Factorisation and Vortices in 3d $\mathcal{N} = 4$ Gauge Theories

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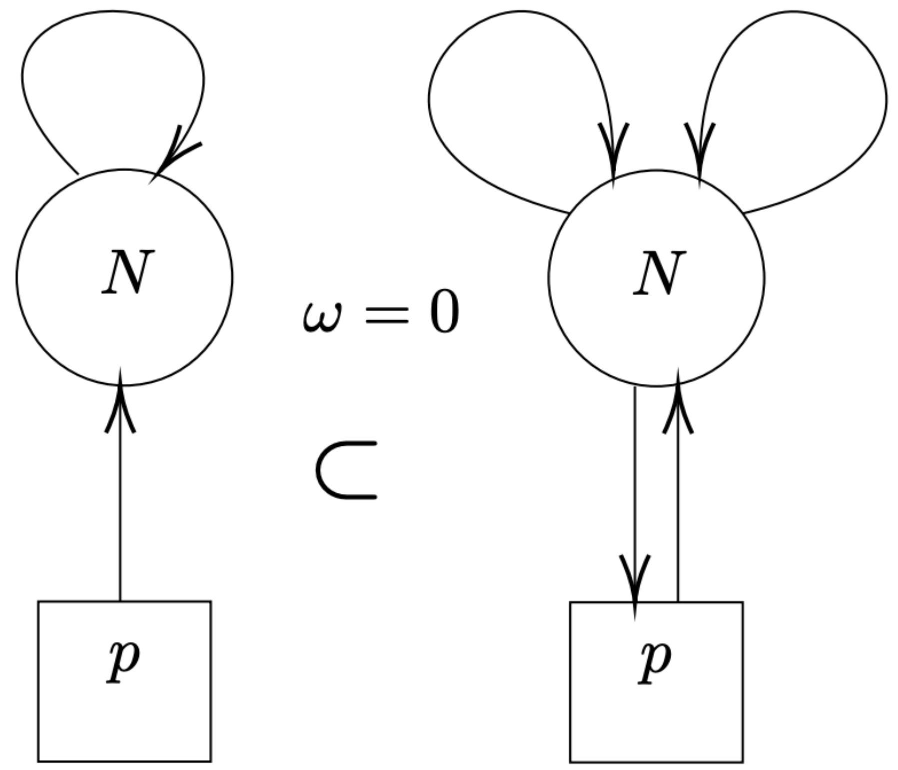
Based on 2002.04573, 2010.09732 and 2010.09741 with M. Bullimore, N. Dorey and D. Zhang

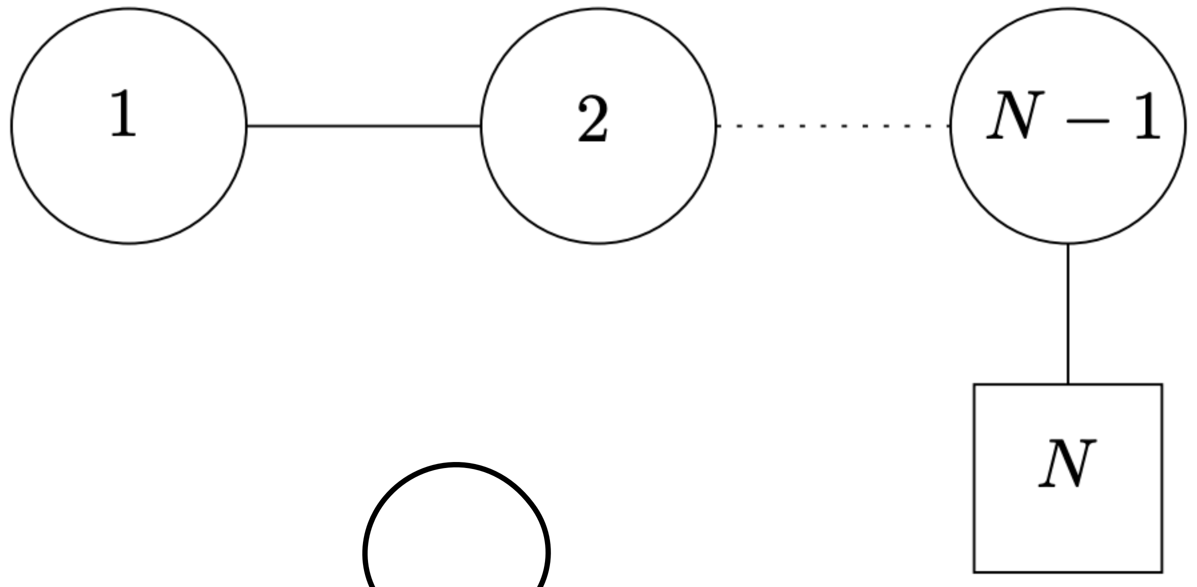
Background

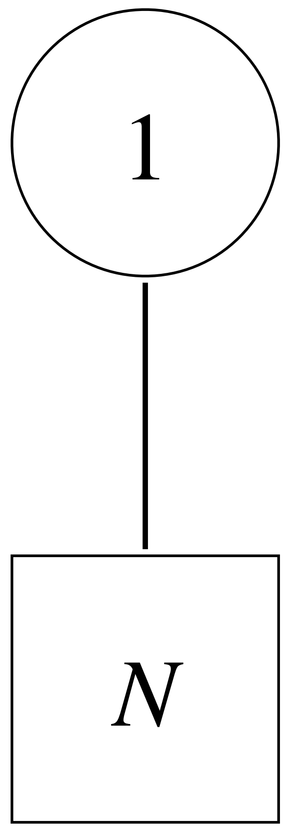
- Extended algebras acting on BPS states of supersymmetric field theories in various dimensions
  - Supersymmetric indices/partition functions as characters
  - Quantum algebras acting on homology, K-theory, elliptic cohomology of quiver varieties
  - 3d mirror symmetry and symplectic duality
- Exponential $N^{3/2}$ growth of states counted by indices. AdS_4 holography — saddle points

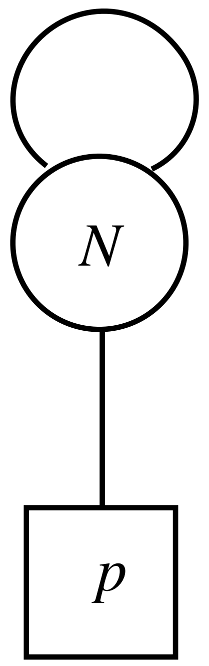
Outline

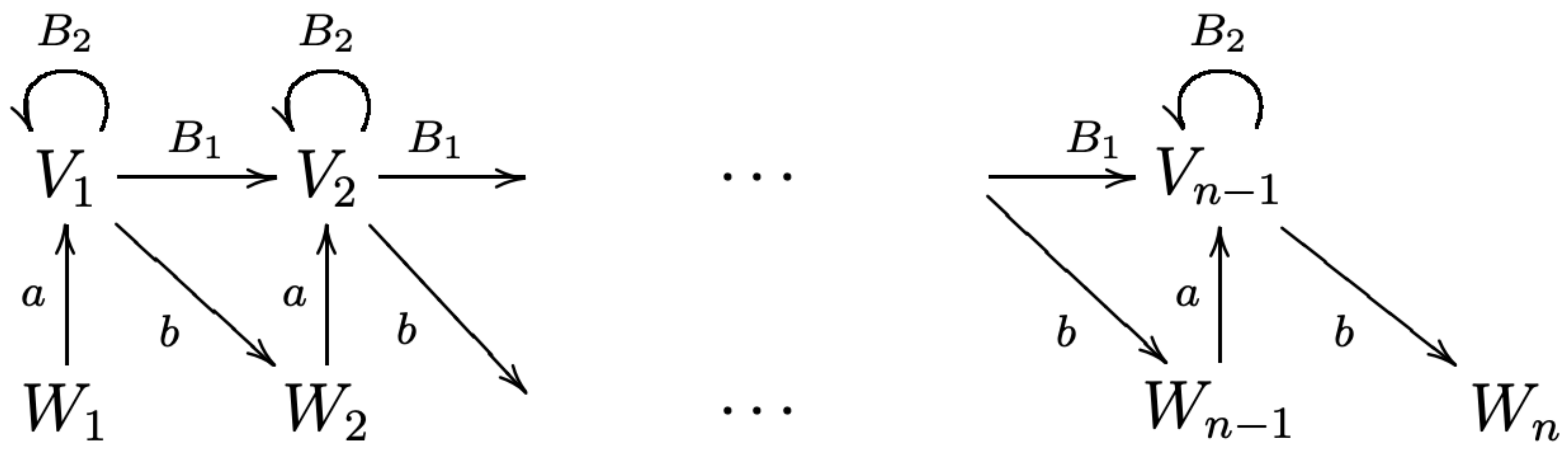
- Quick review of 3d $\mathcal{N} = 4$ gauge theory
- Quantised Higgs and Coulomb branch algebras. Modules induced by boundary conditions.
  - Verma modules and exceptional Dirichlet
- Hemisphere partition functions $S^1 \times H^2$, factorisation and "IR formulae"
  - Concrete examples
- Twisted indices, Hilbert series and Poincaré polynomials
- 3d ADHM theory











