Gopakumar-Vafa type invariants for Calabi-Yau 4-folds

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Background:

on CY3, $GW_{g,\beta} \in \mathbb{Q} \overset{\text{Gopakumar-Vafa conj}}{\longleftrightarrow} BPS_{g,\beta} \in \mathbb{Z}$

GV proposed to use $sl_2 \times sl_2$-action on coh theory of moduli of 1-dim stable sheaves

Mathematically: Katz ($g=0$) Maulik-Toda ($\forall g$) (modified earlier approaches of Hosono-Saito-Takahashi, Kiem-Li).
involves sheaf of vanishing cycles & coho DT invs.

Today, we discussed the story on CY4.

- GW/GV invs on CY_4 :

X sm proj CY_4 ($K_X = \mathcal{O}_X$) / $\mathbb{C}$

$\overline{M}_{g,n}(X,\beta)$: moduli space of genus g n-pted stable maps $f: C \to X$ w/ $f_*[C] = \beta \in H_2(X,\mathbb{Z})$, $p_1, \dots, p_n \in C$.

perf obs theory $\rightsquigarrow$ virtual class

$$v.d_{\mathbb{C}}\, \overline{M}_{g,n}(X,\beta) = 1-g+n.$$

No marked pts $\longrightarrow$ $v.d_{\mathbb{C}} = 1-g < 0$ (if $g \geq 2$).

$g=0$

$$\begin{array}{ccc} \overline{M}_{0,1}(X,\beta) & \xrightarrow{ev} & X \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f: C \to X & \longmapsto & f(p) \\ \rotatebox{90}{$\in$} & & \\ p & & \end{array}$$

$$GW_{0,\beta}(\gamma) = \int_{[\overline{M}_{0,1}(X,\beta)]^{vir}} ev^*(\gamma) \in \mathbb{Q}, \quad \gamma \in H^4(X,\mathbb{Z})$$

$\underline{g=1}$: $\overline{\mathcal{M}}_{1,0}(X,\beta)$. v.$\dim_{\mathbb{C}} = 1-1 = 0$.

$$GW_{1,\beta} := \int_{[\overline{\mathcal{M}}_{1,0}(X,\beta)]^{vir}} 1 \in \mathbb{Q}$$

$\underline{Q}$: Integral curve counting invs ?

Klemm-Pandharipande (2007) Define $n_{0,\beta}(\gamma)$, $n_{1,\beta}$ by

$$GW_{0,\beta}(\gamma) = \sum_{\substack{k \mid \beta \\ k \geq 1}} \frac{1}{k^2} n_{0,\beta/k}(\gamma)$$

$$\sum_{\beta} GW_{1,\beta} q^{\beta} = \sum_{\beta} n_{1,\beta} \sum_{d=1}^{\infty} \frac{\sigma(d)}{d} q^{d\beta} + \frac{1}{24} \sum_{\beta} n_{0,\beta}(c_2(X)) \log(1-q^{\beta})$$

$$- \frac{1}{24} \sum_{\beta_1,\beta_2} \underline{m_{\beta_1,\beta_2}} \log(1-q^{\beta_1+\beta_2})$$

$$\sigma(d) = \sum_{i \mid d} i$$

n_{β_1, β_2} meeting invs inductively defined by $g=0$ invs.

Conj: $n_{0,\beta}(\gamma), n_{1,\beta} \in \mathbb{Z}$!

call them $g=0, 1$ GV type invs of CY_4.

$\underline{\underline{Q}}$: Do we have intrinsic (e.g sheaf theoretical) understanding of them ?

<u>Our proposal</u>:

Motivated by story on CY_3.

Consider M_β : moduli space of 1-dim stable sheaves E.

w/ $[E] = \beta$, $\chi(E) = \underline{\underline{1}}$. (indep of choice of polarization).

