dot{\psi} + \mathcal{D}\psi + e^{2imt}\mathcal{D}^*\psi^* = 0$$ + Einstein's equations

operator involving metric and spatial derivative

Our main focus is the flat perturbed FLRW in Newtonian gauge

PQSB before inflation $\bar{\psi},\ H,\ a,\ \delta\psi,\ \Phi$

For background we get Schrodinger-Friedmann system

$$i\dot{\bar{\psi}} + i\frac{3}{2}H\left(\bar{\psi} - e^{2imt}\bar{\psi}^*\right) = 0$$

$$3{M_{\rm Pl}}^2 H^2 - m\bar{\psi}^*\bar{\psi} = 0$$

Similar set of equations are obtained for linear fluctuations

Smearing and mode expansion

While the main energy in the field ϕ is its mass scale, we are usually interested in low energy processes, i.e. Hubble scale or momentum scale

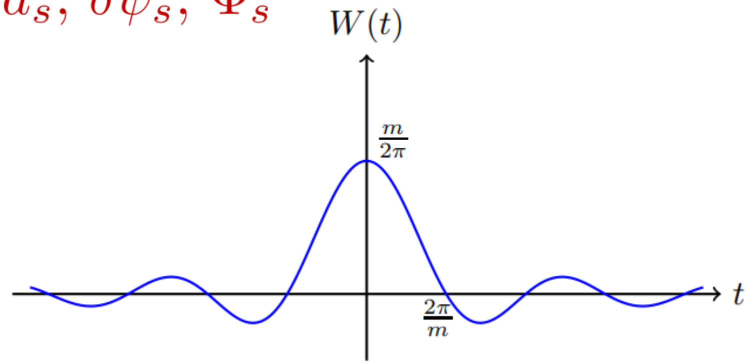
In defining ψ we have already isolated the main time dependence of the original field. However, it still has high-energy remnants produced by pure oscillating factors in the equation of motion