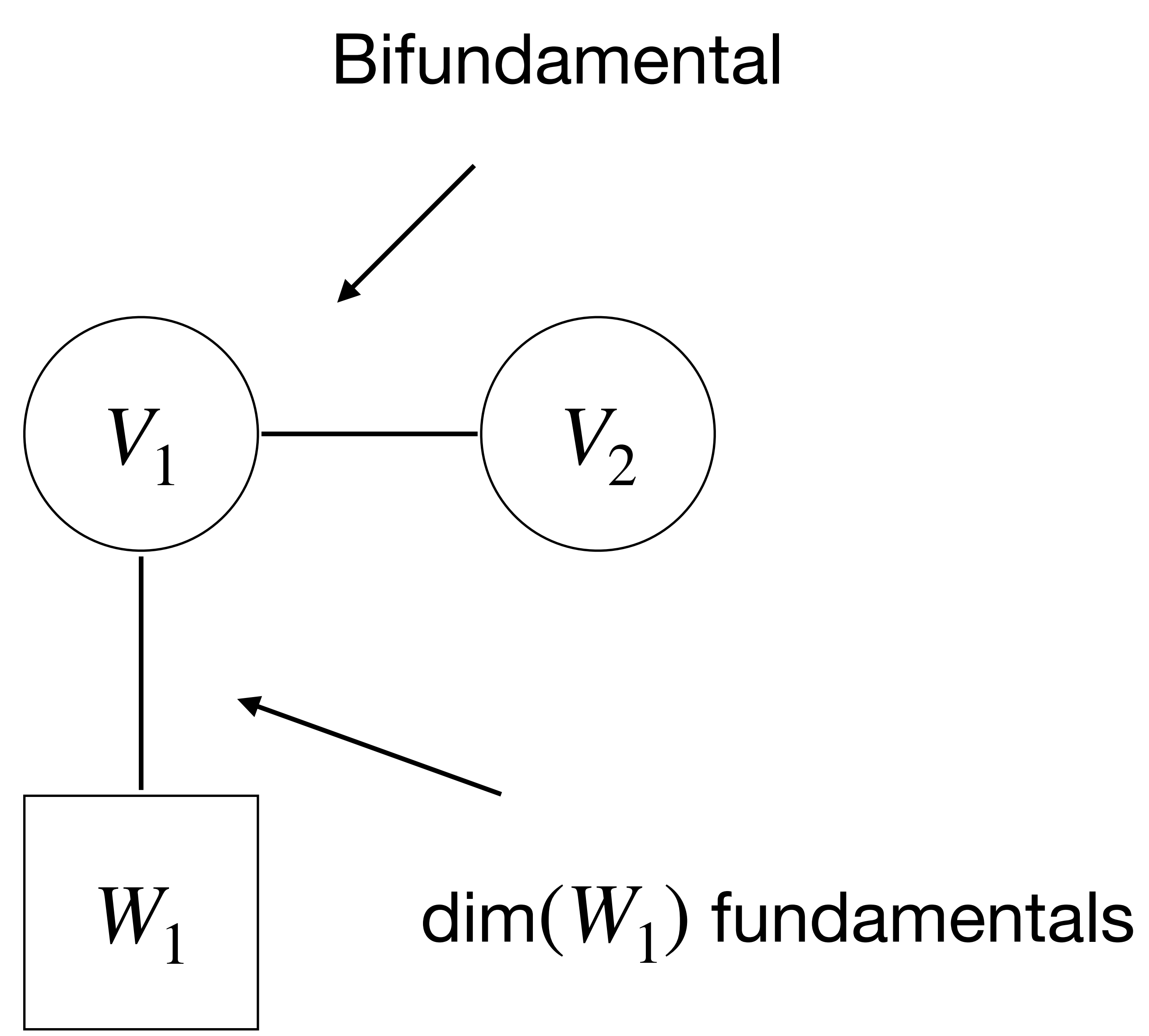
Background on 3d $\mathcal{N} = 4$ theories

- 8 supercharges $\mathcal{Q}^{a\dot{a}}_{\alpha}$
- Gauge group G and representation $\mathscr{R} = R \oplus R^*$
- R-symmetry $SU(2)_H \times SU(2)_C$
- Global symmetry $G_H \times G_C$
- Generic mass and FI deformations
  - $\vec{m} \in (\mathfrak{t}_H)^3$ and $\vec{t} \in (\mathfrak{t}_C)^3$
- Deformation t. Background vev for anti-diagonal R-symmetry combination $\mathcal{N} = 2^*$

3d $\mathcal{N} = 4$ Quiver Lagrangians

- 3d $\mathcal{N} = 4$ vectormultiplet
  - A_μ, σ, φ
- 3d $\mathcal{N} = 4$ hypermultiplets
  - (X, Y)

$$\mathcal{R} = T^*M$$



Moduli spaces of vacua

Classical

- Higgs branch $\mathscr{M}_H$ and Coulomb branch $\mathscr{M}_C$

Vectormultiplet scalars with monopoles $V_\pm$

- $\mathscr{M}_H$ and $\mathscr{M}_C$ hyperkähler
- G_H, G_C are tri-Hamiltonian isometries
- Assumption: flow to superconformal fixed point. Isolated massive vacua α
- $\mathscr{M}_H$ and $\mathscr{M}_C$ are symplectic resolutions with isolated singularities
- $m \in \mathfrak{t}_H$ and $\xi \in \mathfrak{t}_C$ are resolution and deformation parameters.

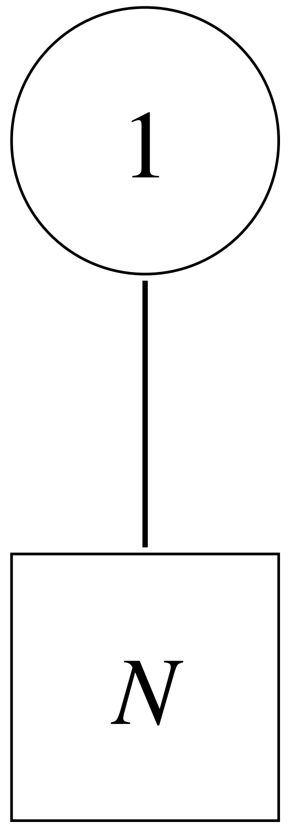
$$\mu_{\mathbb{R}} = X \cdot X^\dagger - Y \cdot Y^\dagger$$

$$\mu_{\mathbb{C}} = X \cdot Y$$

$$\mathscr{M}_H = \{\mu_{\mathbb{C}} = 0\,, \mu_{\mathbb{R}} = \xi\}/G$$

Examples

SQED$[N]$



$$\mathcal{M}_H = T^*\mathbb{P}^N$$

$$\mu_{\mathbb{R}} = \sum |X_i|^2 - |Y_i|^2$$

$$\mu_{\mathbb{C}} = \vec{X} \cdot \vec{Y}$$

$U(1)$ gauge theory.