Want: $[M_\beta]^{vir} \in H_*(M_\beta, \mathbb{Z})$.

deform-obs: $\quad \mathrm{Ext}^1(E,E) \xrightarrow{K} \mathrm{Ext}^2(E,E)$ s.t. $K^{-1}(0) \overset{\text{locally}}{\cong} M_\beta$ (at E).

difficulty: we can not use this local model.

$$\mathrm{ext}^1 - \mathrm{ext}^2 \ \text{Not topo const}$$

$$\underset{\substack{\downarrow \\ \text{topo (only dep on } ch\,E)}}{\chi(E,E)} = \underset{\substack{\| \\ 1}}{\mathrm{ext}^0} - \mathrm{ext}^1 + \mathrm{ext}^2 - \mathrm{ext}^3 + \mathrm{ext}^4.$$

$$\| \quad 2\,\mathrm{ext}^0 - 2\,\mathrm{ext}^1 + \mathrm{ext}^2.$$

$$= 2 - 2\,\mathrm{ext}^1 + \mathrm{ext}^2.$$

$$\Rightarrow \quad \mathrm{ext}^1 - \frac{1}{2}\mathrm{ext}^2 \ \text{topo}.$$

$$Q: \mathrm{Ext}^2(E,E) \otimes \mathrm{Ext}^2(E,E) \to \mathrm{Ext}^4(E,E) \cong \mathbb{C}$$

(Ext^2, Q) has $O(n;\mathbb{C}) \sim O(n;\mathbb{R})$

$\mathrm{Ext}^2_+ \subset \mathrm{Ext}^2$ positive real form

$(Q|_{\mathrm{Ext}^2_+} > 0)$

half dim real subspace

$$\begin{tikzcd} \mathrm{Ext}^1 \arrow[r, "\kappa"] \arrow[dr, "\kappa_+"'] & \mathrm{Ext}^2 = \mathrm{Ext}^2_+ \oplus \sqrt{-1}\,\mathrm{Ext}^2_+ \arrow[d, "\pi_+"] \\ & \mathrm{Ext}^2_+ \end{tikzcd}$$

glue them $\longrightarrow$ $[M_\beta]^{\mathrm{vir}} \in H_2(M_\beta, \mathbb{Z})$

Borisov–Joyce

deps on choice of ori

Need an orientability result

(det line bdl) $\mathcal{L} = \underline{\mathbb{R}}$ (C.–Gross–Joyce)

Insertions:

$$\tau_0 : H^4(X,\mathbb{Z}) \to H^2(M_\beta, \mathbb{Z})$$

$$\gamma \mapsto \pi_{M*}\big(\pi_X^*\gamma \cup \underset{\substack{\shortparallel \\ [\mathbb{F}]}}{ch_3(\mathbb{F})}\big)$$

primary insertion

does not dep on choice of $\mathbb{F}$.

$$\begin{tikzcd} & \mathbb{F} \arrow[d] & \\ & M \times X \arrow[dl, "\pi_M"'] \arrow[dr, "\pi_X"] & \\ M & & X \end{tikzcd}$$

$$\tau_1 : H^2(X,\mathbb{Z}) \to H^2(M_\beta, \mathbb{Q})$$

$$\alpha \mapsto \pi_{M*}\big(\pi_X^*\alpha \cup ch_4(\mathbb{F}_{norm})\big)$$

descendant insertion

$\mathbb{F}_{norm}$: normalized universal sheaf i.e. $\det(R\pi_{M*}(\mathbb{F})) = \mathcal{O}_M$.

$$\langle \tau_0(\gamma)\rangle_\beta := \int_{[M_\beta]^{vir}} \tau_0(\gamma) \in \mathbb{Z}, \quad \langle \tau_1(\alpha)\rangle_\beta := \int_{[M_\beta]^{vir}} \tau_1(\alpha) \in \mathbb{Q}.$$