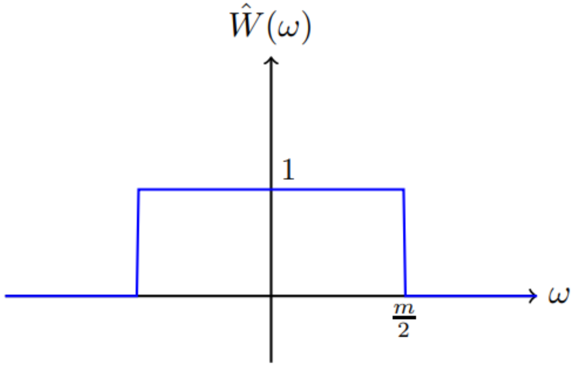
We define the smeared variables

$$\bar{\psi},\ H,\ a,\ \delta\psi,\ \Phi$$

$$\langle X(t)\rangle = \int_{-\infty}^{\infty} \mathrm{d}t'\, X(t')W(t-t')$$

$$\bar{\psi}_s,\ H_s,\ a_s,\ \delta\psi_s,\ \Phi_s$$

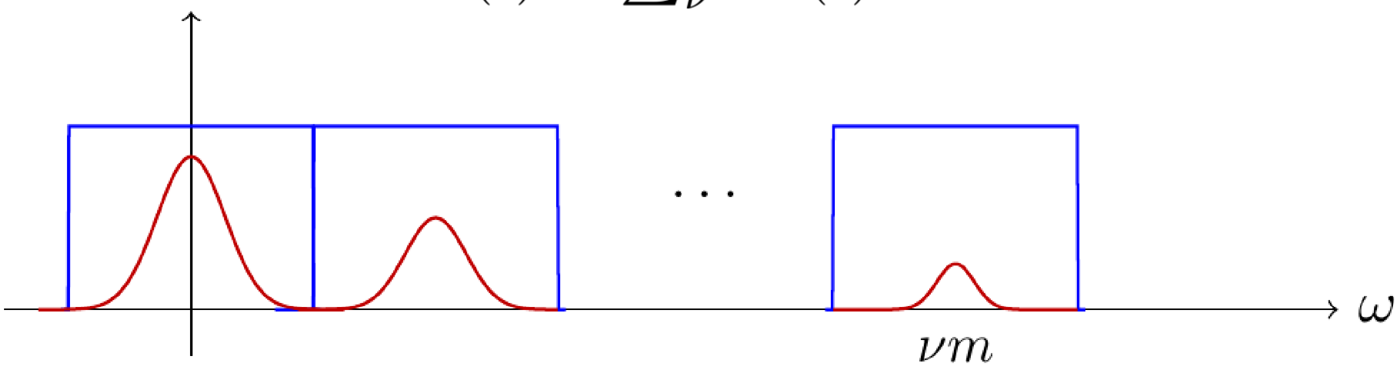




Smearing and mode expansion

Next we write any variable in terms of series of low energy variables times a pure oscillation

$$X(t) = \sum_\nu X_\nu(t) e^{i\nu m t}$$



We will obtain an infinite set of coupled equations for each mode. We are interested in variables with $\nu = 0$ but for that we need to solve nonzero modes. This can be done perturbatively by identifying small parameters in the problem.

$$\bar{\psi}_s,\ H_s,\ a_s,\ \delta\psi_s,\ \Phi_s$$

Namjoo, Guth and Kaiser[2017]

Nonrelativistic effective theory

The small parameters that quantify the amount of being nonrelativistic

$$\epsilon_t \sim \frac{1}{mX}\frac{\partial X}{\partial t}\,, \qquad \epsilon_H \sim \frac{H}{m}\,, \qquad \epsilon_k \sim \frac{k_p^2}{m^2}\,, \qquad \epsilon_\lambda \sim \lambda\frac{\phi^2}{m^2}$$

low energy variable

Nonzero modes can be solved in terms of the slow mode variables order by order in small parameters

$$\psi_2 = \frac{3H_s}{4im}\psi_s^* + \dots \qquad H_2 = \frac{\psi_s^{*\,2}}{8iM_{\rm Pl}{}^2} + \dots \qquad a_2 = -\frac{\psi_s^{*\,2}}{16mM_{\rm Pl}{}^2} + \dots$$

$$\delta\psi_2 = \left[\frac{3H_s}{4im}\delta\psi_s^* + \frac{\nabla^2\delta\psi_s^*}{4m^2a_s^2} - \frac{1}{2}\psi_s^*\Phi_s\right] + \dots \qquad \Phi_2 = \frac{\psi_s^*\delta\psi_s^*}{8mM_{\rm Pl}{}^2} + \dots$$

For real variables $R_{-\nu} = R_\nu^*$

Nonrelativistic effective theory

The effective theory for the slow mode can be obtained

$$i\dot{\bar{\psi}}_s + i\frac{3}{2}H_s\bar{\psi}_s + \frac{9}{16}\frac{|\bar{\psi}_s|^2}{{M_{\rm Pl}}^2}\bar{\psi}_s + i\frac{9}{32}\frac{H_s}{m}\frac{|\bar{\psi}_s|^2}{{M_{\rm Pl}}^2}\bar{\psi}_s + \cdots = 0$$

$$3{M_{\rm Pl}}^2H_s^2 = m|\bar{\psi}_s|^2 + \frac{3}{32{M_{\rm Pl}}^2}|\bar{\psi}_s|^4 + \dots$$

and similarly for linear fluctuations.

