Perverse sheaves and knot contact homology

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The first part is joint work with Yu. Berest and A. Eshmatov

Knot contact homology

Knot contact homology

- An invariant of knots and links (Ng, Ekholm-Etnyre-Ng-Sullivan)
- A special case of Legendrian contact homology (Eliashberg, Chekanov, Eliashberg-Givental-Hofer)
- Defined by counting pseudoholomorphic disks

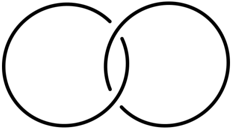
This talk:
A new algebraic construction of knot contact homology

Knots and links

Knots

Trefoil

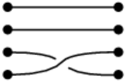
Links



Hopf link

Braid group

The set of braids on *n*-strands up to isotopy forms a group under concatenation.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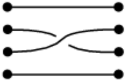
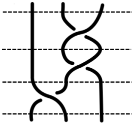

gives


Definition

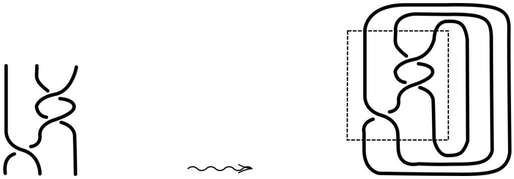
The braid group on *n*-strands is the group

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i \;\; \text{if } |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle$$



$$\beta = \sigma_2^2\sigma_1^{-1} \in B_3$$

Braid closure



$$\beta = \sigma_2^2\sigma_1^{-1} \quad \longmapsto \quad \hat{\beta} \; = \; \text{Hopf link}$$

Braid closure

Alexander's theorem

Every link is the closure of some braid.

Markov's theorem

If two braids $\beta_1 \in B_n$ and $\beta_2 \in B_m$ close to the same link, then they are related by a finite sequence of Markov moves:

(M1) $\alpha \in B_r \quad \sim \quad \gamma\alpha\gamma^{-1} \in B_r$

(M2a) $\alpha \in B_r \quad \sim \quad \alpha\sigma_r \in B_{r+1}$

(M2b) $\alpha \in B_r \quad \sim \quad \alpha\sigma_r^{-1} \in B_{r+1}$

Recipe for producing link invariants

- Construct a sequence of braid group representations
  $B_n \to \mathrm{Aut}(X_n), \qquad n \geq 1$
- Thus each braid $\beta \in B_n$ gives a map $\beta : X_n \to X_n$. Construct an object $P[\beta]$ from this map.
- Show that this construction $P[-]$ is invariant under Markov moves.

Recipe for producing link invariants

Many quantum link invariants can be constructed via this recipe.

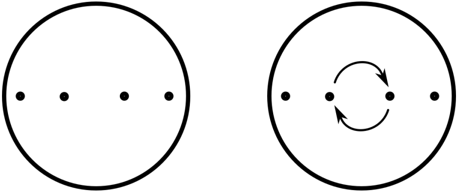
e.g. Jones polynomial,
HOMFLY-PT polynomial,
Reshetikhin-Turaev-Witten invariants,
Khovanov-Rozansky link homology.

This talk:
Many classical invariants can also be constructed via this recipe.

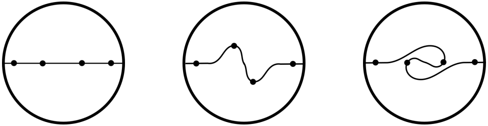
Artin representation

Consider B_n acting on the two-dimensional disk with n marked points $(D^2, \{p_1, \ldots, p_n\})$.

The generator $\sigma_i \in B_n$ acts by

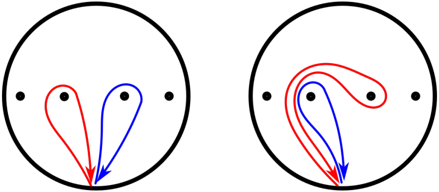


Artin representation



Effect on fundamental group

$$\mathbb{F}_n \cong \pi_1(D^2 \setminus \{p_1, \ldots, p_n\})$$



Artin representation

This induces an action

$$\phi : B_n \to \mathrm{Aut}(\mathbb{F}_n)$$

$$\sigma_k : \begin{cases} x_k & \mapsto \quad x_k \, x_{k+1} \, x_k^{-1} \\ x_{k+1} & \mapsto \quad x_k \\ x_i & \mapsto \quad x_i \quad (i \neq k, k+1) \end{cases}$$

Artin-Birman theorem

Definition

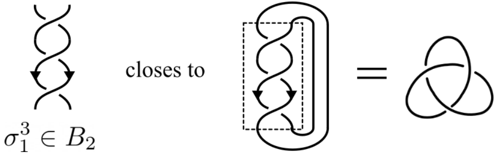
The link group is the fundamental group $\pi_1(\mathbb{R}^3 \setminus L)$ of the link component.

