Integrable Kondo lines, ODE/IM correspondence, 4d Chern Simons

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Integrable Kondo defect

defects in QFTs

- reveal hidden structure of the bulk theory
- part of the definition of a QFT
- interesting on its own: impurity, edge modes, D branes, ...

defects in 2d CFTs

Ward Identity

$$[T - \bar{T}]|_+ - [T - \bar{T}]|_- = 2i\partial_{\|}\hat{t} \quad (1)$$

- topological line defect

$$T|_+ = T|_-, \quad \bar{T}|_+ = \bar{T}|_- \quad (2)$$

- conformal defect

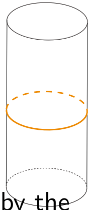
$$[T - \bar{T}]|_+ = [T - \bar{T}]|_- \quad (3)$$

- chiral defect

$$\bar{T}|_+ = \bar{T}|_- \quad (4)$$

  - generically not conformal $\Rightarrow$ defect RG flow
  - chiral + conformal $\Rightarrow$ topological

Kondo defect in chiral $\mathrm{SU}(2)_k$ WZW



$$\hat{T}_{\mathcal{R}_n} := \mathrm{Tr}_{\mathcal{R}_n} \mathcal{P} \exp\left(ig \int_0^{2\pi R} d\sigma \; t_a J^a(\sigma, 0) \right)$$

where t_a are generators of $su(2)$ in the representation $\mathcal{R}$, labeled by the dimension of the representation n.

- physical origin: free fermion with a magnetic impurity in rep of dimension n

$$H_{\text{impurity}} = g\vec{S} \cdot \psi^\dagger(0,t)\frac{\vec{\sigma}}{2}\psi(0,t)$$

- can be quantized nicely [Bachas-Gaberdiel]:

translation	invariant ALONG and NORMAL to the defect
chiral	commutes with $\bar{J}(z)$
XXX	global $\mathrm{SU}(2)$ invariant

- interesting defect RG flow

Kondo defect RG flow

UV: asymptotically free [Bachas-Gaberdiel, Gaiotto-Lee-W]

$$\beta(g) = -g^2 + \frac{k}{2} g^3 + \dots \tag{5}$$

dimensional transmutation: $g \to e^{\theta}$, denoted by $L_n[\theta]$ or $\hat{T}_n[\theta]$
θ, spectral parameter

IR: conjectural IR pattern [Affleck-Ludwig]

- $n \leq k+1$: $L_n \to \mathcal{L}_n$, OVER-SCREENED
- $n > k+1$: $L_n \to \mathcal{L}_{k+1} \otimes L_{n-k}$, UNDER-SCREENED

Recall: there are $k+1$ Verlinde lines in $\mathrm{SU}(2)_k$ WZW, denoted by $\mathcal{L}_n$ for $n = 1, 2, \dots, k+1$

Kondo defect is **Integrable**

Commutativity:

$$\left[\hat{T}_n[\theta], \hat{T}_{n'}[\theta']\right] = 0, \quad \left[\hat{T}_n[\theta], L_0 + \bar{L}_0\right] = 0 \tag{6}$$

Hirota relations:

$$\hat{T}_n\left[\theta - i\frac{\pi}{2}\right]\hat{T}_n\left[\theta + i\frac{\pi}{2}\right] = 1 + \hat{T}_{n-1}[\theta]\hat{T}_{n+1}[\theta] \tag{7}$$

[Kondo, Lesage-Saleur-Kivelson, Bachas-Gaberdiel, Runkel, Andrei,Wiegmann,Destri,Tsvelick-Weigmann, Affleck-Ludwig, ...]

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[Kondo, Lesage-Saleur-Kivelson, Bachas-Gaberdiel, Runkel, Andrei,Wiegmann,Destri,Tsvelick-Weigmann, Affleck-Ludwig, ...]

WHY?

- ODE/IM correspondence
- 4D Chern Simons theory

ODE/IM correspondence

ODE/IM correspondence

Dorey-Tateo, Bazhanov-Lukyanov-Zamolodchikov, Dorey-Dunning-Tateo, Dorey-Dunning-Masoero-Suzuki-Tateo, ...