By comparison to an effective theory for a self-interacting field

$$i\dot{\psi}_s + \frac{1}{m^2}\nabla^2\psi_s - \frac{\lambda}{8m^2}|\psi_s|^2\psi_s + \cdots = 0$$

We can interpret the corrections as effective self-interaction

$$\lambda_{\rm eff} = -\frac{9}{2}\left(\frac{m}{M_{\rm Pl}}\right)^2$$

Nonrelativistic effective theory

The r.h.s of the effective Friedmann equation can be interpreted as the effective energy density in an effective spacetime with H_s as the expansion rate

$$\rho_{\mathrm{eff}} = m|\bar{\psi}_s|^2 + \frac{3}{32{M_{\mathrm{Pl}}}^2}|\bar{\psi}_s|^4 + \ldots\,, \qquad p_{\mathrm{eff}} = \frac{9}{32{M_{\mathrm{Pl}}}^2}|\bar{\psi}_s|^4 + \ldots$$

With the effective equation of state

$$w_{\mathrm{eff}} = \frac{27}{32}\frac{H_s^2}{m^2} + \ldots$$

Nonrelativistic effective theory

For linear fluctuations we have

$$i\delta\dot{\psi}_s + \frac{3i}{2}H_s\delta\psi_s + \frac{\nabla^2\delta\psi_s}{2ma_s^2} - m\bar{\psi}_s\Phi_s$$
$$+\frac{9}{8{M_{\rm Pl}}^2}|\bar{\psi}_s|^2\delta\psi_s - \frac{7}{16{M_{\rm Pl}}^2}\bar{\psi}_s^2\delta\psi_s^* + \frac{\nabla^4\delta\psi_s}{8m^3a_s^4} + 2iH_s\bar{\psi}_s\Phi_s = 0$$

$$\frac{\nabla^2\Phi_s}{a_s^2} = \frac{m}{2{M_{\rm Pl}}^2}(\bar{\psi}_s^*\delta\psi_s + \bar{\psi}_s\delta\psi_s^*) + \frac{3i}{4{M_{\rm Pl}}^2}H_s(\bar{\psi}_s^*\delta\psi_s - \bar{\psi}_s\delta\psi_s^*) - \frac{3}{2}H_s^2\Phi_s$$

In the effective Poisson's equation and interpret the r.h.s as the effective comoving overdensity

$$\frac{\nabla^2\Phi_s}{a_s^2} = \frac{1}{2{M_{\rm Pl}}^2}\rho_{\rm eff}\,\delta_{\rm eff}$$

Nonrelativistic effective theory

By using the effective equations of motion we can find an 2nd order equation for the growth of overdensity

effective coefficient of bulk viscosity

$$\ddot{\delta}_{\rm eff} + 2H_s\dot{\delta}_{\rm eff} - c_{\rm eff}^2\frac{\nabla^2\delta_{\rm eff}}{a_s^2} - \frac{\zeta_{\rm eff}}{\rho_{\rm eff}}\frac{\nabla^2\dot{\delta}_{\rm eff}}{a_s^2} = \frac{\rho_{\rm eff}}{2{M_{\rm Pl}}^2}\delta_{\rm eff}$$

effective sound speed

Nonrelativistic effective theory

The effective sound speed

$$c_{\text{eff}}^2 = \frac{k^2}{4m^2a_s^2} - \frac{k^4}{8m^4a_s^4} + \frac{15}{16}\frac{H_s^2}{m^2} + \dots$$

This is usually written in the literature

Hwang and Noh [2009]

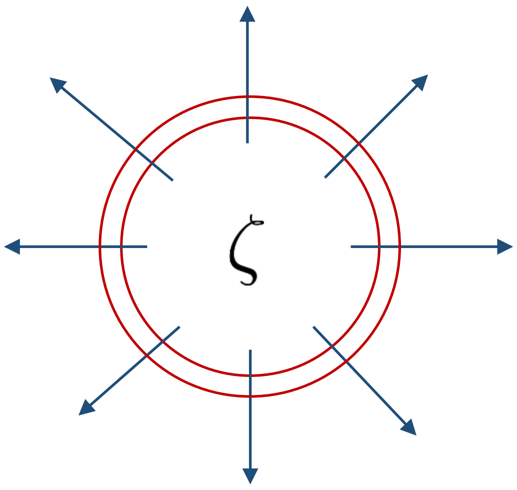
$$c_s^2 = \frac{k^2}{4m^2a^2}\left(1 + \frac{k^2}{4m^2a^2}\right)^{-1} = \frac{k^2}{4m^2a^2} - \frac{k^4}{16m^4a^4} + \dots$$