Artin-Birman Theorem

Suppose that a braid $\beta \in B_n$ closes to a link $L = \hat{\beta}$, then the link group has a presentation

$$\pi_1(\mathbb{R}^3 \setminus L) \cong \langle x_1, \ldots, x_n \mid \beta(x_1) = x_1, \ldots, \beta(x_n) = x_n \rangle$$

Artin-Birman theorem



$$\sigma_1 : \begin{cases} x_1 & \mapsto & x_1\, x_2\, x_1^{-1} \\ x_2 & \mapsto & x_1 \end{cases}$$

$$x_1 \xmapsto{\sigma_1} x_1\, x_2\, x_1^{-1} \xmapsto{\sigma_1} x_1\, x_2\, x_1\, x_2^{-1}\, x_1^{-1} \xmapsto{\sigma_1} x_1\, x_2\, x_1\, x_2\, x_1^{-1} x_2^{-1} x_1^{-1}$$

$$x_2 \xmapsto{\sigma_1} x_1 \xmapsto{\sigma_1} x_1\, x_2\, x_1^{-1} \xmapsto{\sigma_1} x_1\, x_2\, x_1\, x_2^{-1}\, x_1^{-1}$$

$$\pi_1(\mathbb{R}^3 \setminus K) = \langle x_1, x_2 \,|\, x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$$

The Artin representation revisited

$$\phi_n : B_n \to \mathrm{Aut}(\mathbb{F}_n)$$

$$\sigma_k : \begin{cases} x_k & \mapsto \quad x_k\, x_{k+1}\, x_k^{-1} \\ x_{k+1} & \mapsto \quad x_k \\ x_i & \mapsto \quad x_i \quad (i \neq k, k+1) \end{cases}$$

Observe:

- $\mathbb{F}_n$ is the *n*-fold coproduct of $\mathbb{F}_1$.
- Each generator σ_k acts only on two copies of $\mathbb{F}_1$ in $\mathbb{F}_n$
- The generators all act by the same formula

The sequence of representations is generated by a single map $\sigma : \mathbb{F}_2 \to \mathbb{F}_2$.

Cocartesian Yang-Baxter operators

Definition

A cocartesian Yang-Baxter operator on an object A in a category $\mathcal{C}$ is a map

$$\sigma : A \amalg A \to A \amalg A$$

such that

$$\sigma_{12}\,\sigma_{23}\,\sigma_{12} = \sigma_{23}\,\sigma_{12}\,\sigma_{23} : A^{(3)} \to A^{(3)}$$

where we write

$$A^{(n)} := A \amalg \overset{n}{\ldots} \amalg A$$

Artin-Birman Theorem revisited

Recall the Artin-Birman Theorem:

$$\pi_1(\mathbb{R}^3 \backslash L) \cong \langle x_1, \dots, x_n \mid \beta(x_1) = x_1, \dots, \beta(x_n) = x_n \rangle$$

This can be expressed in categorical terms

$$\pi_1(\mathbb{R}^3 \backslash L) \cong \operatorname{coeq} \left[\begin{tikzcd} \mathbb{F}_n \arrow[r, shift left, "\beta"] \arrow[r, shift right, "\mathrm{id}"'] & \mathbb{F}_n \end{tikzcd} \right]$$

Categorical braid closure

Definition

Given a cocartesian Yang-Baxter operator in $\mathcal{C}$

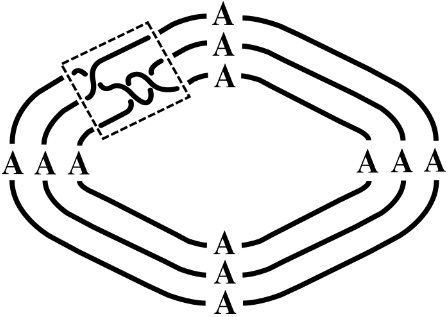
$$\sigma : A \amalg A \to A \amalg A$$

The *categorical closure* of a braid $\beta \in B_n$ is the colimit in $\mathcal{C}$

$$\begin{aligned}\mathcal{L}(A,\sigma)[\beta] &= \operatorname{coeq}\Big[\, A^{(n)} \overset{\beta}{\underset{\mathrm{id}}{\rightrightarrows}} A^{(n)} \,\Big] \\ &= \operatorname{colim}\Big[\, A^{(n)} \xleftarrow{(\beta,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\mathrm{id})} A^{(n)} \Big]\end{aligned}$$

Categorical braid closure

$$\operatorname{colim}\left[\, A^{(n)} \xleftarrow{(\beta,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\mathrm{id})} A^{(n)} \,\right]$$



Recipe for producing link invariants, revisited

- Construct a sequence of braid group representations
  $B_n \to \mathrm{Aut}(X_n), \qquad n \geq 1.$

  ☑ Generated by a cocartesian Yang-Baxter operator.

- Thus each braid $\beta \in B_n$ gives a map $\beta : X_n \to X_n$.
  Construct an object $P[\beta]$ from this map.

  ☑ Categorical braid closure $\mathcal{L}(A, \sigma)[\beta]$

- Show that this construction $P[-]$ is invariant under Markov moves.

Markov invariance

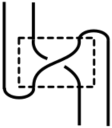
Theorem

Suppose that the cocartesian Yang-Baxter operator $\sigma : A^{(2)} \to A^{(2)}$ is Reidemeister, then the categorical braid closure $\mathcal{L}(A, \sigma)[\beta]$ is invariant under (framed) Markov moves, and hence gives an invariant of (framed) links.

The Reidemeister condition

Represent diagrams by colimits.

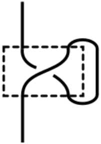
1)



$$\begin{tikzcd} A \amalg A \\ A \amalg A \arrow[u, "\sigma'"'] \end{tikzcd}$$

is invertible

2)



$$\begin{tikzcd} A \\ A \arrow[u, "\chi"'] \end{tikzcd}$$

is invertible and
slides through a crossing

M. Wada studied cocartesian Yang-Baxter operators on the object $A = \mathbb{F}_1$ in the category of groups. He showed Markov invariance of categorical braid closure under the condition that $\chi = \mathrm{id}$. For this reason, we call condition (2) the Wada condition.

Link invariants via categorical braid closure

Artin braid action → knot group

Burau braid action → (unreduced) Alexander module

Gelfand-MacPherson-Vilonen braid action → an extension of knot contact homology

Gelfand-MacPherson-Vilonen (GMV) action

Consider again B_n acting on the 2-dimensional disk D^2 with n mark points $\{p_1, \ldots, p_n\}$.