ODE/IM correspondence in our interpretation

quantum	Expectation values of the Kondo lines in a CFT can be computed by
classical	the Stokes data of an auxiliary ODE

Example

quantum $\widehat{sl}_2$ KdV is described by

$$\psi''(x) = \Big[x^{2\alpha} - E + \ldots\Big]\psi$$

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Physical Meaning? Unknown!

New examples of ODE/IM correspondence

We claim the ODE for chiral $\mathrm{SU}(2)_k$ WZW is

$$\psi''(x) = \Big[e^{2\theta} e^{2x} x^k + t(x)\Big]\psi$$

[Lukyanov-Zamolodchikov, Lukyanov-Werner]

There is a unique solution $\psi_0(x;\theta)$ that decays exponentially as $x \to \infty$

- Vacuum: $t(x) = 0$.

$$\langle 0|\hat{T}_n|0\rangle \triangleq i\left(\psi_0\Big(x;\theta - \frac{i\pi n}{2}\Big), \psi_0\Big(x,\theta + \frac{i\pi n}{2}\Big)\right) =: i(\psi_{-\frac{n}{2}}, \psi_{\frac{n}{2}})$$

  where $\psi_n(x;\theta) := \psi_0(x;\theta + n\pi i)$

- other states: $t(x) = a(x)^2 + \partial_x a(x)$ and

$$a(x) = \alpha - \frac{l}{x} + \sum_i \frac{1}{x - w_i} - \sum_a \frac{1}{x - w_i'} \tag{8}$$

  w_i and w_a' are given by the solutions of a rational equation. [Feigin-Frenkel, Masoero-Raimondo, Feigin-Jimbo-Mukhin, Bazhanov-Lukyanov-Zamolodchikov]

Checks and Remarks

Advantages of ODE/IM [Zamolodchikov]

- *analytic properties of relevant physical quantities is more explicit*
- *the integrable model can be studied uniformly in different parameter regimes*

Practically

1. calculation of $\langle\phi|\hat{T}_n|\phi\rangle$
2. reproduce the IR phase diagram
3. solutions of Bethe equation nicely reproduce all the states in the spectrum[1], including null states
4. Hirota equation given by Plucker relation of the Wronskians
5. reproduce beta function

[1] More precisely, Bethe equation is not complete. We actually need generalized oper.

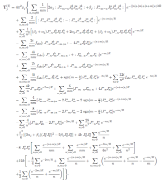
calculation of $\langle \phi | \hat{T}_n | \phi \rangle$

Kondo line $\hat{T}_n$ is only defined in the UV

$$\hat{T}_n := \mathrm{Tr}_{\mathcal{R}_n} \mathcal{P} \exp \left(ig \int_0^{2\pi R} d\sigma \; t_a J^a(\sigma, 0) \right) = \sum_{N=0}^{\infty} (ig)^N \hat{T}_n^{(N)}$$

need a painful renormalization procedure

$$\hat{T}_n^{(0)} = n,$$
$$\hat{T}_n^{(2)} = 2\pi^2 \mathcal{I}_n J_0^a J_0^a,$$
$$\hat{T}_n^{(3)} = -8i\pi^2 \mathcal{I}_n \left\{ \sum_{m>0} \frac{i}{2m} f_{abc} J_{-m}^a J_0^b J_m^c + \sum_{m>0} \frac{2}{m} J_{-m}^a J_m^a - \log R \; J_0^a J_0^a - \frac{k}{2} \right\}$$



- Can be easily reproduced in the ODE side!
- IR expansion($\theta \to \infty$) also accessible using exact WKB analysis

reproduce the IR phase diagram

What is the goal?

reproduce the IR phase diagram

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IR: conjectural IR pattern [Affleck-Ludwig]