Conj: $\exists$ ori s.t

i) (CMT) $\langle \tau_0(\gamma) \rangle_\beta = n_{0,\beta}(\gamma)$

ii) (CT) $\langle \tau_1(\alpha) \rangle_\beta = \dfrac{n_{0,\beta}(\alpha^2)}{2(\alpha.\beta)} - \sum_{\beta_1+\beta_2=\beta} \dfrac{(\alpha.\beta_1)(\alpha.\beta_2)}{4(\alpha.\beta)} m_{\beta_1,\beta_2}$

$$- \sum_{\substack{k \mid \beta \\ k \geq 1}} \frac{(\alpha.\beta)}{k} n_{1,\beta/k}.$$

Rk: from (i) obvious that RHS $\in \mathbb{Z}$.

from (ii) Not obvious why $n_{1,\beta} \in \mathbb{Z} \subseteq \mathbb{Q}$!

Note in order to make sense ii) Need a cancellation of poles, i.e.

$$n_{0,\beta}(\alpha^2) = \frac{1}{2} \sum_{\beta_1+\beta_2=\beta} (\alpha\cdot\beta_1)(\alpha\cdot\beta_2)\, m_{\beta_1,\beta_2}, \quad \text{if } (\alpha\cdot\beta)=0.$$

This is nontrivial constrain on $g=0$ GW/GV invs for CY_4 !

<u>Thm</u> (CT). Above eqn holds for CY_4 so ii) makes sense as an identity of linear fun on $\alpha \in H^2(X)$

Key pt: Some combinatorics reduce to WDVV equation.

