Gromov–Witten invariants and Givental's formalism
Counting curves with differential equations

Alexis Roquefeuil

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Section 1: Defining Gromov–Witten invariants

1.1. Counting curves in $\mathbb{P}^2$

Goal: Give an understandable example to motivate Gromov–Witten invariants

An enumerative geometry question

Question

Let $d \in \mathbb{Z}_{>0}$ be a positive number. Count the number N_d of (complex) curves of degree d in $\mathbb{P}^2_{\mathbb{C}}$ going through $3d-1$ points in general position.

Example

$N_1 = 1, N_2 = 1, N_3 = 12$, $N_4 = 620$ (Zeuthen, 1873), $N_5 = 87304$ (Vainsencher, 1993)

Theorem (Kontsevich–Manin, 1994)

For $d > 1$, the numbers N_d satisfy the recursion relation

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} N_{d_1} N_{d_2} \left(\binom{3d-4}{3d_1-2} d_1^2 d_2^2 - \binom{3d-4}{3d_1-1} d_1^3 d_2 \right)$$

1.2. Stable maps

Goal: Generalise what we are going to be counting in the definition of the previous numbers N_d

Stable maps: generalizing what we are counting

Instead of $\mathbb{P}^2$, we now consider instead X a complex projective variety.

Definition (Stable map, Kontsevich, 1995)

Let $n \in \mathbb{Z}_{\geq 0}$. A ($n$-pointed) *stable map* is the data of a connected compact curve C with n points $p_1, \ldots, p_n$ called markings and a morphism $f : C \to X$ such that

(i) The singularities of C are of nodal type at worst

(ii) The markings $p_1, \ldots, p_n \in C$ are distinct smooth points of the curve.

(iii) The data $(C, p_1, \ldots, p_n, f)$ has a finite number of automorphisms.

Let $(C, p_1, \ldots, p_n, f), (C', p'_1, \ldots, p'_n, f')$ be two stable maps. An isomorphism between these two stable maps is an isomorphism of pointed curves $\varphi : (C, p_1, \ldots, p_n) \to (C', p'_1, \ldots, p'_n)$ such that $f' \circ \varphi = f$

Further definitions on stable maps

Definition (Degree of a stable map)

Let $d \in H_2(X, \mathbb{Z})$. We say that the stable map $\left(f : (C; \underline{p}) \to X\right)$ has class d if the fundamental class $[C] \in H_2(C; \mathbb{Z})$ satisfies $f_*[C] = d$.

Example

Let $f : \mathbb{P}^1 \to \mathbb{P}^3$ defined by $[x : y] \mapsto \left[x^4 : x^3 y : x^2 y^2 : y^4\right]$. Then the degree of f is given by $f_* \left[\mathbb{P}^1\right] = 4 \left[\mathbb{P}^1\right] \in H_2\left(\mathbb{P}^3; \mathbb{Z}\right)$, which is given by the degree of the homogeneous coordinates of $f[x : y]$.

Definition

The genus of a stable map $f : (C, p_1, \ldots, p_n) \to X$ is the genus of the curve C.

Kontsevich's moduli space of stable maps

Theorem (Kontsevich, 1995)

Fix a genus $g \in \mathbb{Z}_{\geq 0}$, a number of markings $n \in \mathbb{Z}_{\geq 0}$ and a degree $d \in H_2(X;\mathbb{Z})$. There exists a Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}(X,d)$ over $\mathbb{C}$ which is compact, and whose set of geometrical points is given by

$$\left\{\begin{matrix}\text{stable maps } f : (C, p_1, \ldots, p_n) \to X \\ \text{with } n \text{ markings of genus } g \text{ and degree } d\end{matrix}\right\} /\text{isomorphism}$$

This space is called the moduli space of stable maps.

Understand: we can define and integrate forms on the space of classes of stable maps up to isomorphism.

Remark

Locally, the topology of $\overline{\mathcal{M}}_{g,n}(X,d)$ looks like the quotient of a complex space by a finite group.

1.3. Gromov–Witten invariants

Aim: Discuss cohomological Gromov–Witten invariants and explain how they are related to the numbers N_d defined before. Briefly mention K-theoretical Gromov–Witten invariants as a more technical analogue.