In the flat spacetime limit we can find a nonlocal field redefinition such that the sound speed can be obtained for arbitrary momentum

$$c_s^2 = \frac{m^2}{k^2}\left(\sqrt{1 + \frac{k^2}{m^2}} - 1\right)^2 = \frac{k^2}{4m^2} - \frac{k^4}{8m^4} + \dots$$

The last term which is k-independent was missing in the literature

Nonrelativistic effective theory



The effective bulk viscosity coefficient

$$\zeta_{\rm eff} = -\frac{H_s}{2m^2}\rho_{\rm eff}$$

The effect of bulk viscosity is to modify the pressure

$$p \to p - \zeta\, u^{\kappa}{}_{;\kappa}$$

For background equations the effect is degenerate with pressure

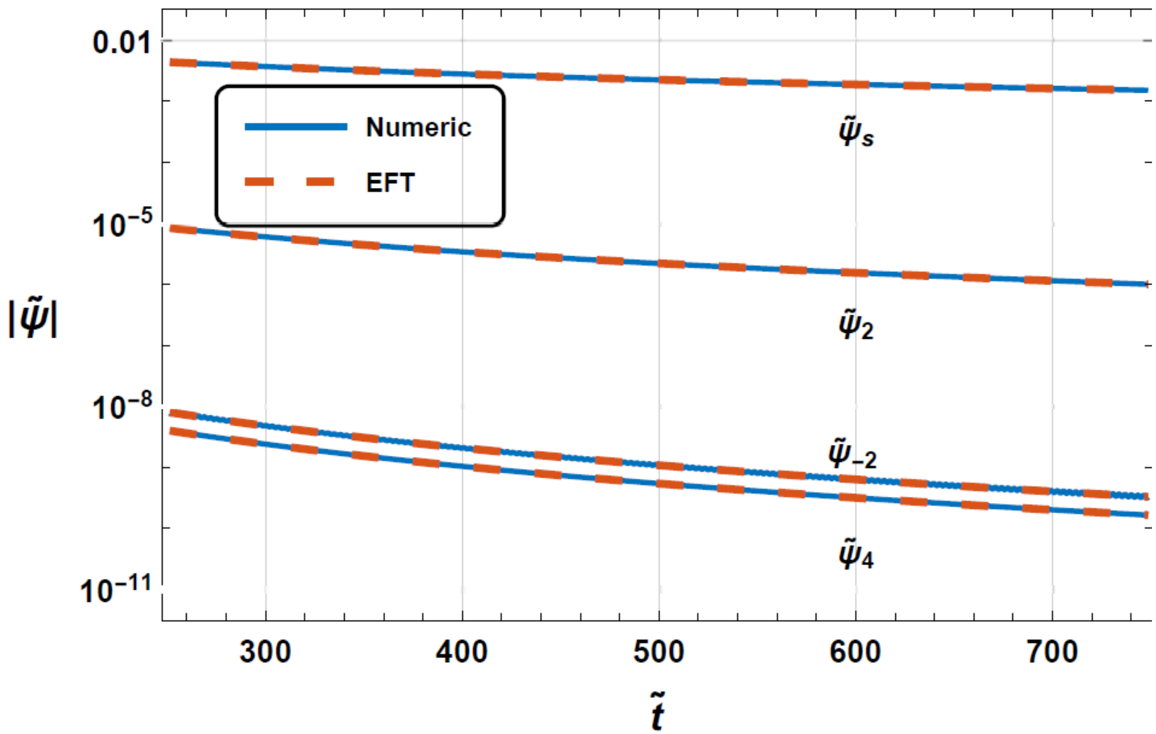
$$p_{\rm eff} = p - 3H\zeta$$

But for fluctuations there are nontrivial contributions for instance

$$\delta p = c_s^2\,\bar\rho\delta_c - \left[3H\dot\zeta - \zeta\frac{3\bar\rho}{2{M_{\rm Pl}}^2}\right]\delta u$$

