Singularities of R-matrices, graded quiver varieties, & generalized quantum affine Schur-Weyl duality

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Introduction

Classical theory

$\mathfrak{g}$: f.d. simple Lie / $\mathbb{C}$, $I = \{$ Dynkin index $\}$

$\mathrm{Rep}(\mathfrak{g})$ = category of f.d. rep's of $\mathfrak{g}$

$\left(\begin{array}{l} \text{semisimple as abelian cat.} \\ \text{symmetric as } \otimes\text{-cat.} \end{array} \right.$ $\quad V \otimes W \cong W \otimes V$

$\quad v \otimes w \leftrightarrow w \otimes v$

$$\rightsquigarrow \underset{\substack{\text{Grothendieck} \\ \text{ring}}}{K(\mathrm{Rep}\,\mathfrak{g})} = \bigoplus_{V:\mathrm{irr}} \mathbb{Z}[V]$$

$$\cong \mathbb{Z}[X_i \mid i \in I] \quad \text{poly ring}$$

$$\underset{\substack{\uparrow \\ \text{fund. rep.}}}{[V(\varpi_i)]} \leftrightarrow X_i$$

Quantum affinization

$$\begin{tikzcd}[column sep=small] & \mathfrak{g} \arrow[dl, squiggly, "\text{quantize}"'] \arrow[dr, squiggly, "\text{affinze}"] & \\ U_q(\mathfrak{g}) \arrow[dr, squiggly] & & L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[z^{\pm 1}] \arrow[dl, squiggly] \\ & U_q(L\mathfrak{g}) & \end{tikzcd}$$

non symm $\otimes$-cat.

non semisimple abel cat

$U_q(L\mathfrak{g})$ quantum loop alg

$/ \ \mathbb{k} := \overline{\mathbb{Q}(q)}$

$\uparrow$ quantization param

$\mathcal{C} := \mathrm{Rep}\, U_q(L\mathfrak{g})$

Not semisimple & Not symmetric

abelian $\otimes$-cat. $V \otimes W \not\cong W \otimes V$ in general

(!) Nevertheless

$K(\mathcal{C}) \cong \mathbb{Z}[X_{i,a} \mid i \in I, a \in \Bbbk^\times]$ comm

$[V_i(a)] \longleftrightarrow X_{i,a}$

fund. rep. spectral param. (~ affinization)

Q. When $V_i(a) \otimes V_j(b) \cong V_j(b) \otimes V_i(a)$?

Rem "Monoidal categorification of cluster alg"

[Hernandez–Leclerc 2010s]

We expect a nice $\otimes$-subcat

$\mathcal{C}' \subset \mathcal{C}$ categorifies a cluster alg $\mathcal{A}$

i.e.
$$\begin{array}{ccc} K(\mathcal{C}') & \cong & \mathcal{A} \\ \cup & & \cup \\ \mathrm{irr}\, \mathcal{C}' & \hookleftarrow & \{\text{cluster variables}\} \end{array}$$

max'l collection of mutually comm irrep $\longleftrightarrow$ cluster

failure of $V \otimes W \cong W \otimes V$ $\longleftrightarrow$ mutation

R-matrix

z_1, z_2 formal spectral param

$$\exists!\ R_{ij}(u) : V_i(z_1) \otimes V_j(z_2) \xrightarrow{\sim} V_j(z_2) \otimes V_i(z_1)$$

$$\begin{array}{ccc} \underbrace{v_i}_{\text{h.wt. vec}} \otimes \underbrace{v_j} & \longmapsto & v_j \otimes v_i \end{array}$$

matrix-valued rat'l function in $u = z_2/z_1$

$\rightsquigarrow$ $d_{ij}(u) \in \Bbbk[u]$ denominator

Ex. $\mathfrak{g} = \mathfrak{sl}_2$, $I = \{1\}$, $V_1(a) \cong \Bbbk^2$ vec. rep.

$$R_{11}(u) = \frac{1}{\underbrace{u - q^2}_{= d_{11}(u)}} \begin{pmatrix} u-q^2 & 0 & 0 & 0 \\ 0 & 1-q^2 & q(u-1) & 0 \\ 0 & q(u-1) & (1-q^2)u & 0 \\ 0 & 0 & 0 & u-q^2 \end{pmatrix}$$

Th'm (Chari, Kashiwara, ...)