Gromov–Witten invariants

Definition (Evaluation maps)

We denote by $ev_i : \overline{\mathcal{M}}_{g,n}(X,d) \to X$ the i^{ith} evaluation map $i \in \{1, \dots, n\}$. This maps takes the stable map $[f : (C; p_1, \dots, p_n) \to X]$ and associates the evaluation of f at the i^{ith} marked point $f(p_i) \in X$.

Idea: Let $Z_1, \dots, Z_n \in H_*(X)$ be homological classes in X (e.g. points, curves...), whose Poincaré duals are $\gamma_1, \dots, \gamma_n \in H^*(X)$. We want the Gromov–Witten invariant $\langle \gamma_1, \dots, \gamma_n \rangle^{\text{coh}}_{g,n,d}$ to count the numbers of stable maps f of genus g and degree d, such that $ev_i(f) = f(p_i) \in Z_i$.

Definition (Cohomological Gromov–Witten invariant)

Let $\gamma_1, \dots, \gamma_n \in H^*(X; \mathbb{Q})$. The associated Gromov–Witten invariant is given by

$$\langle \gamma_1, \dots, \gamma_n \rangle^{\text{coh}}_{g,n,d} = \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{\text{vir}}} ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n) \in \mathbb{Q}$$

Back on the counting of curves in $\mathbb{P}^2$

$$\text{(recall)} \quad \langle \gamma_1, \dots, \gamma_n \rangle_{g,n,d}^{\text{coh}} = \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{\text{vir}}} ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n) \in \mathbb{Q}$$

Example

We have $H(\mathbb{P}^2; \mathbb{Q}) = \mathbb{Q}\langle 1, H, H^2 \rangle$.The Poincaré dual of $[pt] \in H_0(\mathbb{P}^2)$ is H^2. The number N_d of curves in $\mathbb{P}^2$ of degree d going through $3d-1$ points can be defined by

$$N_d = \langle H^2, \dots, H^2 \rangle_{0,3d-1,d}^{\text{coh}} \in \mathbb{Z}$$

$n = 3d-1$ can be found by matching the degree of the cohomological class in the integral with the degree of the virtual fundamental class.

Slogan: Gromov–Witten invariants count the number of curves satisfying some incidence conditions: $f(p_i) \in Z_i$, where $Z_i \in H_*(X)$ is such that its Poincaré dual is γ_i.

K-theoretical analogue

Recall: $K(X) = Vect(X)/(0 \to E' \to E \to E'' \to 0 \iff E = E' + E'')$.
The Euler characteristic of a vector bundle is defined by
$\chi(X; E) = \sum_{i \geq 0} (-1)^i \dim H^i(X; E)$.

Definition (K-theoretical Gromov–Witten invariant, Y.-P. Lee 2004)

Let $\phi_1, \ldots, \phi_n \in K(X)$. The associated Gromov–Witten invariant is

$$\langle \phi_1, \ldots, \phi_n \rangle_{g,n,d}^{K\text{th}} = \chi\left(\overline{\mathcal{M}}_{g,n}(X, d); \mathcal{O}^{\text{vir}} \otimes \phi_1 \otimes \cdots \otimes \phi_n\right) \in \mathbb{Z}$$

Slogan: These new numbers are a K-theoretical analogue of the cohomological Gromov–Witten invariants. They are more complicated to compute (e.g. unlike in cohomology, there is no degree on $K(X)$).

Question (Main motivation for the next section)

How are cohomological and K-theoretical Gromov–Witten invariants related?

Section 2
Computations with Gromov–Witten invariants

Main idea

Proposition (Givental '96, Givental–Lee '03)

Denote by $H \in H^2(\mathbb{P}^N)$ *the hyperplane class, and* $P = \mathcal{O}(1) \in K(\mathbb{P}^N)$*.*

$$j^{\text{coh}}(z,Q) = \sum_{d \geq 0} \frac{Q^d}{\prod_{r=1}^{d} (H + rz)^{N+1}} \in H^*\left(\mathbb{P}^N\right) \otimes \mathbb{C}[z, z^{-1}][\![Q]\!]$$

$$j^{K\text{th}}(q,Q) = \sum_{d \geq 0} \frac{Q^d}{\prod_{r=1}^{d} (1 - q^r P^{-1})^{N+1}} \in K\left(\mathbb{P}^N\right) \otimes \mathbb{C}(q)[\![Q]\!]$$

Theorem (Idea to relate Gromov–Witten invariants)

(i) The limit when $q \to 1$ *of the q-difference equation satisfied by* $j^{K\text{th}}$ *is the differential equation satisfied by* j^{coh}

(ii) The limit when $q \to 1$ *of* $j^{K\text{th}}$ *is* j^{coh}

2.1. Givental's J-function and functional equations

Aim: Explain how cohomological Gromov–Witten invariants can be understood from differential equations

Quantum cohomology

From now on we fix a basis $T_0, \ldots, T_N$ of the cohomology $H^*(X;\mathbb{Q})$ and set $\tau(\underline{t}) = t_0 T_0 + \cdots + t_n T_n$. Also, we assume $H^{2\bullet+1}(X) = 0$.