- $n \leq k+1$: $L_n \to \mathcal{L}_n$, with $\langle 0|\mathcal{L}_n|0\rangle = d_n^{(k)} = \frac{\sin\frac{n\pi i}{k+2}}{\sin\frac{\pi i}{k+2}}$ is the quantum dimension
- $n > k+1$: $L_n \to \mathcal{L}_{k+1} \otimes L_{n-k}$, with $\langle 0|\mathcal{L}_{k+1} \otimes L_{n-k}|0\rangle \sim n-k$

We need to show as $\theta \to \infty$

$$i(\psi_{-\frac{n}{2}}, \psi_{\frac{n}{2}}) = \begin{cases} d_n^k + \ldots, & n \leq k+1 \\ n-k+\ldots, & n > k+1 \end{cases} \tag{9}$$

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Exact WKB analysis!

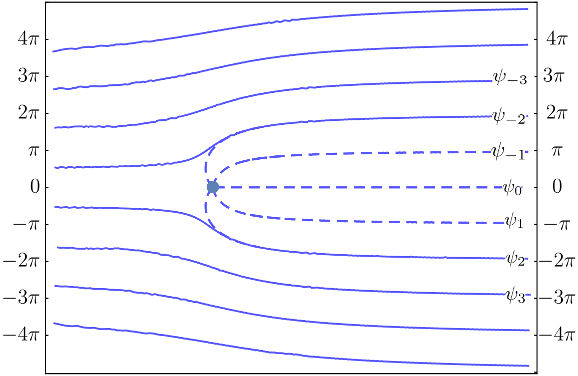
reproduce the IR phase diagram- exact WKB analysis

The ODE on $\mathbb{P}^1$

$$\psi''(x) = e^{2\theta} e^{2x} x^k \psi \tag{10}$$

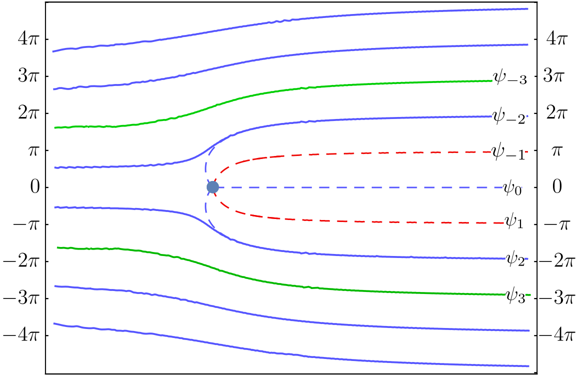
we have $P(x)dx^2 = e^{2\theta} e^{2x} x^k dx^2$. Define WKB diagram by the union of the trajectories

$$\sqrt{P(x)}dx \cdot \partial_t \in \mathbb{R}^+ \tag{11}$$



where $\psi_n(x;\theta) := \psi_0(x;\theta + n\pi i)$

$$i(\psi_{-\frac{n}{2}}, \psi_{\frac{n}{2}}) = \begin{cases} d_n^k + \ldots, & \text{connected at the zero}, \\ n - k + \ldots, & \text{connected at } -\infty, \end{cases}$$



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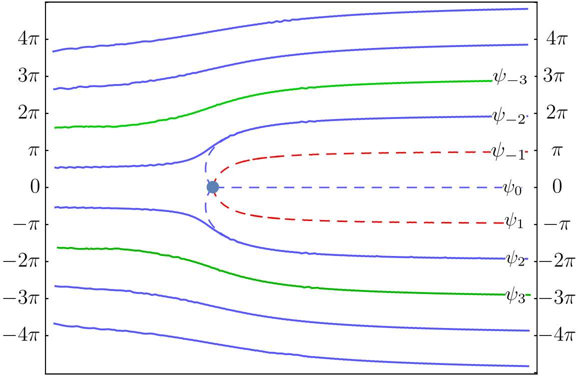

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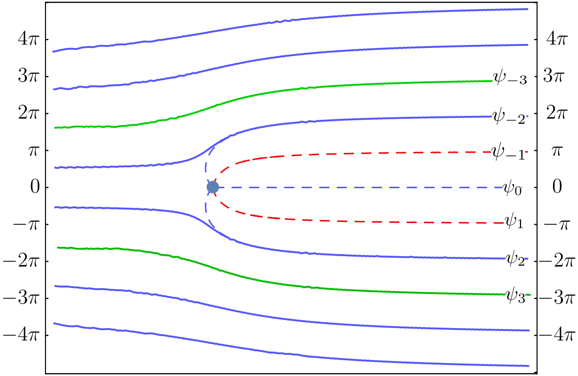


YAY!