This induces an action of B_n on the category

$$\mathrm{Perv}(D^2, \{p_1, \ldots, p_n\})$$

of perverse sheaves on D^2 with singularities at most along $\{p_1, \ldots, p_n\}$.

k-categories

Definition

A k-category (sometimes called a k-linear category) is a category enriched over k-modules.

Think of a k-category as an algebra with several objects.

Associative algebra

- $A = k\langle x, y, z\rangle$
- Elements of A are sums of monomials e.g. $xy^2 + yz^7x$

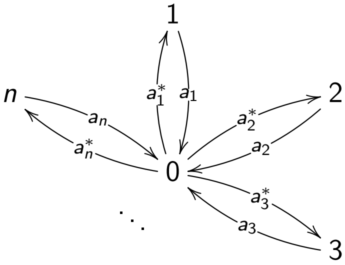
k-category

- $$\mathcal{A} = k\left\langle \begin{tikzcd} 1 \arrow[loop above, "x"] \arrow[r, bend left, "y"] & 2 \arrow[l, bend left, "z"] \end{tikzcd} \right\rangle$$
- Objects of $\mathcal{A}$ are $\{1, 2\}$.
- Morphisms of $\mathcal{A}$ are sums of composable paths e.g. $yx^3 - 2yx^2zy \,:\, 1 \rightarrow 2$

Gelfand-MacPherson-Vilonen (GMV) action

Let $Q^{(n)}$ be the quiver

$$Q^{(n)} =$$



Let $\tilde{A}^{(n)}$ be the k-category

$$\tilde{A}^{(n)} = k\langle Q^{(n)}\rangle[T_1^{-1}, \ldots, T_n^{-1}]$$

where

$$T_i = e_0 + a_i a_i^*$$

Gelfand-MacPherson-Vilonen (GMV) action

Theorem (Gelfand-MacPherson-Vilonen)

There is an equivalence of categories

$$\mathrm{Mod}^{\mathrm{fd}}(\tilde{A}^{(n)}) \simeq \mathrm{Perv}(D^2, \{p_1, \ldots, p_n\})$$

There is a B_n-action on RHS.

Translate that to LHS.

Gelfand-MacPherson-Vilonen (GMV) action

There is an action

$$\phi_n : B_n \to \mathrm{Aut}(\tilde{A}^{(n)})$$

given by explicit formula

$$\sigma_i \ : \ \begin{cases} a_i \mapsto T_i \, a_{i+1} & \\ a_{i+1} \mapsto a_i & \\ a_j \mapsto a_j & (j \neq i, i+1) \\ a_i^* \mapsto a_{i+1}^* \, T_i^{-1} & \\ a_{i+1}^* \mapsto a_i^* & \\ a_j^* \mapsto a_j^* & (j \neq i, i+1) \end{cases}$$

Gelfand-MacPherson-Vilonen (GMV) action

Take $\mathcal{C} = Cat_k^*$ to be the category of all small pointed k-category.

$$\tilde{A}^{(n)} = k\left\langle \begin{tikzcd} & 1 \arrow[d, bend left] & \\ n \arrow[dr, bend left] & & 2 \arrow[dl, bend left] \\ & 0 \arrow[uu, bend left] \arrow[ur, bend left] \arrow[dr, bend left] \arrow[ul, bend left] & \\ \ddots & & 3 \arrow[ul, bend left] \end{tikzcd} \right\rangle [T_1^{-1}, \dots, T_n^{-1}]$$

Then

$$\tilde{A}^{(n)} = \tilde{A}^{(1)} \amalg \dots \amalg \tilde{A}^{(1)} \qquad \text{in } Cat_k^*$$

Gelfand-MacPherson-Vilonen (GMV) action

Observation

The GMV action is generated by a cocartesian Yang-Baxter operator

$$\sigma : \tilde{A}^{(2)} \to \tilde{A}^{(2)}$$

which we call the GMV operator.

Lemma

This cocartesian Yang-Baxter operator is Reidemeister.
Therefore, its categorical braid closure gives an invariant of framed links.

Homotopy braid closure

Recall that, the categorical braid closure is defined as

$$\mathcal{L}(A,\sigma)[\beta] = \operatorname{colim}\left[\, A^{(n)} \xleftarrow{(\beta,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\mathrm{id})} A^{(n)} \,\right]$$

If $\mathcal{C}$ is a model category, we can take

$$\boldsymbol{L}\mathcal{L}(A,\sigma)[\beta] = \operatorname{hocolim}\left[\, A^{(n)} \xleftarrow{(\beta,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\mathrm{id})} A^{(n)} \,\right]$$

Definition

We call $\boldsymbol{L}\mathcal{L}(A,\sigma)[\beta]$ the homotopy closure of the braid $\beta \in B_n$ with respect to the cocartesian Yang-Baxter operator σ.

Link DG category

Theorem (Tabuada)

The category $\mathrm{dg}Cat_k^*$ of (small) pointed DG categories have a model structure where weak equivalences are quasi-equivalences.

Embed the category Cat_k^* into the model category $\mathrm{dg}Cat_k^*$ and take homotopy braid closure there.