TFAE (i) $V_i(a) \otimes V_j(b) \cong V_j(b) \otimes V_i(a)$

(ii) $V_i(a) \otimes V_j(b)$ is irreducible

(iii) $d_{ij}(b/a) \neq 0$ & $d_{ji}(a/b) \neq 0$

Rem All $d_{ij}(u)$ have been computed by many people recently.

From now on, we assume $\mathfrak{g}$ is simply-laced (ADE)

Δ : Dynkin diagram

<u>Relation to quiver rep's</u>

Q : Dynkin quiver $\overset{\text{forget orientation}}{\rightsquigarrow}$ Δ

$\mathrm{Rep}(Q)$: cat of f.d. rep of $Q / \mathbb{C}$

$\mathcal{D}_Q := D^b(\mathrm{Rep}\, Q)$

<u>Th'm</u> (Gabriel & Happel)

(i) $\underset{\text{indecomposable}}{\mathrm{Indec}}\ \mathrm{Rep}(Q) \overset{1:1}{\longleftrightarrow} R^+ = \{\text{positive roots}\}$

$M_\alpha \longleftrightarrow \alpha$

(ii) $\mathrm{Indec}\ \mathcal{D}_Q \overset{1:1}{\longleftrightarrow} R^+ \times \mathbb{Z}$

$M_\alpha[k] \longleftrightarrow (\alpha, k)$

<u>Def</u> (Repetition quiver) $\hat{\Delta} = (\underset{\text{vertices}}{\hat{\Delta}_0}, \underset{\text{arrows}}{\hat{\Delta}_1})$

Fix $\epsilon : I \longrightarrow \{0, 1\}$ s.t. $\epsilon_i \neq \epsilon_j$ if $i - j$ in Δ

$i \longmapsto \epsilon_i$

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$\qquad i \longmapsto \epsilon_i$

$$\hat{\Delta}_0 := \{ (i,p) \in I \times \mathbb{Z} \mid p - \epsilon_i \in 2\mathbb{Z} \}$$

$$\hat{\Delta}_1 := \{ (i,p) \to (j,p+1) \mid i - j \text{ in } \Delta \}$$

Th'm (Happel) $\exists$ isom of quivers

Auslander-Reiten

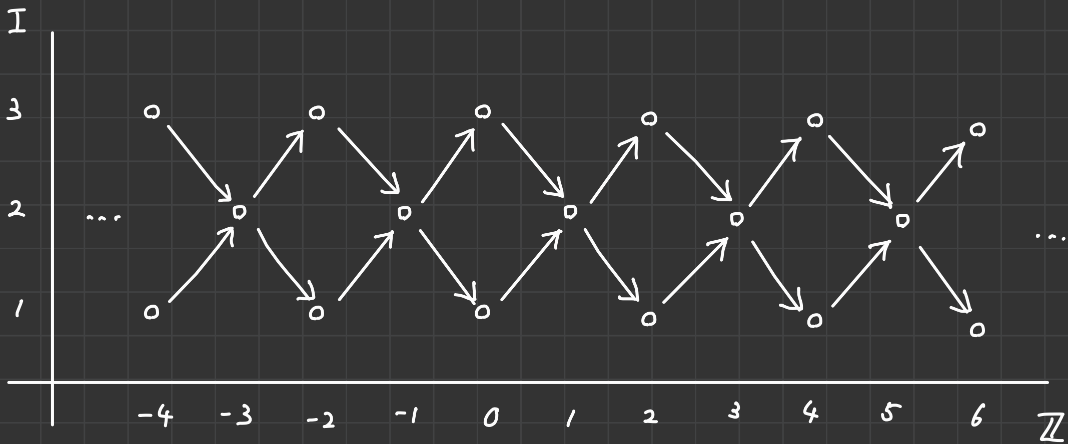
$$H_Q : \hat{\Delta} \xrightarrow{\sim} AR(\mathcal{D}_Q)$$

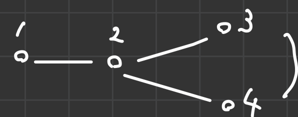
$$\overset{\text{v. sets}}{\rightsquigarrow} \quad \hat{\Delta}_0 \xrightarrow{1:1} \operatorname{Indec} \mathcal{D}_Q = R^+ \times \mathbb{Z}$$