Definition (Genus zero Gromov–Witten potential)

$$\mathcal{F}_{(g=0)}(t_0, \ldots, t_N, Q) = \sum_{\substack{n \geq 0 \\ d \in H_2(X;\mathbb{Z})}} \frac{1}{n!} \langle \tau, \ldots, \tau \rangle_{0,n,d}^{\mathrm{coh}} Q^d \in \mathbb{Q}[\![\underline{t}, Q]\!]$$

We assume there exists an open set $U \in \mathbb{Q}[\![\underline{t}, Q]\!]$ in which the genus zero potential is convergent.

Definition (Quantum product)

We define a product $\bullet_{t,Q}$ on U by

$$T_i \bullet_{t,Q} T_j = \sum_{k=0}^{N} \partial_{t_i} \partial_{t_j} \partial_{t_k} \mathcal{F}(\tau, Q) T^k \in U$$

Dubrovin connection and Givental's J-function

Consider the trivial vector bundle $F = (U \times \mathbb{P}^1) \times H^*(X) \to U \times \mathbb{P}^1$.
Denote by z the local coordinate on $\mathbb{P}^1$ around zero.

Definition (Dubrovin's connection)

$$\nabla_{\partial_{t_i}} T_j = \partial_{t_i} T_j + \frac{1}{z} T_i \bullet_{t,Q} T_j$$
$$\nabla_{\partial_z} T_j = \left(\partial_z - \frac{1}{z^2} \mathfrak{E} \bullet_\tau + \frac{1}{z} \mu \right) T_j$$

Where $\mathfrak{E} \in H^*(X)$ and $\mu \in End(U)$

Slogan: geometrical properties of $\overline{\mathcal{M}}_{g,n}(X,d) \leftrightarrow$ properties of the potential $\mathcal{F} \leftrightarrow$ properties of the differential system given by ∇.
For example, the recursion formula of the numbers N_d of the first section is obtained as a consequence of the WDVV equations satisfied by the potential $\mathcal{F}$, applied to the case of $\mathbb{P}^2$.

Theorem (Givental '96)

The isomorphism $S^{\mathrm{coh}} \in Iso(F)$ *defined by (for a cohom. class* $\alpha \in F$*)*

$$S^{\mathrm{coh}}(\tau, z)\alpha = e^{-\tau_2/z}\alpha - \sum_{\substack{d \in H_2(X;\mathbb{Z})^* \\ l \geq 0}} \sum_{k=0}^{N} \frac{e^{\tau_2(d)}}{l!} \left\langle T_k, \tau', \ldots, \tau', \frac{e^{-\tau_2/z}\alpha}{z+\psi} \right\rangle_{0,l+2,d}^{\mathrm{coh}} T^k$$

satisfies $\nabla_{\partial_{t_i}} S^{\mathrm{coh}}(\tau, z)\alpha = 0$

Definition

Givental's J-function is defined by $J^{\mathrm{coh}}(t, z, Q) = S(t, z)^{-1}\mathbb{1}$

Theorem (Givental quantization, Teleman '07)

If (F, ∇) *is a semisimple Frobenius manifold, then the data of the small* J*-function* $J^{\mathrm{coh}}(t = 0, z, Q)$ *is enough to determine all genera Gromov–Witten invariants.*

2.2. Confluence of quantum K-theory to quantum cohomology for $\mathbb{P}^N$

Aim: Describe Givental's small J-function for projective spaces. Explain how to use the J-functions to obtain a relation between K-theoretical and cohomological Gromov–Witten invariants

Small J-functions for $\mathbb{P}^N$, again

Proposition (Givental '96, Givental–Lee '03)

Denote by $H \in H^2(\mathbb{P}^N)$ the hyperplane class, and $P = \mathcal{O}(1) \in K(\mathbb{P}^N)$.