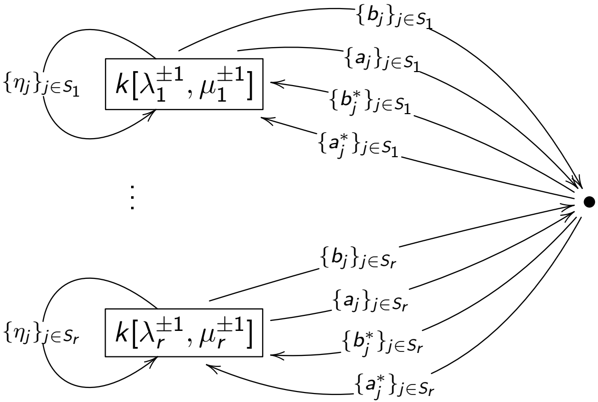
Definition

The link DG category $\tilde{\mathscr{A}}_L$ of a link $L = \hat{\beta}$ is the homotopy braid closure

$$\tilde{\mathscr{A}}_L := \boldsymbol{L}\mathcal{L}(\tilde{A}, \sigma)[\beta]$$

Link DG category

Suppose $L = L_1 \cup \ldots \cup L_r$, then

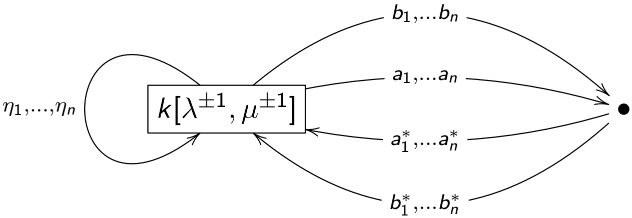


$\deg(a_j) = \deg(a_j^*) = 0, \qquad \deg(b_j) = \deg(b_j^*) = 1, \qquad \deg(\eta_j) = 2$

with differential defined in terms of β.

Link DG category

If $L = \hat{\beta}$ is a knot, then



$$\deg(a_j) = \deg(a_j^*) = 0, \qquad \deg(b_j) = \deg(b_j^*) = 1, \qquad \deg(\eta_j) = 2$$

with differential defined in terms of β.

Link DG category

Theorem

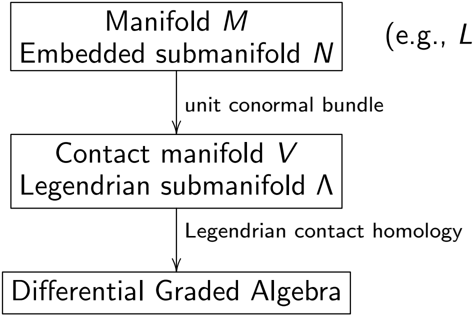
For a knot K, the endomorphism DGA at the LHS object 1 is quasi-isomorphic to the knot DGA

$$\tilde{\mathscr{A}}_K(1,1) \simeq \text{ knot DGA}$$

Therefore

$$H_*(\tilde{\mathscr{A}}_K(1,1)) \cong \text{ knot contact homology}$$

Knot contact homology



Manifold M
Embedded submanifold N

(e.g., $L \subset \mathbb{R}^3$)

unit conormal bundle

Contact manifold V
Legendrian submanifold Λ

Legendrian contact homology

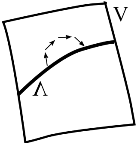
Differential Graded Algebra

Legendrian contact homology

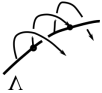
Setting: V contact manifold, Λ Legendrian submanifold

Reeb chords are integral curves of the Reeb vector field that start and end at $\Lambda \subset V$.

e.g.



Generically finite:



Consider the graded associative algebra freely generated by the Reeb cords:

$$\mathcal{A} = k\langle\{\text{ Reeb cords }\}\rangle, \qquad \deg(a) = CZ(a)$$

Legendrian contact homology

$$\mathcal{A} = k\langle\{\text{ Reeb cords }\}\rangle, \qquad \deg(a) = CZ(a)$$

To specify the differentials, we consider

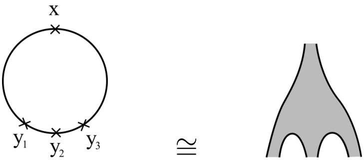
$$d(a_i) = \sum_{(j_1,\ldots,j_r)} \underbrace{c_i^{j_1,\ldots,j_r}}_{\in\mathbb{Z}} a_{j_1}\cdots a_{j_r}$$

$$c_i^{j_1,\ldots,j_r} = \#\ (\ \mathcal{M}(a_i; a_{j_1},\ldots,a_{j_r})/\mathbb{R}\ )$$

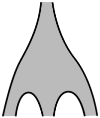
where $\mathcal{M}(a_i; a_{j_1},\ldots a_{j_r})$ is the moduli space of pseudoholomorphic disks that flow from $(a_{j_1},\ldots a_{j_r})$ to a_i.

Legendrian contact homology

Consider the marked disk

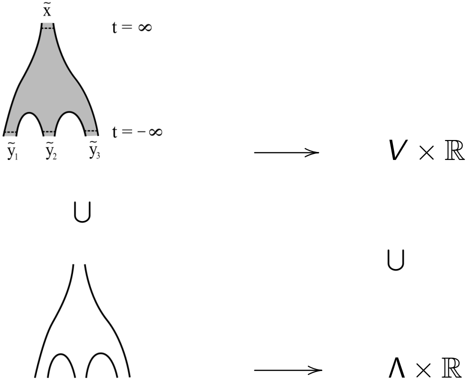


Consider pseudoholomorphic maps



$$\longrightarrow \qquad V \times \mathbb{R}$$

Legendrian contact homology



such that $\tilde{x}_i \mapsto a_i$, and $\tilde{y}_k \mapsto a_{j_k}$.