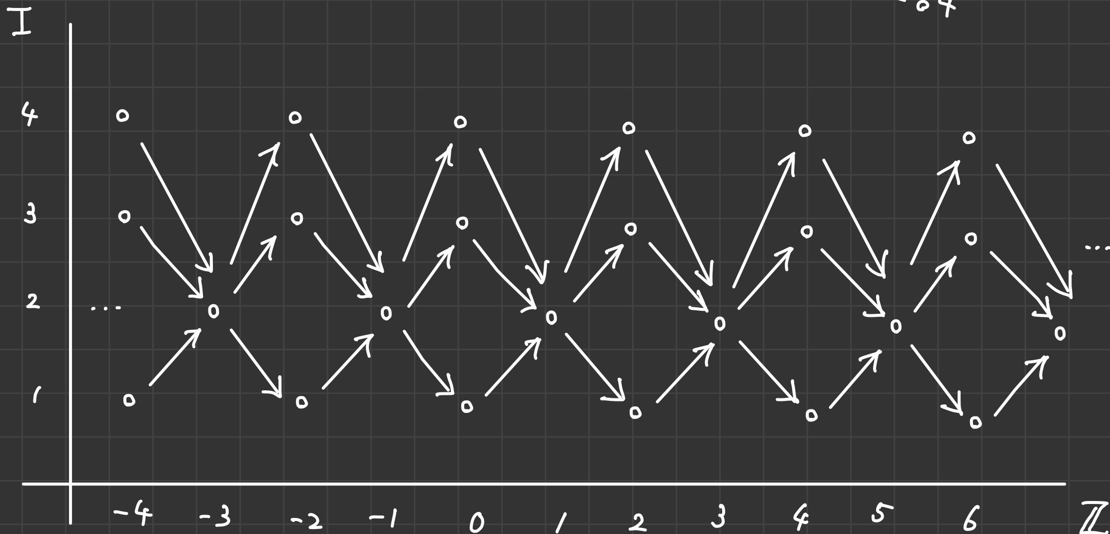
H_Q can be considered as a "coordinate of $AR(\mathcal{D}_Q)$"

Picture of $\hat{\Delta}$

$\mathfrak{g} = \mathfrak{sl}_4$ (type A_3) $\qquad \Delta = ($  $)$

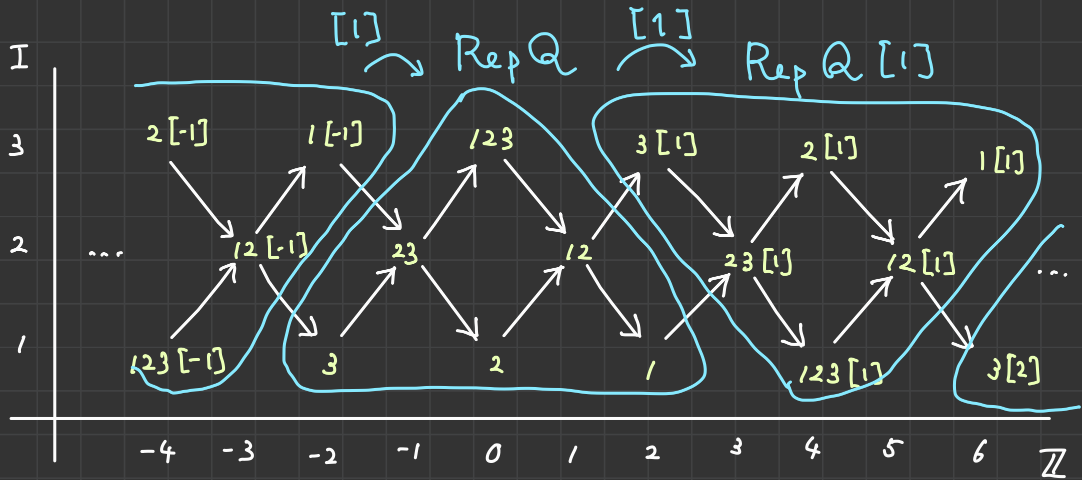


$\mathfrak{g} = \mathfrak{so}_8$ (type D_4) $\qquad \Delta = ($  $)$



Picture of H_Q

$\mathfrak{g} = \mathfrak{sl}_4$ (type A_3) $\qquad Q = \left(\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ} \right)$