Numerical results

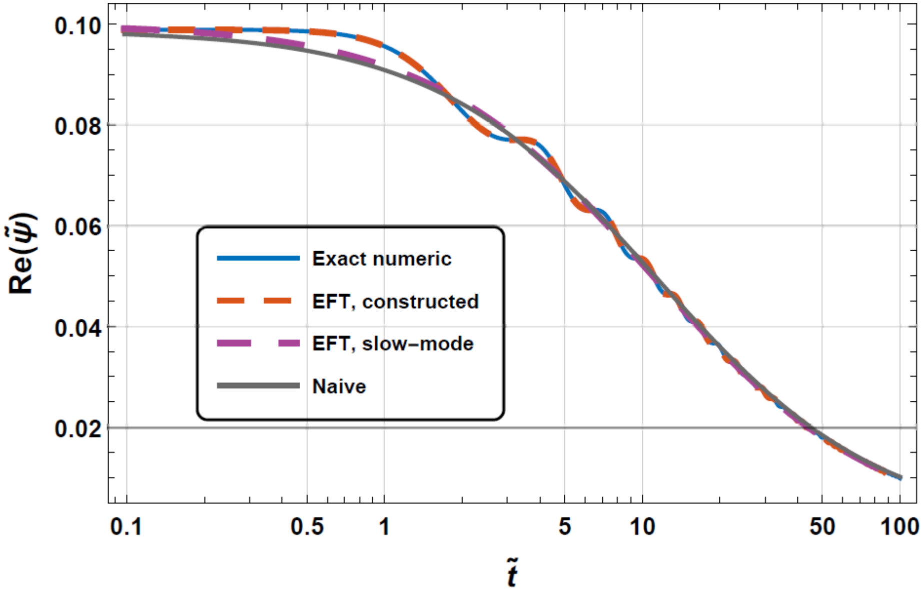
Different modes can be obtained numerically by applying the window function. This can be compared with analytic results.

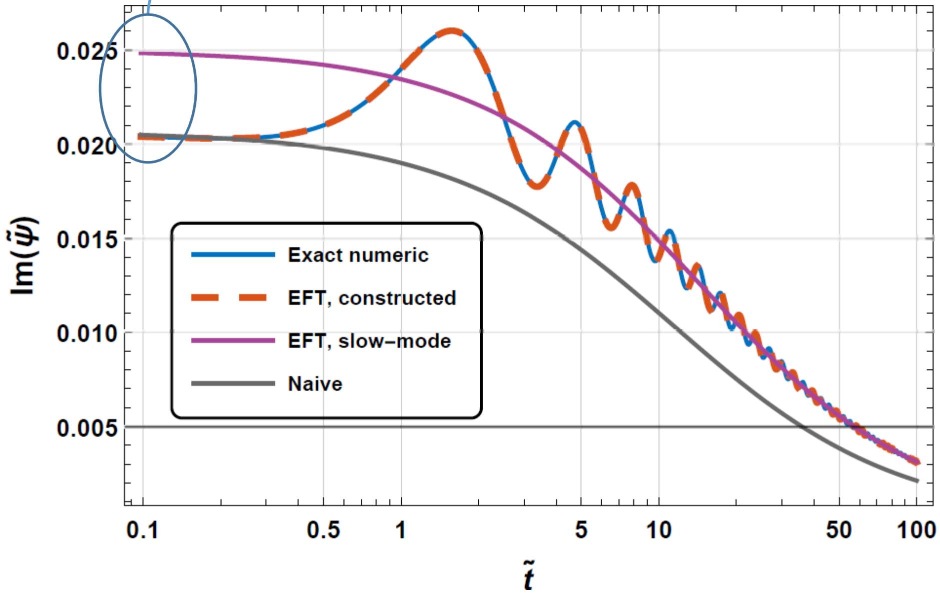


Numerical results

The behavior of the effective theory can be examined compared to the naive and exact equations