Legendrian contact homology

$$\deg(a) = CZ(a), \qquad d(a_i) = \sum_{(j_1,\dots,j_r)} \underbrace{c_i^{j_1,\dots,j_r}}_{\in\mathbb{Z}} a_{j_1}\cdots a_{j_r}$$

$$c_i^{j_1,\dots,j_r} = \#\,(\ \mathcal{M}(a_i; a_{j_1},\dots,a_{j_r})/\mathbb{R}\ )$$

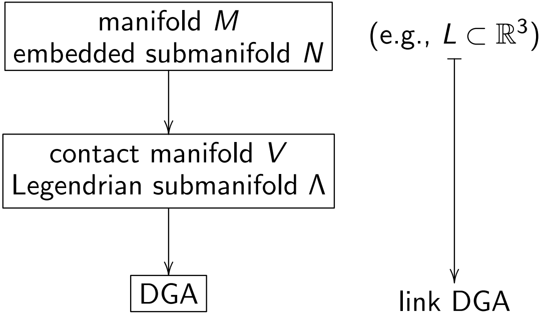
$$\dim\,(\,\mathcal{M}(a_i; a_{j_1},\dots a_{j_r})\,) = CZ(a_i) - CZ(a_{j_1}) - \dots CZ(a_{j_r})$$

Theorem

This differential is well-defined and satisfies $d^2 = 0$. Moreover, the quasi-isomorphism type of the resulting DGA is independent of any choices.

This is by Eliashberg, Chekanov, Eliasherg-Givental-Hofer, Ng, Ekholm-Etnyre-Sullivan, ...

Knot contact homology



manifold M
embedded submanifold N

contact manifold V
Legendrian submanifold Λ

DGA

(e.g., $L \subset \mathbb{R}^3$)

link DGA

Definition (Ng)

$$\text{knot contact homology} := H_*(\text{ link DGA })$$

Knot contact homology

Theorem (Ng, Ekholm-Ng-Etnyre-Sullivan)

If $L = \hat{\beta}$ for some $\beta \in B_n$, then the link DGA has explicit generators

$$\{A_{ij}, B_{ij}, C_{ij}, D_{ij}, e_i\}_{1 \leq i,j \leq n}$$

with differentials defined in terms of β, as follows:

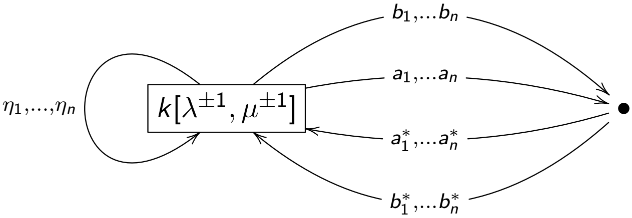
$$\begin{aligned}
d(A) &= 0 \\
d(B) &= (1 - \Lambda \cdot \Phi^L_\beta) \cdot A \\
d(C) &= A \cdot (1 - \Phi^R_\beta \cdot \Lambda^{-1}) \\
d(D) &= (\Lambda \cdot \Phi^L_\beta - 1) \cdot C - B \cdot (\Phi^R_\beta \cdot \Lambda^{-1} - 1) \\
d(e) &= (B + \Lambda \cdot \Phi^L_B \cdot C)_{ii}
\end{aligned}$$

Link DG category, revisited

Recall that we want to show

$$\tilde{\mathscr{A}}_K(1,1) \simeq \text{ link DGA}$$

Recall also that, if $K = \hat{\beta}$, then



$$d(a_i) = d(a_i^*) = 0, \qquad d(b_i) = \beta(a_i) - a_i, \qquad d(b_i^*) = \beta(a_i^*) - a_i^*$$
$$d(\eta_i) = -b_i^* a_i - \beta(a_i^*) b_i$$

Link DG category, revisited

Therefore $\tilde{\mathscr{A}}_K(1,1)$ has generators

$$\begin{aligned} A_{ij} &= -a_i^* a_j \\ B_{ij} &= b_i^* a_j \\ C_{ij} &= a_i^* b_j \\ D_{ij} &= b_i^* b_j \\ e_i &= -\eta_i \end{aligned}$$

Then the generators and differentials match EXACTLY with the one in the theorem of Ng and Ekholm-Ng-Etnyre-Sullivan.

Perverse sheaves and knot contact homology

Theorem

Let k be a field. Then the category of finite dimensional modules over the 0-th homology $H_0(\tilde{\mathscr{A}}_L)$ is equivalent to the category of perverse sheaves on $\mathbb{R}^3$ with singularities at most along L.

$$\mathrm{Mod}^{\mathrm{fd}}\left(H_0(\tilde{\mathscr{A}}_L)\right) \simeq \mathrm{Perv}(\mathbb{R}^3, L)$$

Partially wrapped Fukaya category

Conjecture

The DG category $\tilde{\mathscr{A}}_L$ is derived Morita equivalent to the partially wrapped Fukaya category of the cotangent bundle $T^*\mathbb{R}^3$ with wrapping stopped by the unit conormal bundle $ST^*_L\mathbb{R}^3$ along the link L.

Remark

One should be able to prove this using the work of Ganatra-Pardon-Shende.

Computation of the link DG category

The link DG category is defined as

$$\boldsymbol{L}\mathcal{L}(A,\sigma)[\beta] = \mathrm{hocolim}\,[\,A^{(n)} \xleftarrow{(\beta,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\mathrm{id})} A^{(n)}\,]$$

Compute it by a “minimal” resolution

$\Rightarrow$ the above presentation, where the endomorphism DGA is *isomorphic* to Ng’s presentation.

Compute it by a more symmetrical resolution

$\Rightarrow$ a “topological” presentation which is manifestly a link invariant.