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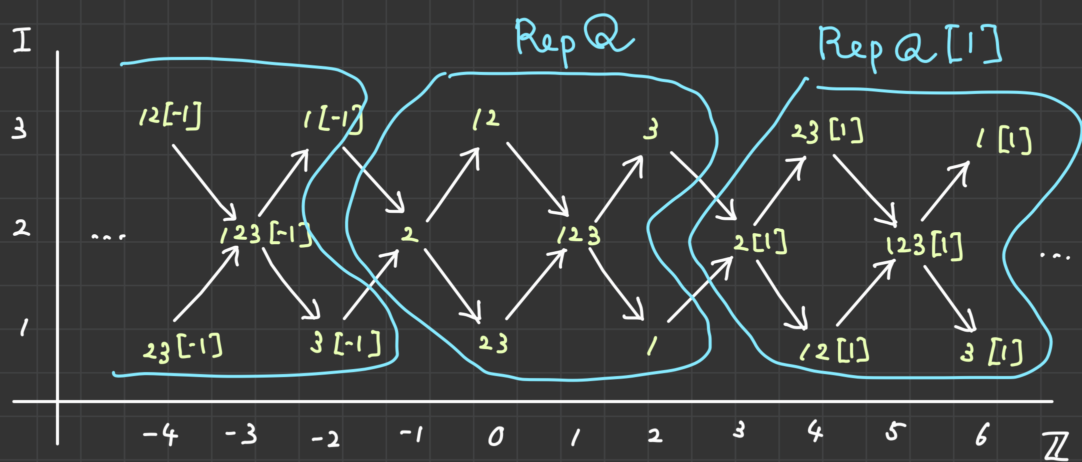
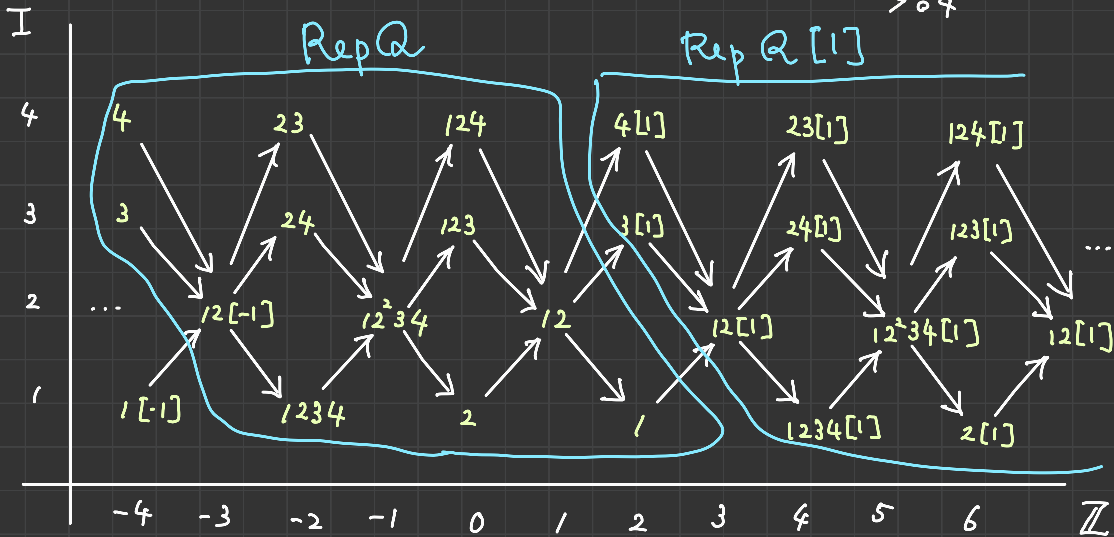


$\mathfrak{g} = \mathfrak{so}_8$ (type D_4)

$$Q = \left(\begin{tikzcd}[row sep=small] & & \overset{3}{\circ} \\ \overset{1}{\circ} \arrow[r] & \overset{2}{\circ} \arrow[ur] \arrow[dr] & \\ & & \circ\, 4 \end{tikzcd} \right)$$



Th'm 1 (F.) $a \in \mathbb{k}^\times$

easier to compute

$$\operatorname*{zero}_{u=a} d_{ij}(u) = \begin{cases} \dim \operatorname{Ext}^1_{\mathcal{D}_Q}(H_Q(j,r), H_Q(i,p)) & \text{if } a = q^r/q^p, \exists (i,p),(j,r) \in \hat{\Delta}_0 \\ 0 & \text{otherwise} \end{cases}$$

//

Notation $x = (i,p) \in \hat{\Delta}_0$

$V_x = V_i(q^p) \in \operatorname{irr} \mathcal{C}$

$H_Q(x) \in \operatorname{Indec} \mathcal{D}_Q$

Cor. $x, y \in \hat{\Delta}_0$

$$V_x \otimes V_y \cong V_y \otimes V_x$$

$$\Longleftrightarrow \operatorname{Ext}^1(H_Q(x), H_Q(y)) = \operatorname{Ext}^1(H_Q(y), H_Q(x)) = 0$$