<u>Heuristic argument:</u>

consider ideal CY_4: curves deform in families of exp dim

w/ expected generic properties

For i) primary insertion

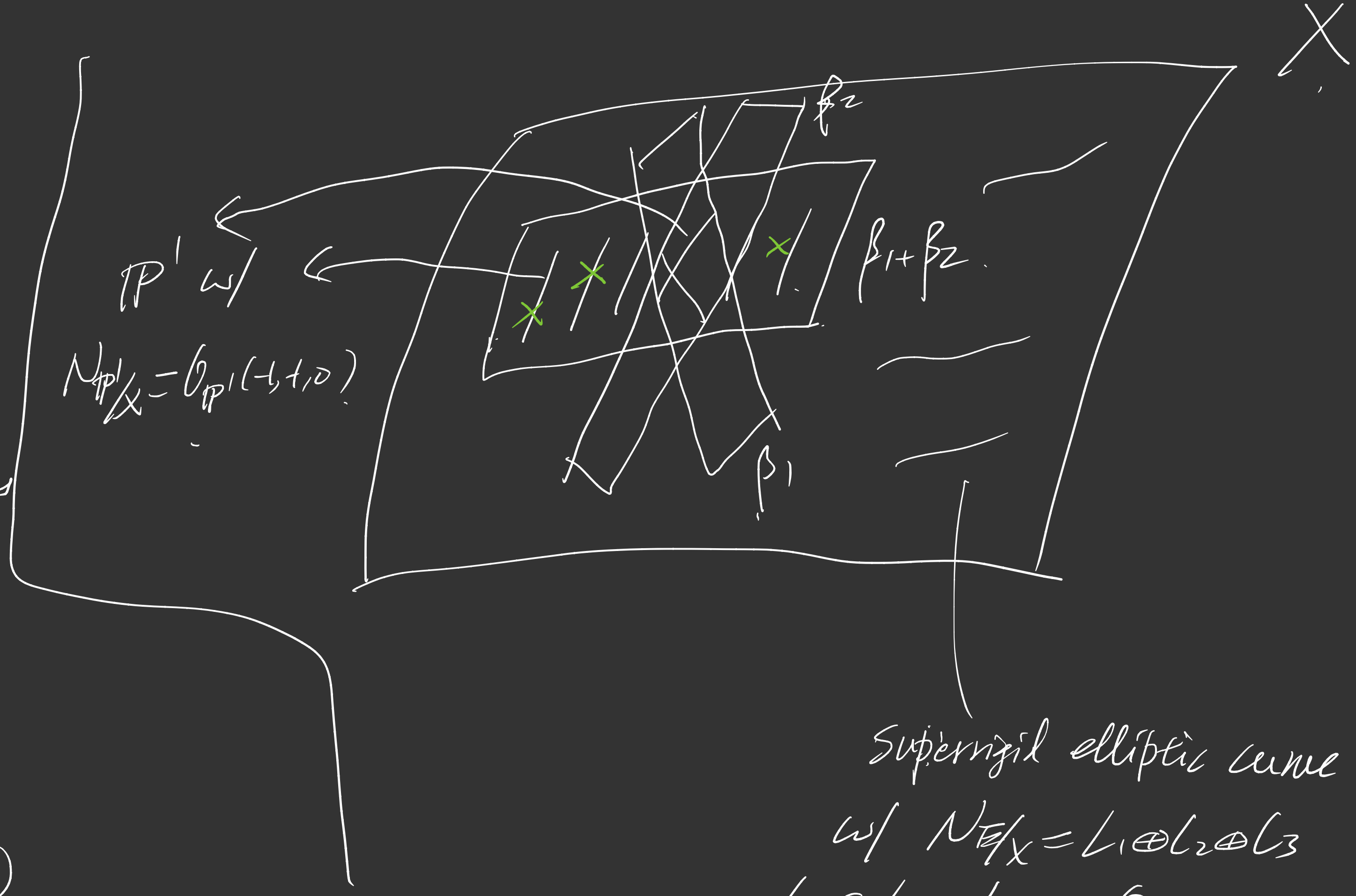
for $\gamma \in H^4(X, \mathbb{Z})$

does not intersect elliptic curves

& intersects  in

finite number of pts

Count pts $\longrightarrow$ $n_{0,\beta}(\gamma)$.



$$\int_{[M_\beta]^{vir}} \tau_0(\gamma) \underset{\substack{\uparrow \\ \text{by} \\ (N_{\mathbb{P}^1/X} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))}}{=} \int_{M_\beta} \tau_0(\gamma) = n_{0,\beta}(\gamma).$$

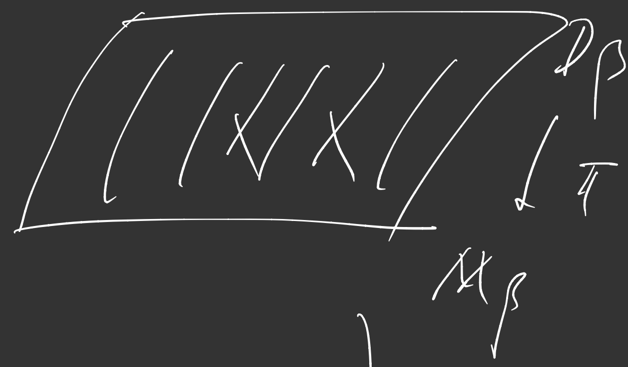
$M_\beta \longleftarrow E \in M_\beta$

$\mathrm{Ext}^1(E,E) = \mathbb{C}$

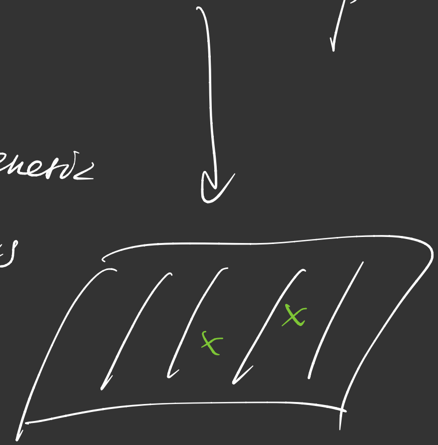
$\mathrm{Ext}^2(E,E) = 0.$

For ii) fix β consider $D_\beta \longrightarrow M_\beta$

total space of 1-dim family of rational curves



D_β modelled on blow-up of $\mathbb{P}^1$-bdl over f.m. generic pts



Consider its contribution to $\langle \tau_1(\alpha) \rangle_\beta$. $\boxed{|K|}$

$$\begin{array}{ccc} D_\beta & \overset{j}{\hookrightarrow} & X \times M_\beta \\ \downarrow & \swarrow & \\ M_\beta & & \end{array}$$

Universal family $j_* \mathcal{O}_{D_\beta}$ (normalized!)