Topological description

One can construct $\tilde{\mathscr{A}}_L$ via a gluing construction from two ingredients:

- The peripheral map $\phi : C_*(\Omega(\partial X)) \to C_*(\Omega(X))$ for the link complement $X = \mathbb{R}^3 \setminus \nu(L)$.
- The DG category

$$\mathscr{I}_3(S^1) = \left[\begin{tikzcd} \boxed{k[\lambda^{\pm}, \mu^{\pm}]} \arrow[loop above, "\xi"] \arrow[loop below, "\eta"] \arrow[r, bend left=40, "b"] \arrow[r, bend left=15, "a"] & \boxed{k[\lambda'^{\pm}, \mu'^{\pm}]} \arrow[loop above, "\xi'"] \arrow[loop below, "\eta'"] \arrow[l, bend left=15, "a^*"] \arrow[l, bend left=40, "b^*"] \end{tikzcd}\right]$$

with explicit differentials.

An explicit DG category

$$\mathscr{I}_3(S^1) = \left[\begin{tikzcd} \boxed{k[\lambda^{\pm}, \mu^{\pm}]} \arrow[loop above, "\xi"] \arrow[loop below, "\eta"] \arrow[r, bend left=40, "b"] \arrow[r, bend left=15, "a"] & \boxed{k[\lambda'^{\pm}, \mu'^{\pm}]} \arrow[loop above, "\xi'"] \arrow[loop below, "\eta'"] \arrow[l, bend left=15, "a^*"] \arrow[l, bend left=40, "b^*"] \end{tikzcd}\right]$$

$$d(b) = \lambda'^{-1} \cdot a - a \cdot \lambda^{-1} \qquad d(b^*) = a^* \lambda^{-1} - \lambda'^{-1} a_i^*$$

$$d(\xi) = -a^* a + \mu - 1 \qquad d(\xi') = -aa^* + \mu_i' - 1$$

$$d(\eta) = \xi \lambda^{-1} - \lambda^{-1} \xi + b^* a + a^* b$$

$$d(\eta') = \xi' \lambda'^{-1} - \lambda'^{-1} \xi' + ab^* + ba^*$$

Goal : Give a conceptual framework that produces these formulas.

What we get: replace S^1 by any manifold N.
Use the same gluing construction to define $\mathscr{A}(M, N)$ for any embedded submanifold $N \subset M$ (generalizing $L \subset \mathbb{R}^3$).

Calabi-Yau algebras

Some notations:

- Let A be a DG algebra.
- Let $D(A^e)$ be the derived category of DG bimodules of A.
- For $M \in D(A^e)$, let $M^! = \boldsymbol{R}\underline{\mathrm{Hom}}_{A^\mathrm{e}}(M, A^\mathrm{e})$ be its derived bimodule dual.

Definition [Ginzburg]

A (homologically smooth) DG algebra A is said to be n-Calabi-Yau if there is an isomorphism

$$A \cong A^![n]$$

in the derived category $D(A^e)$ of bimodules of A.

Calabi-Yau algebra

Example : Ginzburg DG algebras are 3-Calabi-Yau.

$$\mathscr{D} = k\langle x_1, \ldots, x_m, \theta_1, \ldots, \theta_m, c\rangle$$

$$\deg(x_j) = 0, \qquad \deg(\theta_j) = 1, \qquad \deg(c) = 2$$

with differentials given by

$$d(\theta_j) = \frac{\partial \Phi}{\partial x_j} \qquad \text{and} \qquad d(c) = \sum_{j=1}^{m} [x_j, \theta_j]$$

Here, $\Phi \in k\langle x_1, \ldots, x_m\rangle$ is called the superpotential.

The 0-th homology is the Jacobian algebra

$$H_0(\mathscr{D}) = k\langle x_1, \ldots, x_m\rangle / (\partial \Phi / \partial x_j)$$

Why is the Ginzburg DG algebra 3-Calabi-Yau?

Given any semi-free DG algebra $B = k\langle y_1, \ldots, y_n \rangle$, how can we show $B \cong B^![n]$?

Need a resolution of B as a bimodule.
Use the Cuntz-Quillen resolution.

Consider the short exact sequence

$$0 \to \Omega^1(B) \xrightarrow{\alpha} B \otimes B \xrightarrow{m} B \to 0$$

so that we have a semi-free bimodule resolution

$$\mathrm{Res}(B) = \mathrm{cone}\,[\,\Omega^1(B) \xrightarrow{\alpha} B \otimes B\,] \xrightarrow{\sim} B$$

Notice that $\mathrm{Res}(B)$ is semi-free with basis

$$\{\, E_0\,,\, sDy_1\,, \ldots,\, sDy_n \,\}$$

Why is the Ginzburg DG algebra 3-Calabi-Yau?

$$\mathscr{D} = k\langle x_1, \ldots, x_m, \theta_1, \ldots, \theta_m, c\rangle$$

Semi-free bimodule resolution $\mathrm{Res}(\mathscr{D})$ with basis

$$\{\, (E_0)^{(0)}, (sDx_1)^{(1)}, \ldots, (sDx_m)^{(1)}, (sD\theta_1)^{(2)}, \ldots, (sD\theta_m)^{(2)}, (sDc)^{(3)} \,\}$$

There is a perfect pairing of degree 3 on this bimodule resolution:

$$sDx_i \longleftrightarrow sD\theta_i \qquad\qquad E_0 \longleftrightarrow sDc$$

This gives an isomorphism of graded bimodules
$\mathrm{Res}(B) \cong \mathrm{Res}(B)^![3]$.
Check that it commutes with differential $\Rightarrow$ 3-CY property.