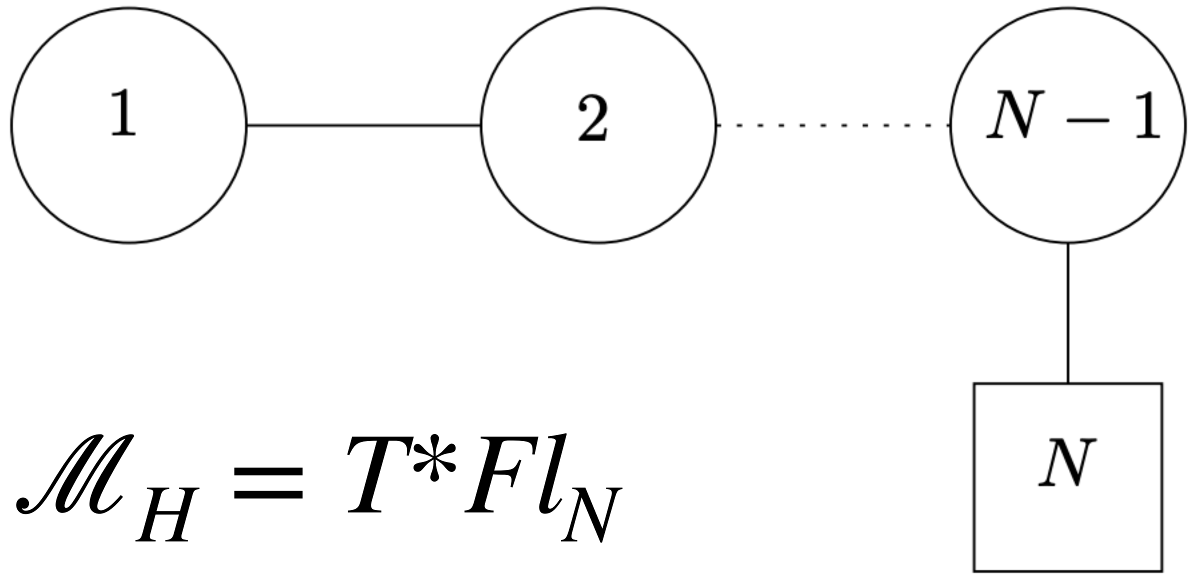
Fundamental hypermultiplets (X_i, Y_i) with $i = 1,\dots,N$

$G_H = SU(N)$ and $G_C = U(1)$

Masses $m_1, \dots, m_N$ and FI parameter η

N isolated vacua

$T[SU(N)]$



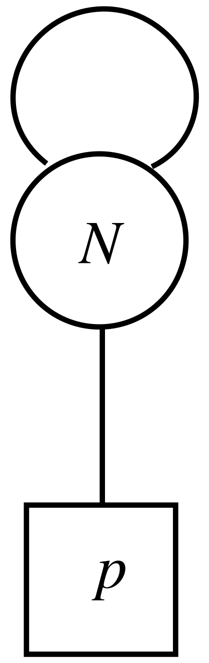
$$\mathcal{M}_H = T^*Fl_N$$

$G = U(1) \times \dots \times U(N-1)$

$G_H = G_C = SU(N)$

Masses $m_1, \dots, m_N$ and FI parameters $\eta_1, \dots, \eta_N$

Vacua labelled by $\sigma \in S_N$



$x = e^m$ and $\zeta = e^\eta$

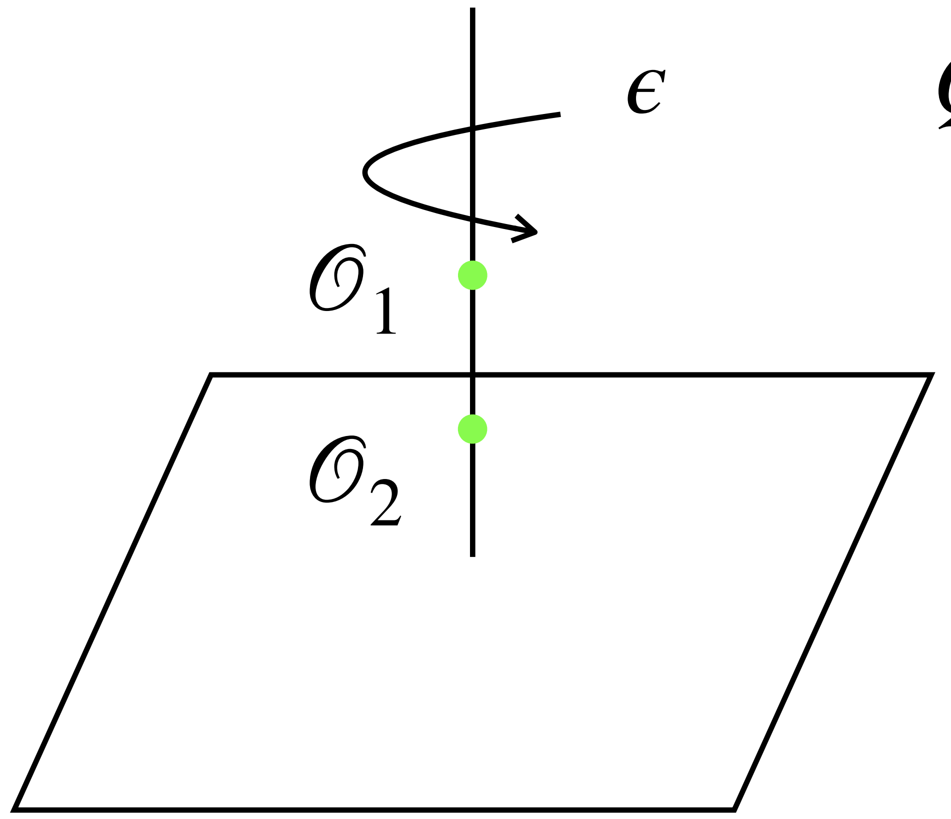
Higgs and Coulomb algebras

- Fix $\mathcal{N} = 2$ subalgebra $U(1)_H \times U(1)_C \subset SU(2)_H \times SU(2)_C$
- Ring of chiral operators/holomorphic functions $\mathbb{C}[\mathcal{M}_H]$ and $\mathbb{C}[\mathcal{M}_C]$
- Hyperkähler geometry equips with Poisson bracket

Quantisation

Ω background quantises chiral rings

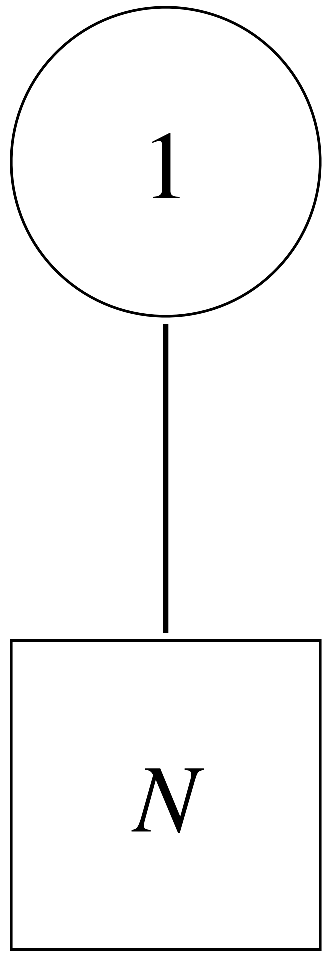
$\hat{\mathbb{C}}[\mathcal{M}_H]$ and $\hat{\mathbb{C}}[\mathcal{M}_C]$



$Q^2_{H/C} = \epsilon \mathcal{L}_V$

$q = e^{\epsilon}$

Example: SQED$[N]$



Higgs algebra $\hat{\mathbb{C}}[\mathscr{M}_H]$

Generated by X_i and Y_i

P.B. quantised: $[\hat{Y}_i, \hat{X}_j] = \epsilon \delta_{ij}$

Moment map $\sum_{i=1}^{N} : \hat{X}_i \hat{Y}_i := t_{\mathbb{C}}$

Central quotient of $U(\mathfrak{sl}_N)$

$$e_{ij} = \hat{X}_i \hat{Y}_j, \quad i < j$$

$$f_{ij} = \hat{X}_i \hat{Y}_j, \quad i > j$$

$$h_j = \hat{X}_j \hat{Y}_j - \hat{X}_{j+1} \hat{Y}_{j+1} \quad j = 1, \ldots, N-1$$

Coulomb algebra $\hat{\mathbb{C}}[\mathscr{M}_C]$

Generated by complex scalar φ and monopole operators $v_\pm$

$$[\hat{\varphi}, \hat{v}_\pm] = \pm \epsilon \hat{v}_\pm$$

$$\hat{v}_+ \hat{v}_- = \prod_{i=1}^{N} (\hat{\varphi} + m_{i,\mathbb{C}} - \frac{\epsilon}{2})$$

$$\hat{v}_- \hat{v}_+ = \prod_{i=1}^{N} (\hat{\varphi} + m_{i,\mathbb{C}} + \frac{\epsilon}{2})$$

Spherical rational Cherednik algebra — finite W algebra

We will discuss Verma modules later!