GRR. $\quad ch_4(j_* \mathcal{O}_{D_\beta}) = -\frac{1}{2} c_1(\omega_\pi)$

So $\quad \displaystyle\int_{M_\beta} \tau_1(\alpha) = -\frac{1}{2} \int_{D_\beta} c_1(\omega_\pi) \cdot \alpha|_{D_\beta}$

Note $\quad H^2(D_\beta, \mathbb{Q}) = \mathbb{Q}\langle \beta, \psi, \beta_1^{(1)}, \cdots, \beta_1^{(k)} \rangle$

$\psi = c_1(\omega_\pi)$, $\quad \beta_1^{(i)} =$ classes of excep curves

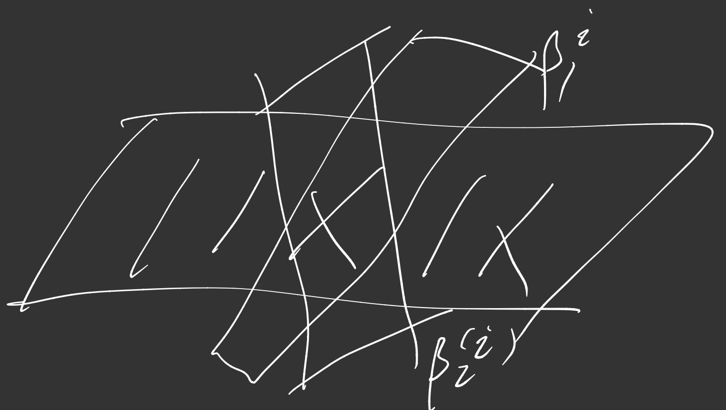
w/ $\quad \beta \cdot \beta = \beta_1^{(i)} \cdot \beta = 0$. $\quad \beta \cdot \psi = -2$, $\quad \beta_1^{(i)} \cdot \psi = (\beta_1^{i})^2 = -1$

$$\alpha|_{D_\beta} = a\beta + b\psi + \sum_{i=1}^{k} d_i\, \beta_1^{(i)}, \quad a, b, d_i \in \mathbb{Q}$$

$\alpha \cap$ w/ other divs

$$\begin{cases} (\alpha|_{D_\beta})^2 = b^2\psi^2 - \sum_{i=1}^{k} d_i^2 - 4ab - 2b\sum_{i=1}^{k} d_i \\ -\frac{1}{2}\psi\,\alpha|_{D_\beta} = a - \frac{1}{2} b\psi^2 + \frac{1}{2}\sum_{i=1}^{k} d_i \\ (\alpha\cdot\beta) = -2b, \qquad \alpha\cdot\beta_1^{(i)} = -b - d_i \end{cases} \qquad \beta_2^{(i)} = \beta - \beta_1^{(i)}$$

$$-\frac{1}{2}\psi\cdot\alpha|_{D_\beta} = \frac{(\alpha|_{D_\beta})^2}{2(\alpha\cdot\beta)} + \frac{(\alpha\cdot\beta)}{8}\psi^2 + \sum_{i=1}^{k} \frac{(\alpha\beta_1^{(i)} - \alpha\beta_2^{(i)})^2}{8(\alpha\cdot\beta)}$$



# of times such decomposition appears is meeting lines of $\beta_1^{(i)}$ $\beta_2^{(i)}$!

$\Rightarrow$ Above formula adjusted to

$$-\frac{1}{2}\psi\cdot\alpha|_{D\beta} = \frac{\alpha^2|_{D\beta}}{2(\alpha\cdot\beta)} + \frac{(\alpha\cdot\beta)}{8}\psi^2 + \sum_{\beta_1+\beta_2=\beta} \frac{(\alpha\beta_1-\alpha\beta_2)^2}{8\alpha\cdot\beta}\frac{m_{\beta_1,\beta_2}}{2}$$

$=$

mod sym between $\beta_1 \leftrightarrow \beta_2$.