Initial conditions are dictated by mode expansion

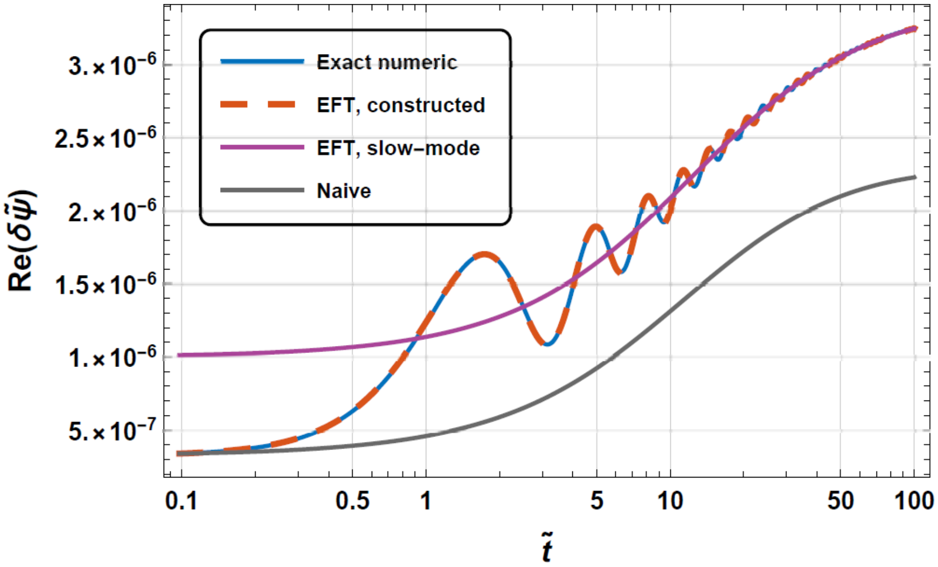


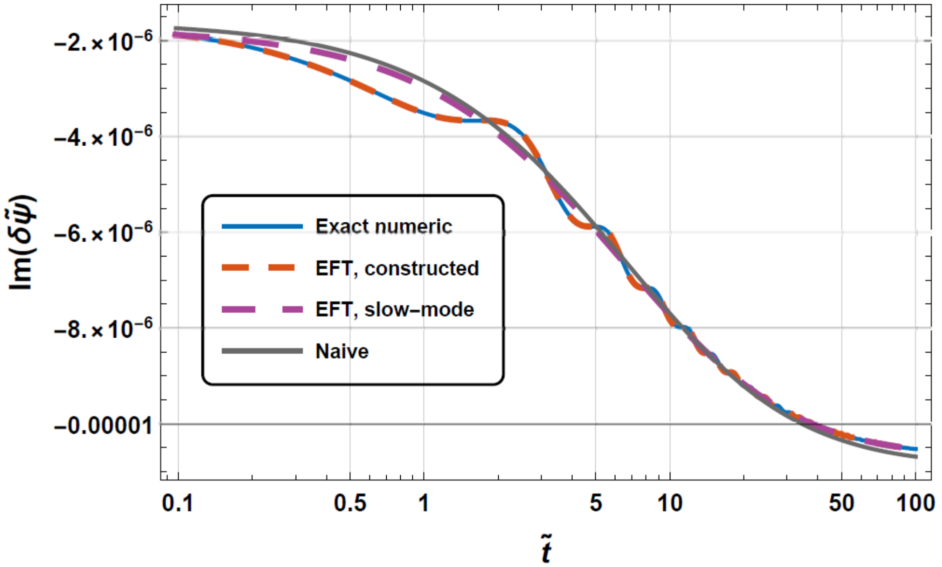


As a by product, we can reconstruct the exact result by solving the slow mode.