Calabi-Yau completion

Reinterpretation of the Ginzburg DG algebra:

Start with the data $A = k\langle x_1, \dots, x_m\rangle$ and Φ, add some "dual generators" $\theta_1, \dots, \theta_m$ and c to form $\mathscr{D}$, with a carefully prescribed differential so that we have a perfect duality on $\mathrm{Res}(\mathscr{D})$.

Calabi-Yau completions

$A = k\langle x_1, \ldots, x_m \rangle$.

- $\mathrm{Res}(A)$ is semi-free with basis of degree $(0, 1, \ldots, 1)$.

  Hence, $\mathrm{Res}(A)^!$ is semi-free with basis of degree $(-1, \ldots, -1, 0)$.

- Therefore, the tensor algebra $T_A(\mathrm{Res}(A)^![2])$ has generators of the degrees $(0, \ldots, 0, 1, \ldots, 1, 2)$.

- In fact, we have

$$T_A(\mathrm{Res}(A)^![2]) = \mathscr{D} = k\langle x_1, \ldots, x_m, \theta_1, \ldots, \theta_m, c \rangle$$

  is the Ginzburg DG algebra with zero potential $\Phi = 0$.

Calabi-Yau completions

In general, one has the construction

$$\begin{array}{ccc} A & \rightsquigarrow & \Pi_n(A) := T_A(A^![n-1]) \\ k\langle x_1, \ldots, x_m \rangle & \underset{n=3}{\rightsquigarrow} & \mathscr{D}|_{\Phi=0} = k\langle x_1, \ldots, x_m, \theta_1, \ldots, \theta_m, c\rangle \end{array}$$

$\exists$ deformations $\Pi_n(A) \rightsquigarrow \Pi_n(A;\eta)$ parametrized by $\eta \in HH_{n-2}(A)$

$$\begin{array}{ccc} A & \rightsquigarrow & \Pi_n(A;\eta) \\ k\langle x_1, \ldots, x_m \rangle & \underset{n=3}{\rightsquigarrow} & \mathscr{D}|_{\text{arbitrary } \Phi} \quad (\text{for } \eta = B(\Phi)) \end{array}$$

Same graded algebra $\Pi_n(A,\eta) = \Pi_n(A) = T_A(A^![n-1])$,
with differential deformed as $d = d_0 + d_\eta$.

Calabi-Yau completions

Definition [Keller '09]

The DG category $\Pi_n(A, \eta)$ is called the deformed n-Calabi-Yau completion of A with deformation parameter $\eta \in HH_{n-2}(A)$.

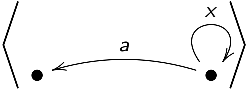
Theorem [Keller '09, Y. '16]

If the deformation parameter $\eta \in HH_{n-2}(A)$ has a negative cyclic lift $\tilde{\eta} \in HC^{-}_{n-2}(A)$, then the deformed n-Calabi-Yau completion $\Pi_n(A, \eta)$ is n-Calabi-Yau.

Examples

Let

$A = k$



Want to compute $\Pi_2(A) = T_A(A^![1])$.

Think of A to have
a generator x of degree 0;
a generator a of degree 0; and
two generators $\bullet$, each of degree -1.

Add $(n-2)$-shifted duals to these generators

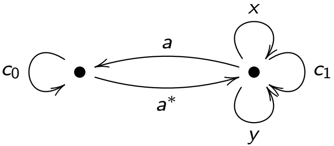
Take $n = 2$.

$$A = k\left\langle \begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "a"'] \arrow[loop above, "x"] \end{tikzcd} \right\rangle$$

Add extra generators:
a^* of degree 0 (dual to a)
y of degree 0 (dual to x)
c_0 of degree 1 (dual to the left vertex)
c_1 of degree 1 (dual to the right vertex)

Examples

Take $n = 2$. The result is

$\Pi_2(A;\epsilon) = k \Big\langle$  $\Big\rangle$

$$deg(a) = deg(a^*) = deg(x) = deg(y) = 0$$

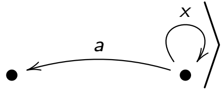
$$deg(c_1) = deg(c_0) = 1$$

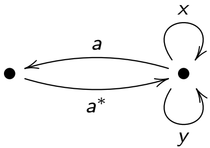
with differentials

$$d(c_1) = xy - yx - a^* a - \epsilon_1$$

$$d(c_0) = a^* a - \epsilon_0$$

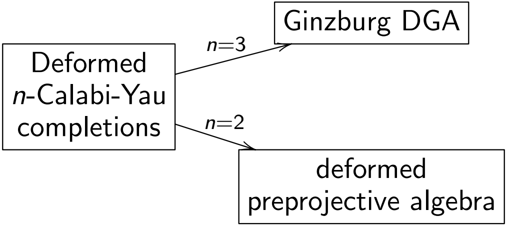
Examples

$A = k \Bigg\langle$  $\Bigg\rangle$

$H_0(\Pi_2(A;\epsilon)) = k \Bigg\langle$  $\Bigg\rangle \Big/ \left(\begin{matrix} xy - yx - a^* a - \epsilon_1 = 0 \\ a^* a - \epsilon_0 = 0 \end{matrix} \right)$

which is the deformed preprojective algebra.

Examples



Deformed n-Calabi-Yau completions

n=3

Ginzburg DGA

n=2

deformed preprojective algebra

Non-commutative analogue of symplectic varieties

Consider the analogy

Non-commutative	Commutative
n-Calabi-Yau completions $T_A(A^![n-1])$	cotangent bundle $\mathrm{Sym}_B(\Omega^1(B)^\vee)$
n-Calabi-Yau structure	(shifted) symplectic structure

Stretch this analogy

Non-commutative	Commutative
relative n-Calabi-Yau completions	conormal bundle
relative n-Calabi-Yau structure	(shifted) Lagrangian structure

Definition [Pantev-Toen-Vaquie-Vezzosi]

A map $f : X \to Y$ of derived stacks (locally of finite type) is said to have an m-shifted Lagrangian structure if there is an isomorphism in the derived category $D_{\mathrm{perf}}(X)$:

$$\mathrm{cone}\,(\,(\mathbb{L}_X)^\vee \to f^*(\mathbb{L}_Y)^\vee\,)\,[-m] \cong \mathbb{L}_X$$

(together with a certain closedness and compatibility condition).