$$j^{\mathrm{coh}}(z,Q) = \sum_{d \geq 0} \frac{Q^d}{\prod_{r=1}^d (H+rz)^{N+1}} \in H^*\left(\mathbb{P}^N\right) \otimes \mathbb{C}[z,z^{-1}][\![Q]\!]$$

$$j^{K\mathrm{th}}(q,Q) = \sum_{d \geq 0} \frac{Q^d}{\prod_{r=1}^d (1-q^r P^{-1})^{N+1}} \in K\left(\mathbb{P}^N\right) \otimes \mathbb{C}(q)[\![Q]\!]$$

Denote by $q^{Q\partial_Q}$ the q-difference operator $f(Q) \mapsto f(qQ)$. We have the functional equations

$$\left[(H + zQ\partial_Q)^{N+1} - Q\right] j^{\mathrm{coh}}(z,Q) = 0$$

$$\left[\left(1 - P^{-1}q^{Q\partial_Q}\right)^{N+1} - Q\right] j^{K\mathrm{th}}(q,Q) = 0$$

Confluence of q-difference equations

Remark

$$\frac{q^{Q\partial_Q} - \mathrm{Id}}{q-1} \cdot Q^k = \frac{q^k - 1}{q-1} Q^k \to_{q\to 1} kQ^k = Q\partial_Q \cdot Q^k$$

Denote by $\delta_q = \frac{q^{Q\partial_Q} - \mathrm{Id}}{q-1}$. The formal limit of this q-difference operator when $q \to 1$ is therefore the Euler derivative $Q\partial_Q$.

Example

The q-exponential ($q \in \mathbb{C}$)

$$e_q(Q) := \sum_{d\geq 0} Q^d \prod_{r=1}^{d} \frac{1-q}{1-q^r} \to_{q\to 1} e^Q$$

Satisfies the q-difference equation

$$\delta_q f(Q) = Qf(Q)$$

Confluence applied to K-theoretical Gromov–Witten of $\mathbb{P}^N$

Theorem (R., '19)

(i) The limit when $q \to 1$ of the q-difference equation satisfied by $j^{K\text{th}}$ is the differential equation satisfied by j^{coh}

(ii) The limit when $q \to 1$ of $j^{K\text{th}}$ is j^{coh}

Conclusion: through the point of view of the "simple" principle $\delta_q \to_{q \to 1} Q\partial_Q$, it turns out we can find a relation between our technical Gromov–Witten invariants.

2.3. Gromov–Witten theory and Stokes phenomena

Stokes phenomena

Theorem (Riemann–Hilbert–Birkhoff corresp.,Mochizuki '11)

There is an equivalence of categories

(Differential equations with irregular singularity)

$$\leftrightarrow$$

(Stokes-filtered local systems)

Question: How to interpret the Stokes matrices of the differential system of quantum cohomology as some geometrical data?

Theorem (Dubrovin's conjecture)

(i) The quantum cohomology ring $QH(X)$ is semisimple iff the category $\mathcal{D}^b(X)$ admits a full exceptional collection $(E_1, \ldots, E_k)$

(ii) The Stokes matrices are given by the Gram matrix of the f.e.c. $S_{ij} = \chi(E_i, E_j)$.

Fundamental solutions for q-difference equations

We consider the equivariant version of the q-difference equation for the small J-function. Denoting by $\Lambda_0, \ldots, \Lambda_N$ the equivariant parameters resulting from the action of $(\mathbb{C}^*)^{N+1}$ acting on $\mathbb{P}^N$, we consider

$$\left[\left(1 - \Lambda_0 q^{Q\partial_Q}\right) \cdots \left(1 - \Lambda_N q^{Q\partial_Q}\right) - Q\right] y_q(Q) = 0$$

This q-difference equation has a fundamental solution at $Q = 0$ given by Givental's small J-function.