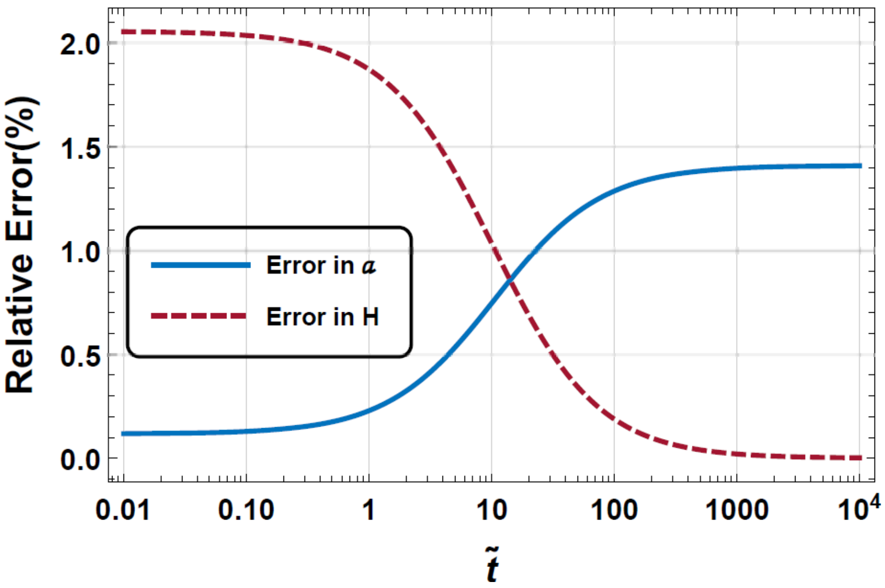
$$\bar{\psi}(t) = \bar{\psi}_s + \bar{\psi}_2 e^{2imt} + \bar{\psi}_4 e^{4imt} + \bar{\psi}_{-2} e^{-2imt} + \dots$$

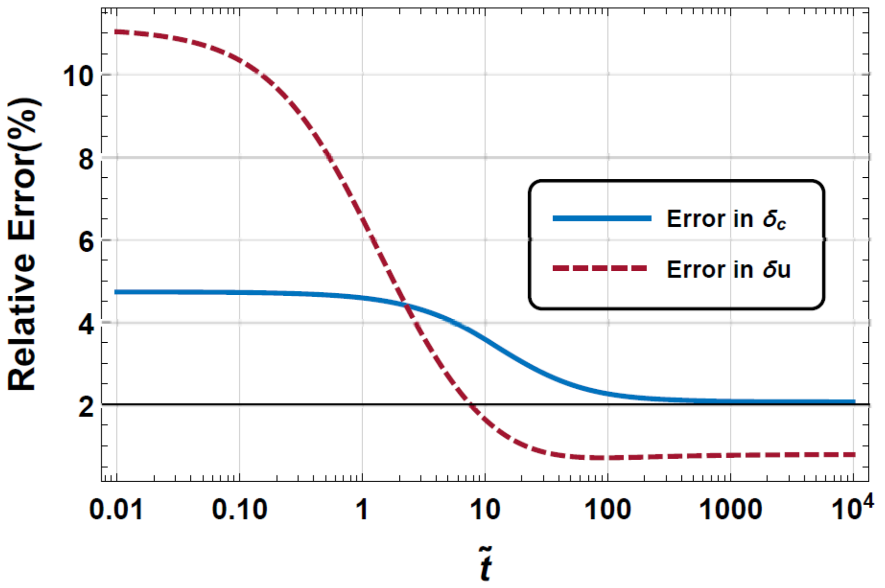
Numerical results





Numerical results





Summary

- We have explored a consistent EFT for nonrelativistic scalar field in FLRW universe.
- We showed that the system could be described by an imperfect fluid and obtained nontrivial effective pressure, sound speed and bulk viscosity.

- Observational signature (like CMB) of deviation from naive theory in realistic situations must be explored.
- Implications for structure formation (nonlinear regime) might be interesting.
- Coupling to gauge boson can be considered.

Thank you