It is easy to compute $\psi^2 = -\frac{1}{2}\sum_{\beta_1+\beta_2=\beta} m_{\beta_1,\beta_2}$

$$\Rightarrow \quad -\frac{1}{2}\psi\cdot\alpha|_{D_\beta} = \frac{\alpha^2|_{D\beta}}{2(\alpha\cdot\beta)} - \sum_{\beta_1+\beta_2=\beta}\frac{(\alpha\beta_1)(\alpha\beta_2)}{4(\alpha\cdot\beta)} m_{\beta_1,\beta_2}.$$

$$\begin{array}{cc} \| & \| \\ \langle\tau_1(\alpha)\rangle_\beta & \dfrac{n_{0,\beta}(\alpha^2)}{2(\alpha\cdot\beta)} \end{array}$$

Finally consider elliptic curves contributions.

for $\forall\ r \mid \beta$ & E w/ $[E] = \beta/r$.

moduli of 1-dim stable sheaves supp on E is $M_E(r,1)$ moduli of rk r, deg 1 stable bdl on E

$$\begin{array}{c} M_E(r,1) \\ \big\downarrow \det \\ \mathrm{Pic}^1(E) \end{array}$$

normalized $\mathbb{F}$ s.t. $[\mathbb{F}] = [\hat{j}_* \mathcal{O}_{E\times E}(\Delta)] + (r-1)[\hat{j}_* \mathcal{O}_{E\times E}]$

$$\Longrightarrow \quad \int_{M_E(r,1)} \tau_1(\alpha) = \frac{(\alpha\cdot\beta)}{r}.$$

So all superrigid elliptic curves

$$\longrightarrow \quad \underbrace{\sum_{r \mid \beta} \frac{(\alpha \cdot \beta)}{r} n_{1, \beta/r}}$$

(In conj formula. comes w/ minus sign in front)

Verifications:

1) Elliptic fib.

$$\begin{array}{c} X \\ \downarrow \pi \\ \mathbb{P}^3 \end{array}$$

general fibers are elliptic curves

sing fibers nodal / cuspidal plane curve.

<u>Thm</u>: Conj hold for $\beta = r[f]$. ($r \geq 1$)

(pf: $M_\beta \simeq X$ w/ virtual class $\pm PD(\underline{C_3(X)})$)

2) $X = Y \times E$ $\qquad \beta \in H_2(Y) \subseteq H_2(X)$

$\quad$ CY_3 elliptic curve.

$\underline{\text{Thm}}$: Conj i) holds iff Katz's conj holds for Y.

Conj ii) holds.

$$\underline{\int_{[M_\beta(Y)]^{vir}}} I \overset{\text{Katz's conj}}{=} \underline{n_{0,\beta}(Y)}$$

pf: $M_\beta(X) \cong M_\beta(Y) \times E$

w/ $[M_\beta(X)]^{vir} = \underline{\underline{[M_\beta(Y)]^{vir}}} \otimes [E]$)

Rk: Katz's conj is far from being proved!

holds

1). Local toric del Pezzo surface $\forall$ curve class

2). $CICY_3 \hookrightarrow \prod \mathbb{P}^*$ $\forall$ primitive class

3) Local CY4:

Thm: Conj holds for $\mathrm{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1)\oplus\mathcal{O}(-2))$ $\quad d\le 3$,

$\mathrm{Tot}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}(-1,-1)^{\oplus 2})$ $\quad d_1, d_2\le 2$

$K_{\mathbb{P}^3}$ $\quad d\le 3$

In above cases $X = K_Y$ $\quad Y$: Fano 3-fold.

$$M_\beta(X)\cong M_\beta(Y), \quad [M_\beta(X)]^{vir} = (-1)^{c_1(Y)\cdot\beta+1}[M_\beta(Y)]^{vir}.$$