Relation to graded quiver varieties

$\cong$ $\mathbb{C}^\times$-fixed pts of usual Nakajima quiver variety.

$$W = \bigoplus_{x \in \hat{\Delta}_0} W_x \quad \hat{\Delta}_0\text{-gr v.sp.}$$

$$\rightsquigarrow \left[\begin{array}{ll} \mathfrak{M}^\bullet(W) & \text{smooth} \\ \quad \downarrow \pi & \text{proper} \\ \mathfrak{M}_0^\bullet(W) & \text{affine} \end{array} \right] \circlearrowleft G_W = \prod_x GL(W_x)$$

Th'm (Nakajima) $\exists$ alg hom

completion

$$U_q(L\mathfrak{g}) \longrightarrow \hat{K}^{G_W}\left(\mathfrak{M}^\bullet(W) \times_{\mathfrak{M}_0^\bullet(W)} \mathfrak{M}^\bullet(W)\right)_{\mathbb{k}}$$

convolution alg

w./ good properties.

Def. (Ext^1 - quiver) $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1)$

$\mathcal{E}_0 = \hat{\Delta}_0$

$\mathcal{E}_1$: $\#\{x \to y\} = \dim \mathrm{Ext}^1(H_Q(x), H_Q(y))$

$=: \mathcal{E}(x, y)$

//

Th'm (Keller - Scherotzke '16)

G_W - equiv. emb.

$$\mathfrak{M}_0^{\bullet}(W) \hookrightarrow \mathrm{rep}(\mathcal{E}, W) = \bigoplus_{(x \to y) \in \mathcal{E}_1} \mathrm{Hom}(W_x, W_y)$$

$\rightsquigarrow$ proof of Th'm 1.

idea $x, y \in \hat{\Delta}_0$, assume $\mathcal{E}(x, y) > 0$

Take W s.t. $\dim W_z = \delta_{zx} + \delta_{zy}$

$\Rightarrow$ $\mathfrak{M}_0^{\bullet}(W) \cong \mathbb{A}^{\mathcal{E}(x,y)}$

$$\boxed{\begin{array}{ccc} K(\mathfrak{M}^{\bullet}(W)) & \cong & V_x \otimes V_y \\ i_* \uparrow & \circlearrowleft & \uparrow \text{"}R\text{"} \\ K(\pi^{-1}(0)) & \cong & V_y \otimes V_x \end{array}}$$

analyze equiv. ver of this.

Generalized Schur-Weyl duality.

Original construction (Kang-Kashiwara-Kim '18)

Given $\{V^j\}_{j\in J}$ family of irrep in $\mathcal{C}$

any index set

$\rightsquigarrow$ Define quiver $\Gamma = (\Gamma_0, \Gamma_1)$

- $\Gamma_0 = J$
- Γ_1 : $\#\{i \to j\} = \underset{u=1}{\text{pole}}\ R_{V^j,V^i}(u)$

$\rightsquigarrow$ $^\forall \nu \in \mathbb{N}J$ a.k.a. KLR alg (a variant of affine Hecke alg)

Quiver Hecke alg $\mathcal{H}_\nu(\Gamma)$ is defined as a $\mathbb{Z}$-gr $\Bbbk$-alg by gen & rel.

Th'm (KKK) $\exists$ bimodule

$$\mathcal{U}_q(L\mathfrak{g}) \curvearrowright \hat{V}^{\otimes\nu} \overset{R\text{-mat}}{\curvearrowleft} \hat{\mathcal{H}}_\nu(\Gamma)$$

$$\hat{V}^{\otimes\nu} = \bigoplus_{j_1+\cdots+j_n=\nu} \hat{V}^{j_1}\otimes\cdots\otimes\hat{V}^{j_n}$$

"infinitesimal deformation"

$$\rightsquigarrow F^J : \underbrace{\bigoplus_\nu \hat{\mathcal{H}}_\nu(\Gamma)\text{-mod}_{fd}}_{} \longrightarrow \mathcal{C}$$

$$M \longmapsto \hat{V}^{\otimes\nu} \otimes_{\mathcal{H}} M$$

$\otimes$-functor

$\otimes$-cat.
w.r.t. parab. induction.

Assume $\mathfrak{g}$: ADE,

$\exists \ x : J \hookrightarrow \hat{\Delta}_0$

s.t. $V^j = V_{x(j)}$ fundamental rep.