Definition [Brav-Dyckerhoff]

A DG functor $F : A \to B$ is said to be relative n-Calabi-Yau if there is an isomorphism in the derived category $D(B^{\mathrm{e}})$:

$$\mathrm{cone}\,(\,B^! \to \boldsymbol{L}F_!(A)^!\,)\,[n-1] \cong B$$

(together with a certain closedness and compatibility condition).

Non-commutative conormal bundle

(shifted) conormal bundle has a (shifted) Lagrangian structure.

Want: a noncommutative analogue.

For varieties:

- A subscheme $\mathrm{Spec}(D) \hookrightarrow \mathrm{Spec}(C)$
- $g : C \twoheadrightarrow D$
- $g_* : \Omega^1(C) \otimes_C D \twoheadrightarrow \Omega^1(D)$
- $M = \ker(g_*)$
- Total space of the conormal bundle is the Spec of $\mathrm{Sym}_D(M^\vee)$

Non-commutative conormal bundle

Commutative algebras:

- $g : C \twoheadrightarrow D$
- $g_* : \Omega^1(C) \otimes_C D \twoheadrightarrow \Omega^1(D)$
- $M = \ker(g_*)$.
- $\mathrm{Sym}_D(M^\vee)$

DG categories:

- DG functor $F : A \to B$.
- $\gamma_F : \boldsymbol{L}F_!(A) \to B$
- $\Theta = \mathrm{cone}\,(\gamma_F)$
- $T_A(\Theta^![n-1])$

Non-commutative conormal bundle

$$\Pi_n(B, A) = T_B(\Theta^![n-1])$$

$\exists$ deformations $\Pi_n(B, A) \rightsquigarrow \Pi_n(B, A; \eta)$ parametrized by relative Hochschild homology classes $\eta \in HH_{n-2}(B, A)$.

Again, $\Pi_n(B, A; \eta) = \Pi_n(B, A)$ as graded algebra (or category).

Differential is deformed: $d = d_0 + d_\eta$.

Relative Calabi-Yau completions

Definition [Y. '16]

The DG functor

$$\Pi_{n-1}(A; \eta_A) \to \Pi_n(B, A; \eta)$$

is called the deformed relative n-Calabi-Yau completion of $F : A \to B$

Theorem [Y. '16]

If the deformation parameter $\eta \in HH_{n-2}(B, A)$ has a negative cyclic lift, then the deformed relative n-Calabi-Yau completion has a canonical relative n-Calabi-Yau structure.

Remark

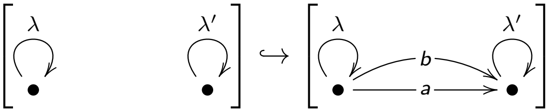
Recently, T. Bozec, D. Calaque, and S. Scherotzke used this notion of deformed relative Calabi-Yau completion to study a Lagrangian in the Hilbert scheme of points on $\mathbb{C}^2$.

Relation with knot contact homology

Start with

$$A = k[\lambda^{\pm 1}] \otimes [\,\bullet \qquad \bullet\,] \hookrightarrow \left[\,\bullet \xrightarrow{a} \bullet\,\right] \otimes k[\lambda^{\pm 1}] = B$$

Find semi-free resolution.

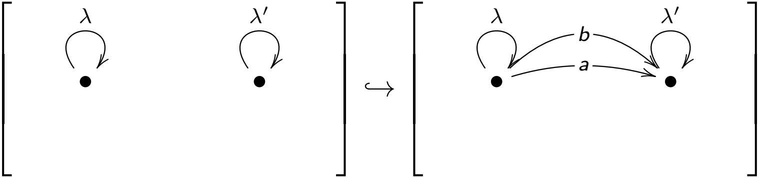


with differential

$$d(b) = \lambda' a - a\lambda$$

and with λ and λ' inverted.

Relation with knot contact homology



Take the (deformed) relative 3-Calabi-Yau completion.

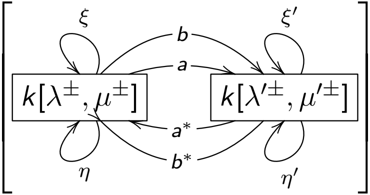
LHS: perform the ordinary 2-Calabi-Yau completion.
$k\langle\lambda\rangle \implies k[\lambda, \mu]$

RHS: add extra generators:
a^* dual to a
b^* dual to b
ξ dual to λ
η dual to the vertex

Relation with knot contact homology

The result (after localizing at μ and μ') is precisely

$\mathscr{I}_3(S^1) =$



with differentials

$$d(b) = \lambda'^{-1} \cdot a - a \cdot \lambda^{-1} \qquad d(b^*) = a^*\lambda^{-1} - \lambda'^{-1}a_i^*$$
$$d(\xi) = -a^*a + \mu - \epsilon \qquad d(\xi') = -aa^* + \mu_i' - \epsilon'$$
$$d(\eta) = \xi\lambda^{-1} - \lambda^{-1}\xi + b^*a + a^*b$$
$$d(\eta') = \xi'\lambda'^{-1} - \lambda'^{-1}\xi' + ab^* + ba^*$$