Vortex moduli spaces

- Theory admits $\frac{1}{2}$ BPS vortex solutions

- Hilbert space of Ω-deformed theory in plane with mass deformations = Equivariant homology of VMS.
  [Bullimore, Dimofte, Gaiotto, Hilburn, Kim]

- Relation between vortices and quasi-maps.

- Kähler manifolds with isometries x , q

Vortices are labelled by $\boldsymbol{d} = \frac{1}{2\pi}\int_{S^2} \mathrm{tr} F_i$ $\qquad X, Y \to \mathscr{M}_H$ at infinity

Identify $\boldsymbol{d} \in H_2(\mathscr{M}_H, \mathbb{Z})$ ← Quasimap degree

Algebraic description ⟶ $\mathrm{QM}^{\boldsymbol{d}}_{\alpha}(\mathbb{P}^1 \to \mathscr{M}_H)$ [Okounkov]

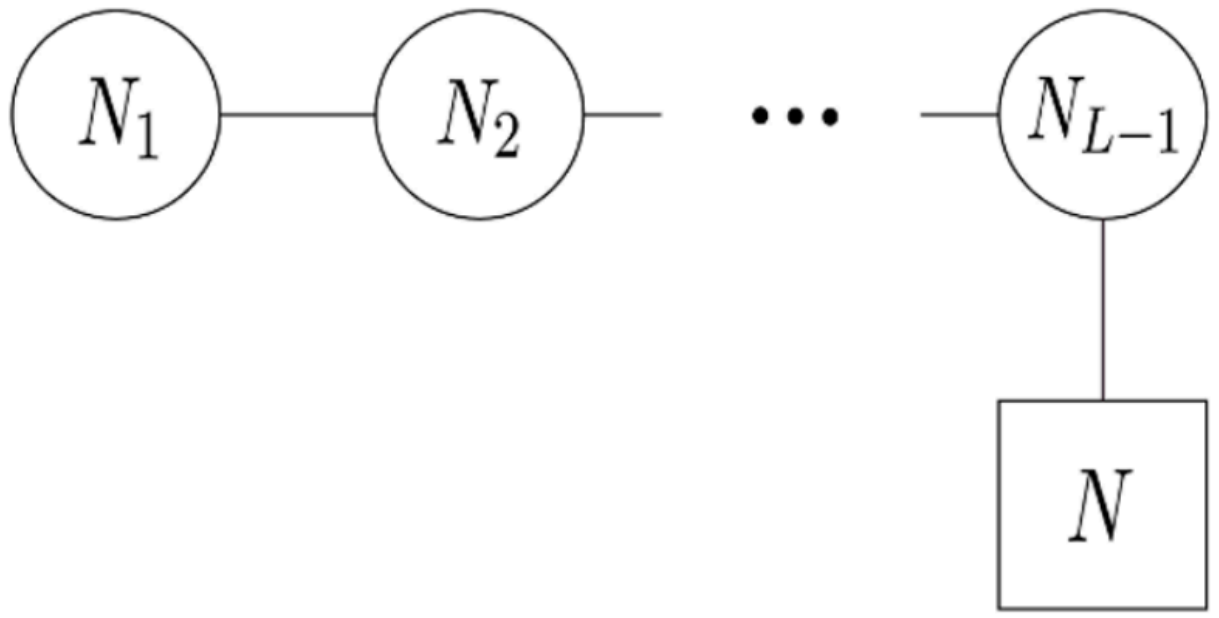
K-theoretic vertex functions

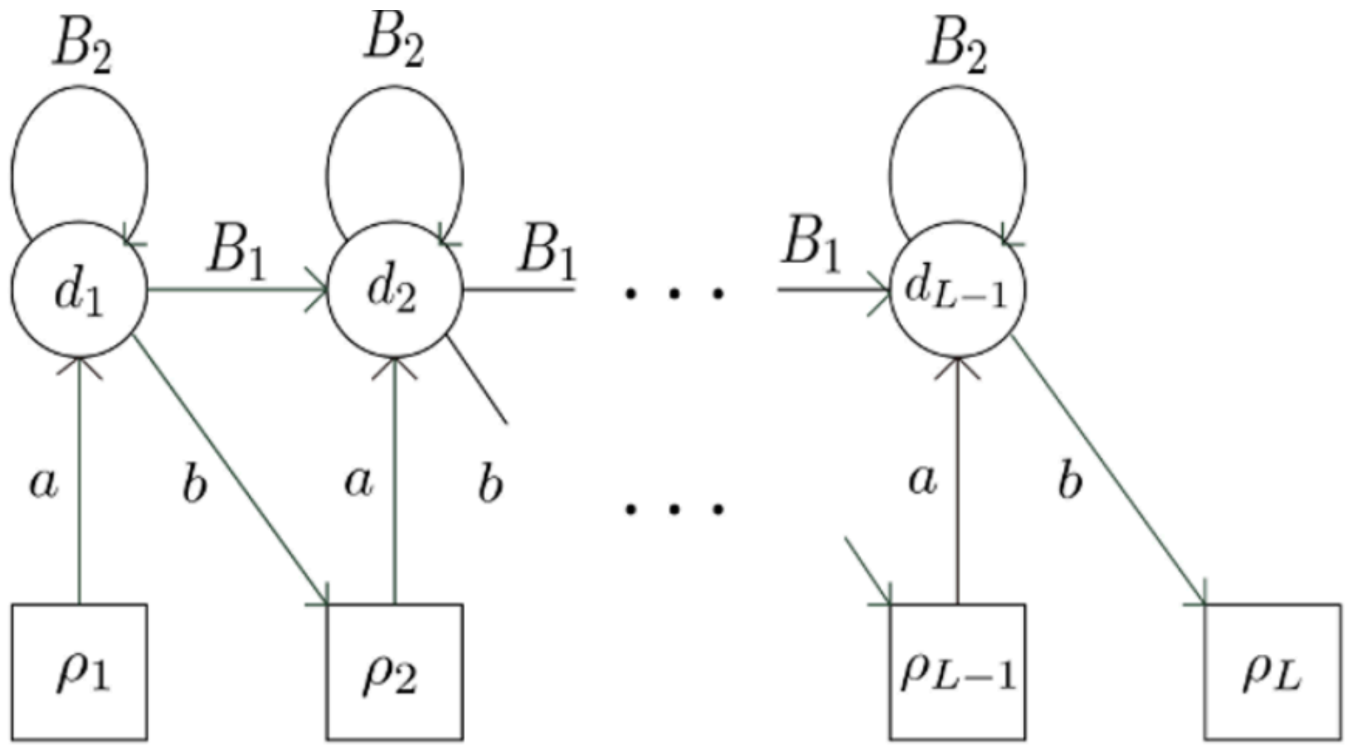
q rotates $\mathbb{P}^1$
G_H global symmetries
Isolated fixed points

Example: Laumon space/Handsaw quiver

Laumon spaces resolution of singularities $\mathfrak{Q}^{\boldsymbol{d}}_{\alpha} = \mathrm{QM}^{\boldsymbol{d}}_{\alpha}(\mathbb{P}^1 \to \text{flag})$

Realisation as handsaw quiver variety [Nakajima]