Proposition (R., 20)

Denote by $w = Q^{-1}$ the local coordinate at ∞. We have a basis of solutions (f_i) at ∞ given by the formal q-hypergeometric series $(\alpha \in \mathbb{C}^* - q^{\mathbb{Z}}, i \in \{0, \ldots, N\})$

$$\frac{\theta_q(w)}{\theta_q(\alpha^{-1}\Lambda_i)}\, {}_{N+1}\psi_0 \left(\begin{matrix} \Lambda_0^{-1}\alpha^{-1}\Lambda_i & \cdots & \Lambda_N^{-1}\alpha^{-1}\Lambda_i \\ & - & \end{matrix} \middle| q, \left(\alpha\Lambda_i^{-1}\right)^{N+1} \Lambda_0 \cdots \Lambda_N w \right)$$

q-Borel–Laplace transform

Definition (Di Vizio–Zhang '09)

The q-Borel transform of a power series $f(w) = \sum_{d \geq 0} f_d w^d$ is given by

$$\mathcal{B}_q f(\xi) := \sum_{d \geq 0} q^{\frac{d(d-1)}{2}} f_d \xi^d$$

The q-Laplace transform a function g along the q-spiral $[\lambda; q] \in \mathbb{C}^*/q^{\mathbb{Z}}$ is given by

$$\mathcal{L}_q^{[\lambda;q]} g(w) := \sum_{m \in \mathbb{Z}} \frac{g(\lambda q^m)}{\theta_q\left(\frac{\lambda q^m}{w}\right)}$$

Proposition

The solutions f_i to the q-difference equation satisfied by the equivariant J-function of $\mathbb{P}^N$ are q^{N+1}-Borel–Laplace summable, therefore their q^{N+1}-Borel–Laplace sums define a basis of analytic solutions.

Connection numbers for $\mathbb{P}^N$

Theorem (R., 20)

Let $\alpha \in \mathbb{C}^* - q^{\mathbb{Z}}$ and let $\left[\lambda; q^{N+1}\right] \in \mathbb{C}^*/q^{(N+1)\mathbb{Z}}$ be a q^{N+1}-spiral. Write $g_k = \mathcal{L}_{q^{N+1}}^{\left[\lambda; q^{N+1}\right]} \mathcal{B}_{q^{N+1}} f_k$. Then, our fundamental solutions are related by

$$g_k^{\left[\lambda; q^{N+1}\right]}(w) = \sum_{j=0}^{N} R_{k,j}^{\left[\lambda; q^{N+1}\right]}(q,w) j^{K\text{th, eq}}_{|P=\Lambda_j}\left(q, \frac{1}{w}\right)$$

$$R_{k,j}^{\left[\lambda; q^{N+1}\right]}(q,w) = \frac{\left(q, \frac{\alpha^{-1}\Lambda_k}{\underline{\widehat{\Lambda}}^j}; q\right)_\infty \theta_q\left((-1)^N \frac{\lambda \alpha^{-1}\Lambda_k}{\Lambda_j}\right) \theta_{q^{N+1}}\left(\frac{\lambda \Lambda_j^{N+1}}{\pi(\underline{\Lambda})w}\right)}{\left(q\alpha\Lambda_k^{-1}\Lambda_j, \frac{\Lambda_j}{\underline{\widehat{\Lambda}}^j}; q\right)_\infty \theta_q\left((-1)^N\lambda\right) \theta_{q^{N+1}}\left(\frac{\lambda}{\left(\alpha\Lambda_k^{-1}\right)^{N+1}\pi(\underline{\Lambda})w}\right)} \times$$

$$\times\, \theta_q(-w)\theta_q(\alpha\Lambda_k^{-1}w)^{-1}\Lambda_j^{\ell_q\left(\frac{1}{w}\right)}$$

2.4. Oscillatory integrals

Oscillatory integrals (WIP, joint with T. Milanov)

Proposition (Givental, '15)

For $X = \mathbb{P}^N$, the oscillatory integral

$$I(q, Q) = \int_{\{x_0 \ldots x_N = Q\}} \exp\left(\sum_{k>0} \sum_j \frac{x_j^k}{k(1-q^k)}\right) \frac{dx_0 \cdots dx_n}{d(x_0 \cdots x_n)}$$

Satisfies the q-difference equation of the small K-theoretical J-function

Proposition

$$I(q, Q) = \int_{\mathbb{P}^N} \widehat{\gamma}_q(T\mathbb{P}^N) \cup \mathrm{ch}_q(j^{K\mathrm{th}}(q, Q))$$

Where $\quad \mathrm{ch}_q(E) = \prod_i q^{x_i} \quad \widehat{\gamma}_q(E) = \prod_i x_i \int_0^\infty e_q\left(\frac{y}{1-q}\right) y^{x_i - 1} dy$

Thank you for your attention!