By Th'm 1, $\quad \Gamma = \mathcal{E}|_J$

$\rightsquigarrow \quad W$: J-gr v. sp. $\qquad \circlearrowright G_W$

$$\left[\begin{tikzcd} \mathcal{M}^\bullet(W) \arrow[d] & & & \mathcal{F}\ell \text{ Lusztig's quiver-flag variety} \arrow[dl] \\ & \mathcal{M}_0^\bullet(W) \arrow[r, hook, "KS"'] & E_W = \mathrm{rep}(\Gamma, W) & \end{tikzcd}\right]$$

Th'm 2. (F.) $\exists$ comm diag

$$\begin{tikzcd} \hat{K}^G(\mathcal{M}^\bullet \times_E \mathcal{M}^\bullet)_{\Bbbk} \arrow[r, phantom, "\circlearrowright"] & \hat{K}^G(\mathcal{M}^\bullet \times_E \mathcal{F}\ell)_{\Bbbk} \arrow[r, phantom, "\circlearrowleft"] & \hat{K}^G(\mathcal{F}\ell \times_E \mathcal{F}\ell)_{\Bbbk} \\ U_q(L\mathfrak{g}) \arrow[u] \arrow[r, phantom, "\circlearrowright"] & \hat{V}^{\otimes \nu} \arrow[u, phantom, "\wr"] \arrow[r, phantom, "\circlearrowleft"] & \hat{\mathcal{H}}_\nu(\Gamma) \arrow[u, phantom, "\wr \leftarrow \text{[Varagnolo-Vasserot]}"] \end{tikzcd}$$

$\nu = \underline{\dim} W$

Ex 1. $J = I \xleftrightarrow{x} \hat{\Delta}_0$

$x(i) := H_Q^{-1}(M_{\alpha_i})$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \Uparrow$

$\quad\quad\quad\quad\quad\quad\quad\quad \operatorname{Rep} Q$

$\rightsquigarrow \quad \Gamma \cong Q$

$\quad\quad \mathcal{M}_0^\bullet(W) \cong \operatorname{rep}(Q, W)$

"supp on $\operatorname{Rep} Q \subset \mathcal{D}_Q$"

$\rightsquigarrow \quad F^J : \hat{\mathcal{H}}(Q)\text{-mod}_{fd} \xrightarrow{\sim} \mathcal{C}_Q \subset \mathcal{C}$

$\overset{K_0}{\rightsquigarrow} \quad \mathcal{U}_q^+(\mathfrak{g})^*|_{q=1} \simeq K(\mathcal{C}_Q)$

dual canonical basis $\longleftrightarrow$ irr $\mathcal{C}_Q$

Ex 2. Assume $Q' = (1 \to 2 \to \cdots \to N-1) \subset Q$

e.g. $Q = \underbrace{\circ \to \circ \to \circ \to \cdots \to \circ \nearrow \circ}_{= Q'} \searrow \circ$

$$\rightsquigarrow \quad J = \mathbb{Z} \xhookrightarrow{\ X\ } \hat{\Delta}_0$$

$$X(j) = \begin{cases} H_Q^{-1}(M_{\alpha_i}[-2k]) & \text{if } j = i + kN,\ 1 \leq i \leq N-1 \\ H_Q^{-1}(M_\theta[-2k+1]) & \text{if } j = kN \end{cases}$$

where $\theta = \alpha_1 + \alpha_2 + \cdots + \alpha_{N-1}$

$\rightsquigarrow \quad \Gamma = (\cdots \to -1 \to 0 \to 1 \to 2 \to \cdots)$ type A_∞

$\hat{\mathcal{H}}_\nu(\Gamma) \cong$ affine Hecke $\hat{\mathcal{H}}^{\mathrm{aff}}(\mathfrak{S}_{|\nu|})$

$\mathfrak{M}_0^\bullet(W) \cong$ graded nilpotent orbits in $\mathfrak{gl}(W)_-$

$\rightsquigarrow \quad F^J : \underbrace{\hat{\mathcal{H}}^{\mathrm{aff}}(A_\infty)\text{-mod}_{fd}}_{\text{well-studied}} \longrightarrow \mathcal{C}' \subset \mathcal{C}$

("supp on $\mathcal{D}_{Q'} \subset \mathcal{D}_Q$")

Using [Kashiwara-Kim-Oh-Park '19]

we see $\underline{\mathcal{C}' \text{ categorifies a cluster alg.}}$