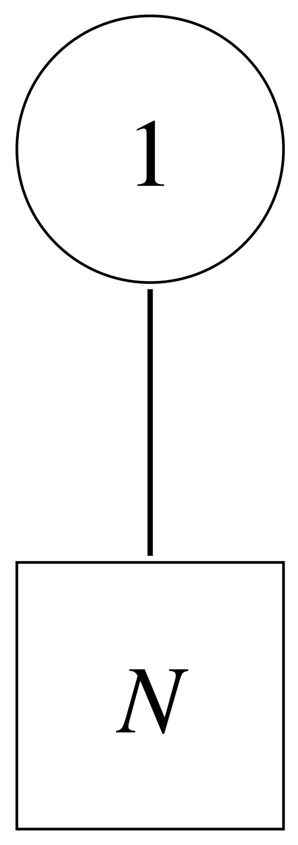


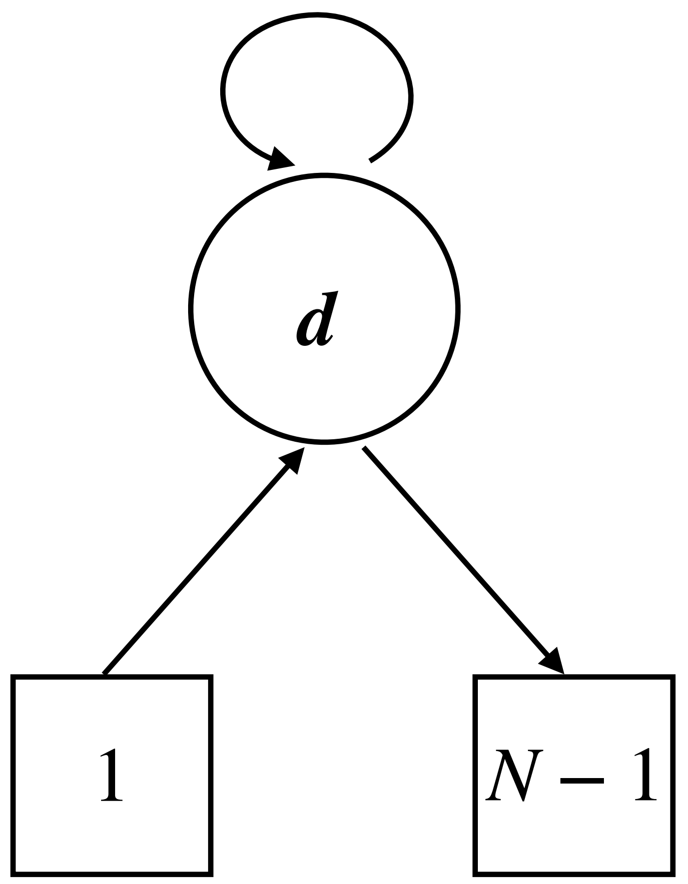


Will discuss χ_t genera and Poincaré polynomials

Example

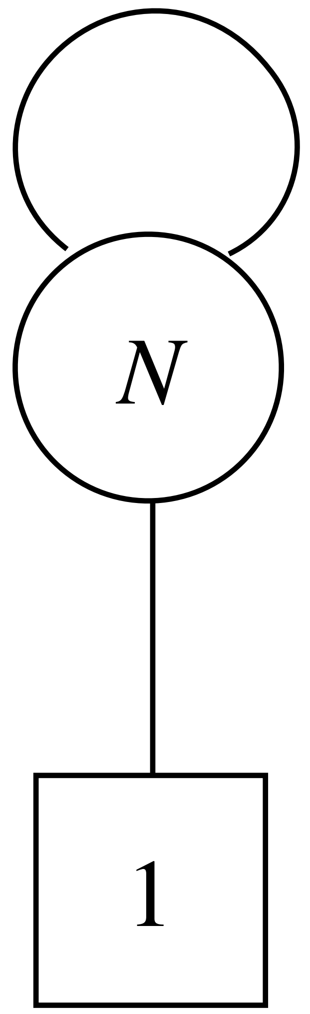
- SQED$[N]$ vortex moduli space





$\mathbb{C}^{dN}$

Another Example



$\mathscr{M}_H = \text{Hilb}^N(\mathbb{C}^2)$

- $\text{QM}^{\boldsymbol{d}}_{\lambda}(\mathbb{P}^1 \to \text{Hilb}^N(\mathbb{C}^2))$
- Smooth quiver description?
- $\chi(\hat{\mathscr{O}}_{\text{Vir}})$ —Localisation formula

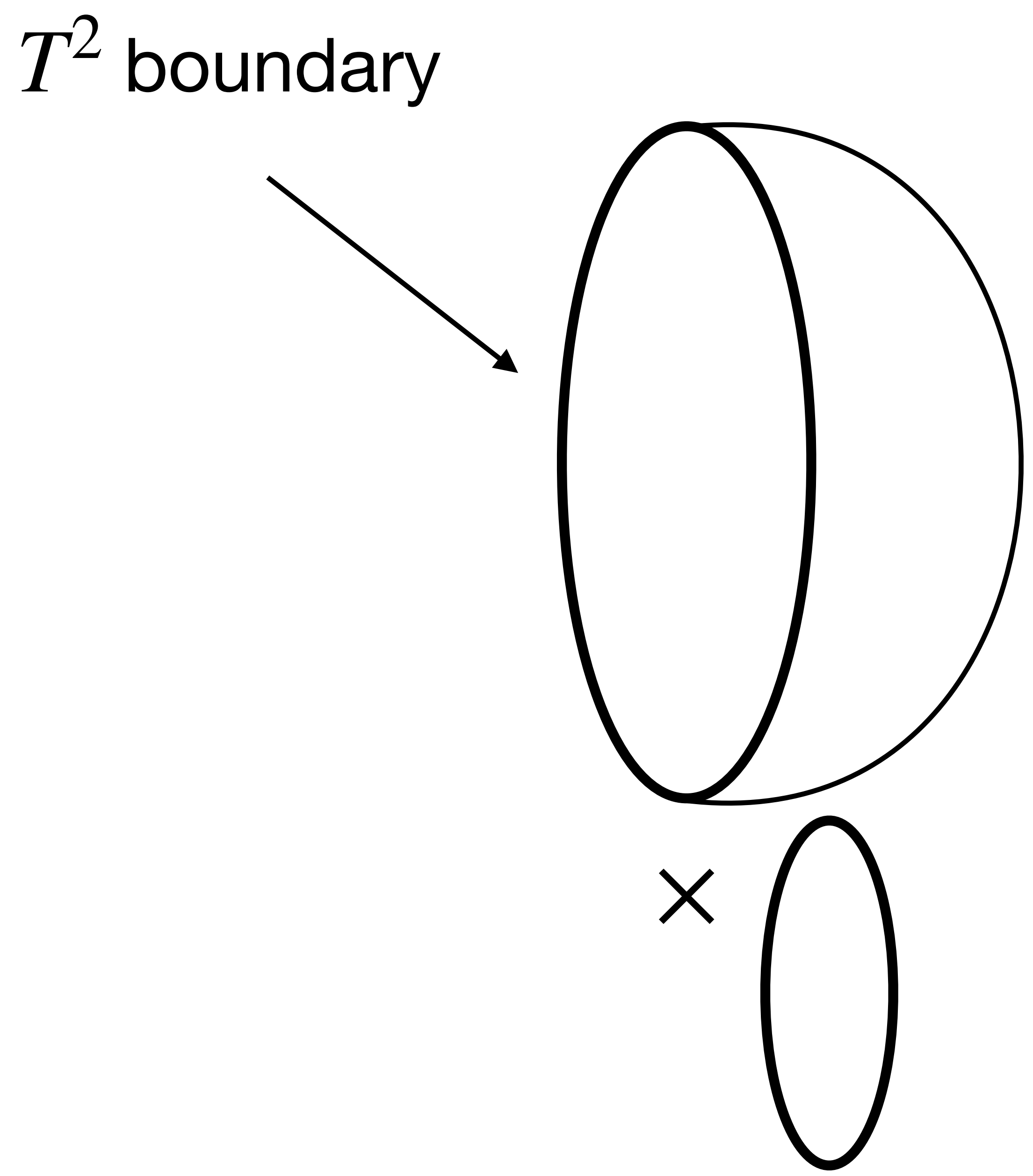
Hemisphere partition function

- We compute partition function on hemisphere $S^1 \times H^2$
- $\mathscr{N} = (2{,}2)$ boundary condition $\mathscr{B}$ on T^2

$$\mathcal{Z}_{S^1\times H^2} = \mathcal{Z}_{\text{Classical}}\mathcal{Z}_{\text{1-loop}}\mathcal{Z}_{\text{Vortex}}$$