Explain ODE/IM correspondence?

4D Chern Simons?

Construction in 4D Chern Simons

Lighting Review of 4D Chern Simons

4D cousin of the three dimensional Chern Simons [Costello, Costello-Witten-Yamazaki, Nekrasov, Vicedo, ...]

$$S = \int_{\mathbb{R}^2 \times \mathbb{C}} dz \wedge \mathrm{CS}[A] \tag{12}$$

- partial connection: $A = A_x dx + A_y dy + A_{\bar{z}} d\bar{z}$
- topological in $\mathbb{R}^2$, holomorphic in $\mathbb{C}$
- simplest operator: Wilson line
- only defined perturbatively, need string theory for non-perturbative statement

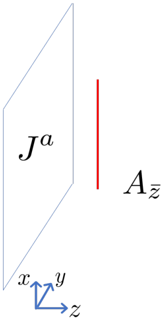
Construction of Kondo problems in 4D CS

- Let's put a chiral WZW on $\mathbb{R}^2 \times \{z_0\}$ and couples to the bulk via $J^a_w A^a_{\bar{w}}$
- gauge anomaly[Costello]:

$$dz \to \omega(z)dz \equiv \left(1 + \frac{k}{2}\frac{1}{z - z_0}\right) dz \qquad (13)$$

- RG flow implemented by shifting θ [Costello]

$$d\theta = \omega(z)dz \qquad (14)$$



Construction of Kondo problems in 4D CS

CLAIM: reduce to 2d, we get Kondo line in chiral WZW

- commutativity of the line
- Hirota relation: fusion of the line
- classically the Wilson line becomes [Costello-Witten-Yamazaki]

$$\int d\sigma \frac{1}{z - z_0} t^a J^a \tag{15}$$

If we identify $g = \frac{1}{z-z_0}$, we obtain the beta function

$$\partial_\theta g = \frac{g^2}{1 + \frac{k}{2} g} = g^2 - \frac{k}{2} g^3 + \dots \tag{16}$$

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WHAT ABOUT ODE/IM CORRESPONDENCE?

$$w(z) = 1 + \frac{k}{2} \frac{1}{z - z_0} \tag{17}$$

$$\frac{1}{2} \frac{\partial P(x)}{P(x)} = 1 + \frac{k}{2} \frac{1}{x} \tag{18}$$

Generalizations: multichannel Kondo

chiral $\prod_i \mathrm{SU}(2)_{k_i}$ WZW

Kondo line

$$\hat{T}_{\mathcal{R}}(\{g_i\}) := \mathrm{Tr}_{\mathcal{R}} \mathcal{P} \exp\left(i \int_0^{2\pi} d\sigma \; g_i t^a J_i^a(\sigma) \right) \tag{19}$$

ODE

$$P(x) = e^{2x} \prod_i (x - x_i)^{k_i} \tag{20}$$

4d CS multiple surface defect at $\mathbb{R}^2 \times \{z_1, z_2, \dots\}$

$$w(z) = 1 + \sum_i \frac{k_i}{2} \frac{1}{z - z_i} \tag{21}$$

Conclusions and Future directions

Conclusions

- study the relation between Kondo defect in CFT, affine oper, 4D Chern Simons, (affine Gaudin model)
- new examples of ODE/IM correspondence for the chiral $\prod_i \mathrm{SU}(2)_{k_i}$ WZW
- strong hints that we can explain the physical origin of ODE/IM from (string embedding of) 4d Chern Simons.

Future directions

- higher rank Lie algebra, coset
- anisotropic
- massive ODE/IM
- explain ODE/IM correspondence, string theory construction?

Thank You!