$$\mathcal{Z}_{\text{Vortex}} \sim \chi(\hat{\mathcal{O}}_{\text{Vir.}})$$

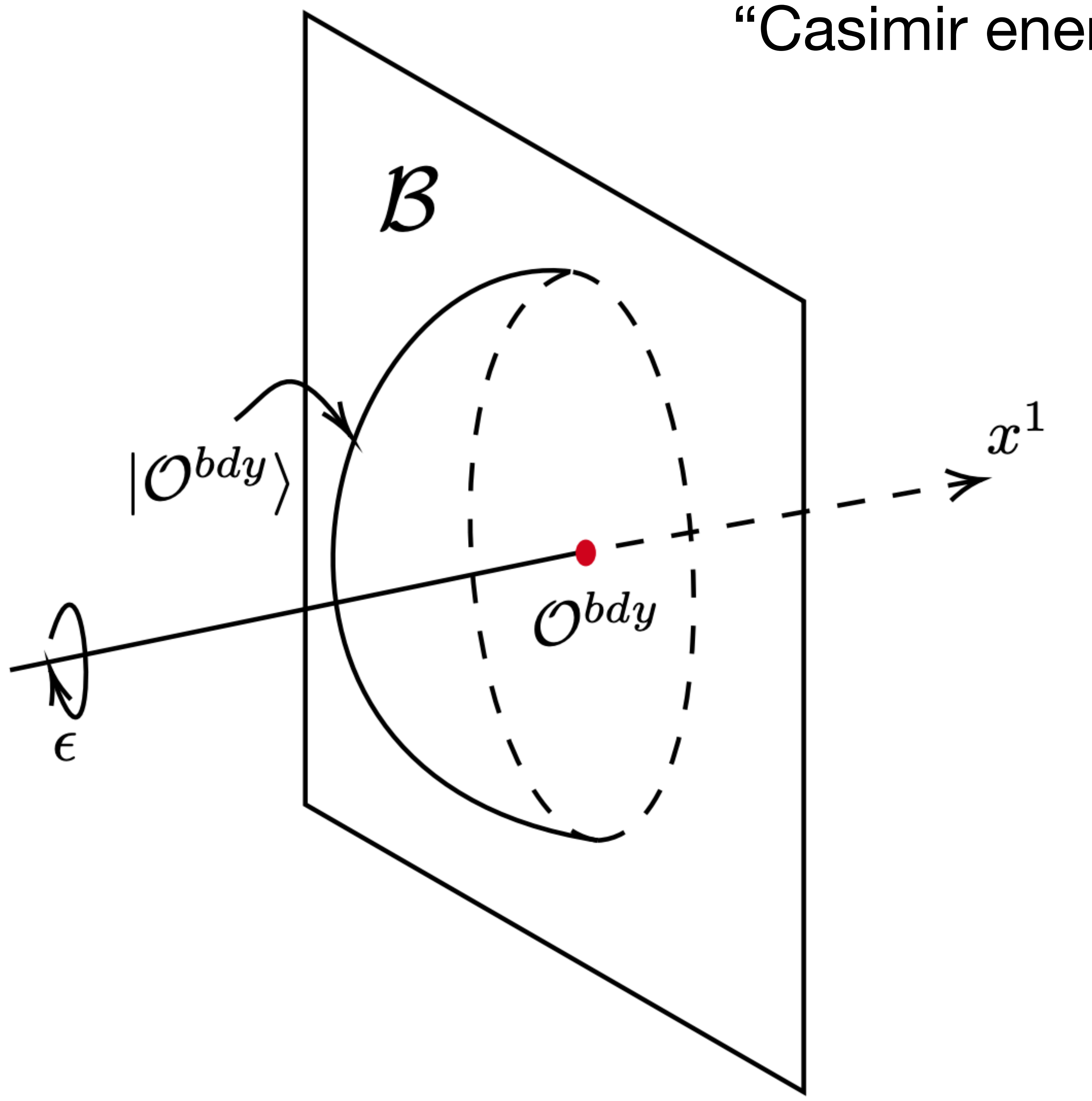
[Benini and Peelaers] [Fujitsuka, Honda and Yoshida]
Modification of [Yoshida and Sugiyama]—with particular B.C.



$$ds^2 = d\tau^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Twisted boundary conditions $\tau \sim \tau + \beta r$

State-operator map



"Casimir energy"

- Relate the hemisphere partition function to count of local operators.

$$\mathcal{Z}_{S^1\times H^2} = e^{\phi}\mathcal{I}$$

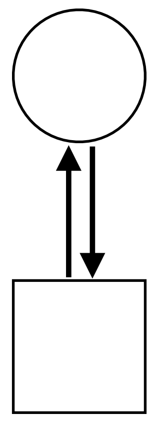
Half index

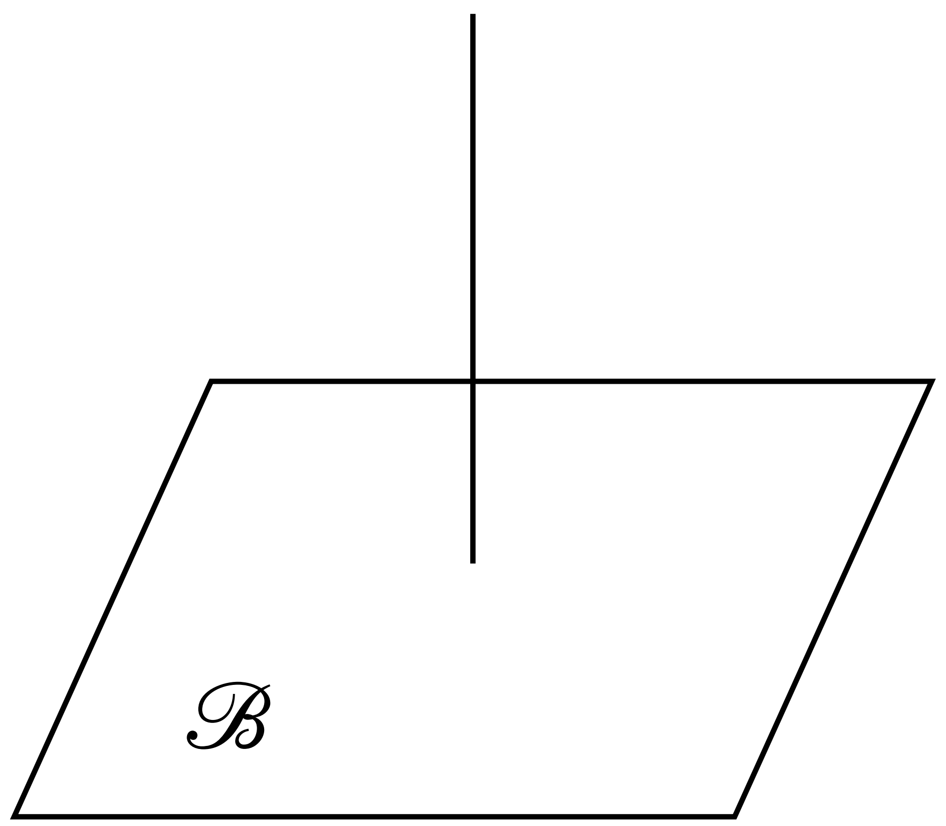
- $\mathrm{Tr}_{\mathscr{H}_{\mathscr{B}}}(-1)^F q^{J+\frac{R_V+R_A}{2}} t^{\frac{R_V-R_A}{2}} x^{F_H} \xi^{F_C}$

- We will compute half index

Exceptional Dirichlet

- Boundary conditions $\mathscr{B}_\alpha$ associated to each vacua α
- Dirichlet $\mathscr{D}$ for $\mathscr{N} = 4$ vector multiplet
- Lagrangian splitting of the hypers $L \oplus L^*$
- $Y_L|_\partial = c_L$
- Fully breaks G, preserves T_H and T_C
- L chosen to give $\mathscr{L}_\alpha \subset \mathscr{M}_H$





Recipe for computing half index [Dimofte, Gaiotto and Paquette]

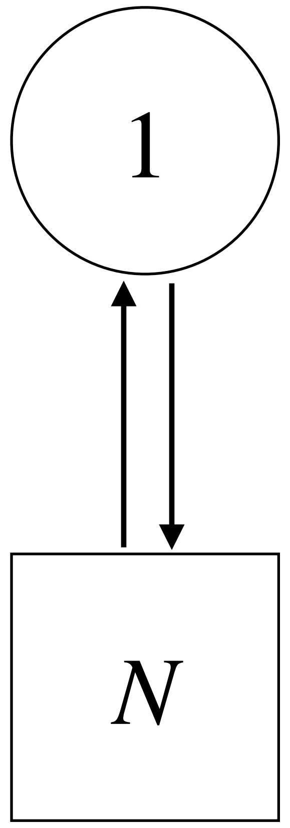
Vector multiplet

$$\frac{(q^{\frac{1}{2}}t^2;q)_\infty}{(q)_\infty}\sum_{m\in\mathbb{Z}} y^{k_{\text{eff}}m} \times [\text{matter index}](q^m u)$$

Boundary monopoles

Boundary hypers

Example: SQED$[N]$



Fix chamber $\mathfrak{C}_H = \{m_1 < \ldots < m_N\}$ and $\mathfrak{C}_C = \{\xi > 0\}$

i specifies particular vacuum

$$\begin{aligned} \partial_\perp Y_j &= 0, & X_j &= c\delta_{ij} & j &\le i \\ \partial_\perp X_j &= 0, & Y_j &= 0 & j &> i \end{aligned}$$

Boundary symmetry $U(1)_\partial$

First compute Dirichlet half-index:

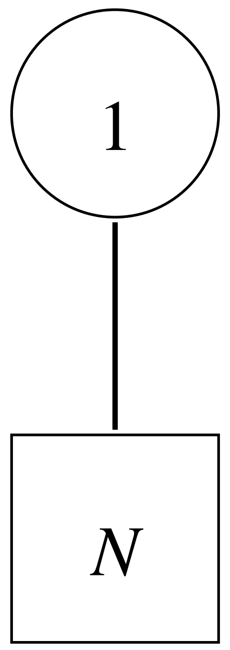
$$\sum_{m\in\mathbb{Z}} \left(\xi \left(q^{\frac{1}{4}} t^{-\frac{1}{2}}\right)^{2i-N}\right)^m \frac{(tq^{\frac{1}{2}};q)_\infty}{(q;q)_\infty} \prod_{j\le i} \frac{\left(q^{\frac{3}{4}+m} t^{-\frac{1}{2}} z^{-1} x_j^{-1};q\right)_\infty}{\left(q^{\frac{1}{4}+m} t^{\frac{1}{2}} z^{-1} x_j^{-1};q\right)_\infty} \prod_{j>i} \frac{\left(q^{\frac{3}{4}-m} t^{-\frac{1}{2}} z x_j;q\right)_\infty}{\left(q^{\frac{1}{4}-m} t^{\frac{1}{2}} z x_j;q\right)_\infty} .$$

Effective CS level

Expansions in q spin

Specialise fugacity $z = x_i^{-1} t^{-\frac{1}{2}} q^{-\frac{1}{4}}$ for non-zero chiral breaking combination of flavour, gauge and R-symmetry

Result: SQED$[N]$



From the hemisphere localisation

$$\mathcal{Z}_{\mathcal{B}_i} = \mathcal{Z}_i^{\text{Cl}} \mathcal{Z}_i^{\text{1-loop}} \mathcal{Z}_i^{\text{Vortex}}$$

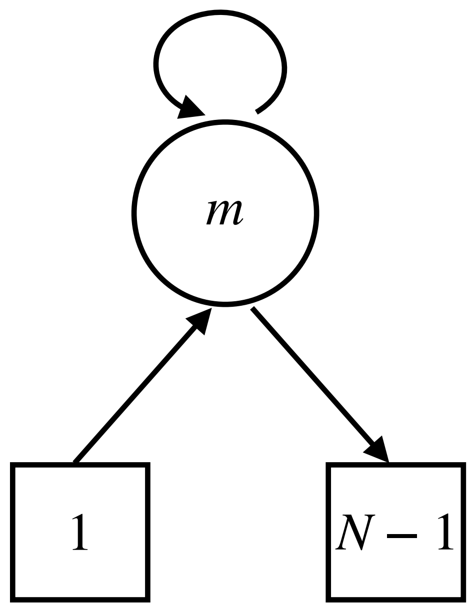
Different

$$\mathcal{Z}_i^{\text{Cl}} = e^{\phi_i} \,,$$

$$\mathcal{Z}_i^{\text{1-loop}} = \prod_{j=1}^{i-1} \frac{\left(q\frac{x_i}{x_j};q\right)_\infty}{\left(q^{\frac{1}{2}}t\frac{x_i}{x_j};q\right)_\infty} \prod_{j=i+1}^{N} \frac{\left(q^{\frac{1}{2}}t^{-1}\frac{x_j}{x_i};q\right)_\infty}{\left(\frac{x_j}{x_i};q\right)_\infty} \,,$$

$$\mathcal{Z}_i^{\text{Vortex}} = \sum_{m\geq 0} \left(\left(q^{\frac{1}{4}}t^{-\frac{1}{2}}\right)^N \xi\right)^m \prod_{j=1}^{N} \frac{\left(q^{\frac{1}{2}}t\frac{x_i}{x_j};q\right)_m}{\left(q\frac{x_i}{x_j};q\right)_m} \,.$$

- Vortex moduli space is handsaw quiver



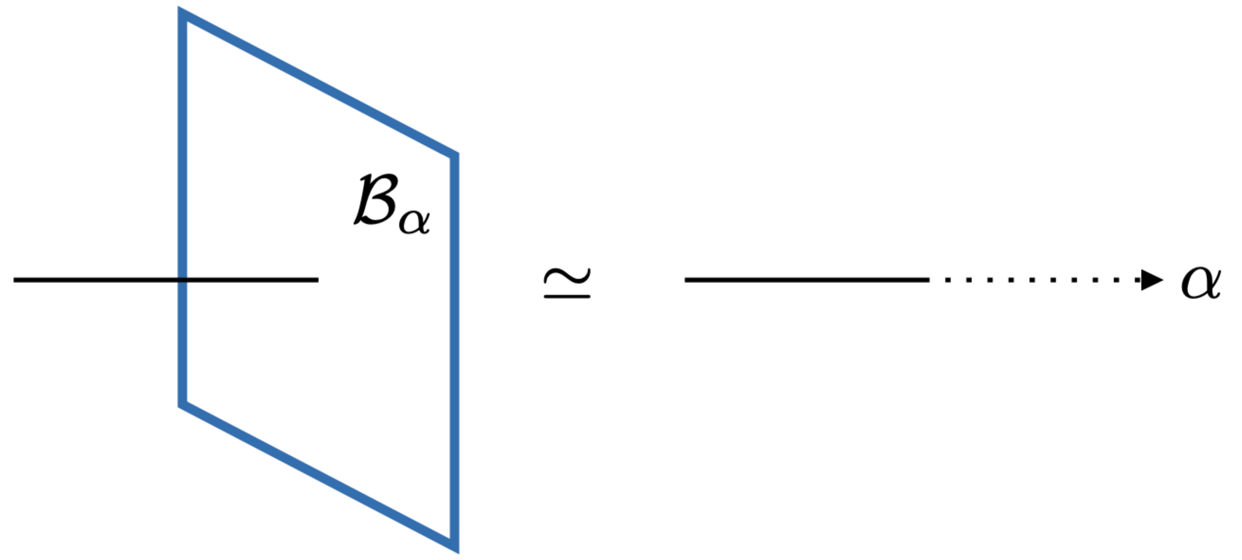
$\mathbb{C}^{mN}$

1 fixed point

$$\mathcal{Z}_i^{\text{Vortex}} = \sum_{d\geq 0} \zeta^d \chi_t(\mathfrak{Q}_\alpha^d)$$

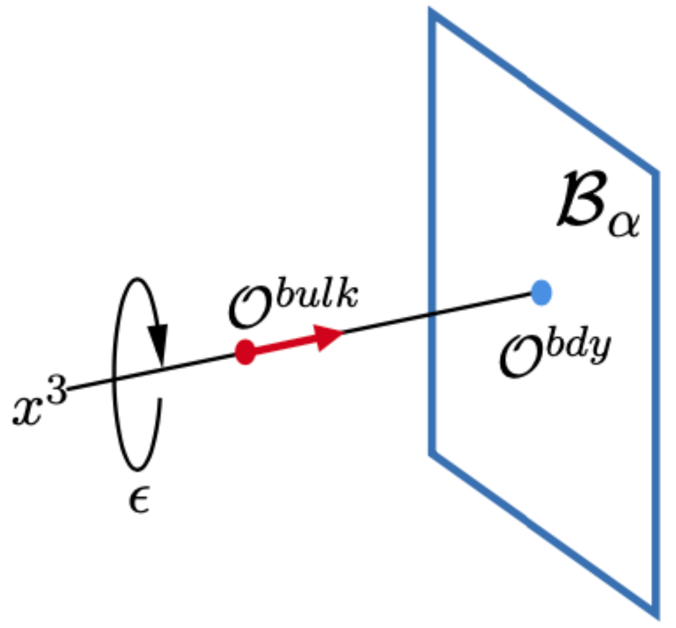
Why exceptional Dirichlet?

- Exceptional Dirichlet mimics vacuum at infinity—factorisation



- Flow to thimbles in IR RW sigma model—associated to vacua

- General principle [Bullimore, Dimofte, Gaiotto and Hilburn] that $\mathscr{B}$ yields modules for $\hat{\mathbb{C}}[\mathscr{M}_H]$ and $\hat{\mathbb{C}}[\mathscr{M}_C]$



- Exceptional Dirichlet $\mathscr{B}_\alpha$ yields Verma modules

"Casimir" e^ϕ	$\longleftrightarrow$	Highest weight of Verma

- $\mathcal{N} = 4$ limits of hemisphere partition functions are Verma module characters

Action on boundary operators $\longleftrightarrow$ Equivariant homology VMS

Specialised limits

$$\mathcal{I}_{\mathcal{B}} = \mathrm{Tr}_{\mathcal{H}_{\mathcal{B}}}(-1)^F q^{J+\frac{R_V+R_A}{4}} t^{\frac{R_V-R_A}{2}} x^{F_H} \xi^{F_C}$$

B-limit: $t \to q^{-\frac{1}{2}}$

A-limit: $t \to q^{\frac{1}{2}}$

$$\mathcal{I}_{\mathcal{B}}^{(B)} := \lim_{t^{\frac{1}{2}} \to q^{-\frac{1}{4}}} \mathcal{I}_{\mathcal{B}} = \mathrm{Tr}_{\mathcal{H}_{\mathcal{B}}^{(B)}} x^{F_H}$$

$$\mathcal{I}_{\mathcal{B}}^{(A)} := \lim_{t^{\frac{1}{2}} \to q^{\frac{1}{4}}} \mathcal{I} = \mathrm{Tr}_{\mathcal{H}_{\mathcal{B}}^{(A)}} \xi^{F_C}$$

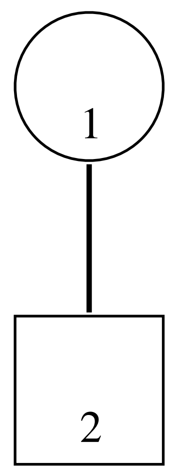
4 supercharges: Higgs operators (1-loop only)

4 supercharges: Coulomb operators (vortex only)

e^{ϕ_α} ⟶ "Equivariant vacuum energy"

Example: $T[SU(2)]$

Coulomb side



$T^*\mathbb{P}^1$
Vacua $\alpha = 1,2$

$$[\hat{\varphi}, \hat{v}_\pm] = \pm\, \epsilon \hat{v}_\pm$$

$$\hat{v}_+\hat{v}_- = (\hat{\varphi} + m_{1,\mathbb{C}} - \frac{\epsilon}{2})(\hat{\varphi} + m_{2,\mathbb{C}} - \frac{\epsilon}{2})$$

$$\hat{v}_-\hat{v}_+ = (\hat{\varphi} + m_{1,\mathbb{C}} + \frac{\epsilon}{2})(\hat{\varphi} + m_{2,\mathbb{C}} + \frac{\epsilon}{2})$$

$$(\hat{\varphi} + m_\alpha + \frac{\epsilon}{2})\,|\,\mathscr{B}_\alpha\rangle = 0\,, \quad \hat{v}^+\,|\,\mathscr{B}_\alpha\rangle = 0$$

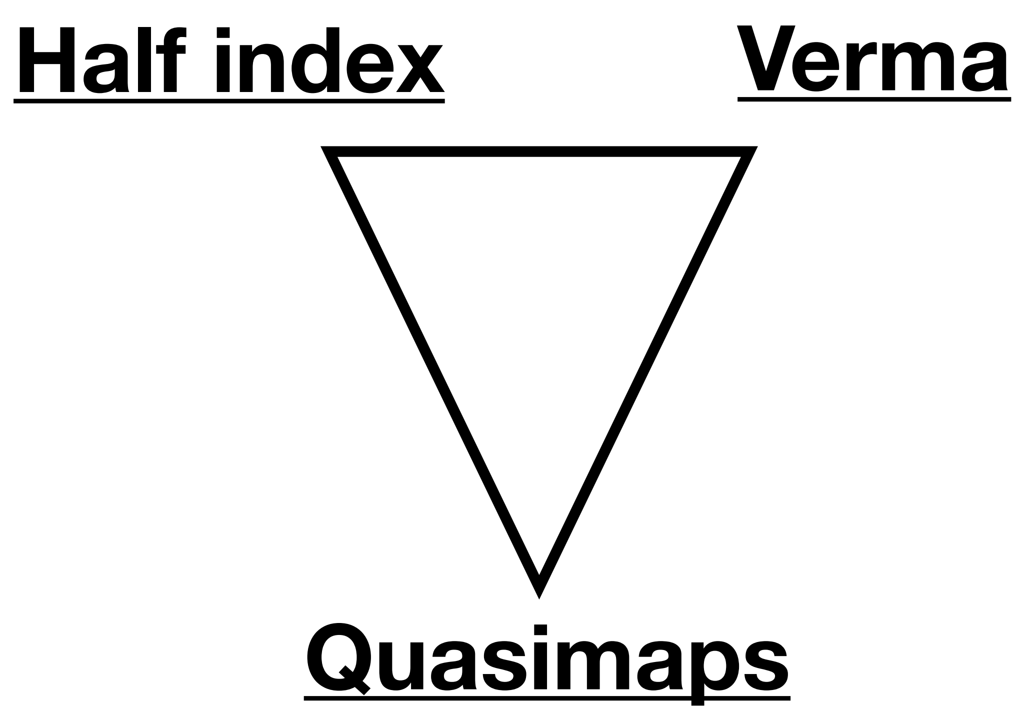
Lower with $\hat{v}_-$

$$\mathcal{Z}_{\text{Vortex},\alpha} = \sum_{m\geq 0} \left(q^{\frac{1}{2}} t^{-\frac{1}{4}} \xi\right)^m \prod_{j=1}^{2} \frac{(q^{\frac{1}{2}} t x_\alpha / x_j; q)_m}{(q x_\alpha / x_j; q)_m}$$

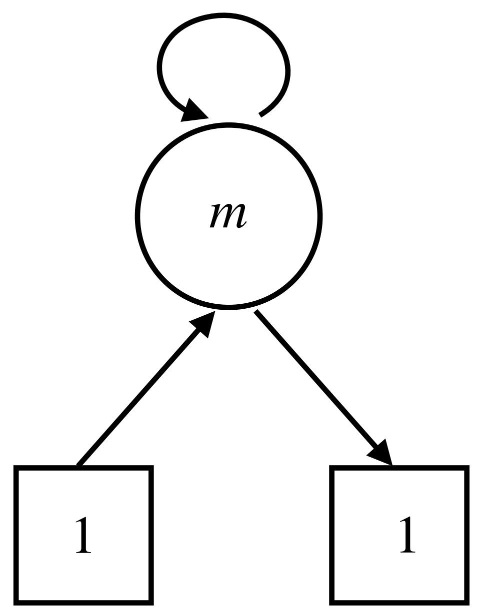
↓ A-limit

$$\chi_\alpha^C = e^{\frac{\log(\xi)\log(x_\alpha)}{\log(q)} + \frac{1}{2}} \frac{1}{1-\xi}$$

Highest weight



$$\mathcal{Z}_{\text{Vortex},\alpha} = \sum_{m\geq 0} \xi^m \chi_t(\mathfrak{Q}_\alpha^m) \xrightarrow{\text{A-limit}} \mathcal{Z}_{\text{Vortex},\alpha} = \sum_{m\geq 0} \xi^m\, (\#\ \text{f.p.})$$



Recap so far

- Compute hemisphere partition functions and half indices with exceptional Dirichlet boundary conditions

- Specialised limits give Verma modules of $\hat{\mathbb{C}}[\mathscr{M}_H]$ and $\hat{\mathbb{C}}[\mathscr{M}_C]$

$$\lim_{t \to q^{\pm\frac{1}{2}}} \mathcal{Z}_\alpha(x, \xi; q, t) = \chi_\alpha^{\mathrm{H,C}}(x \,\mathrm{or}\, \xi)$$

Factorisation

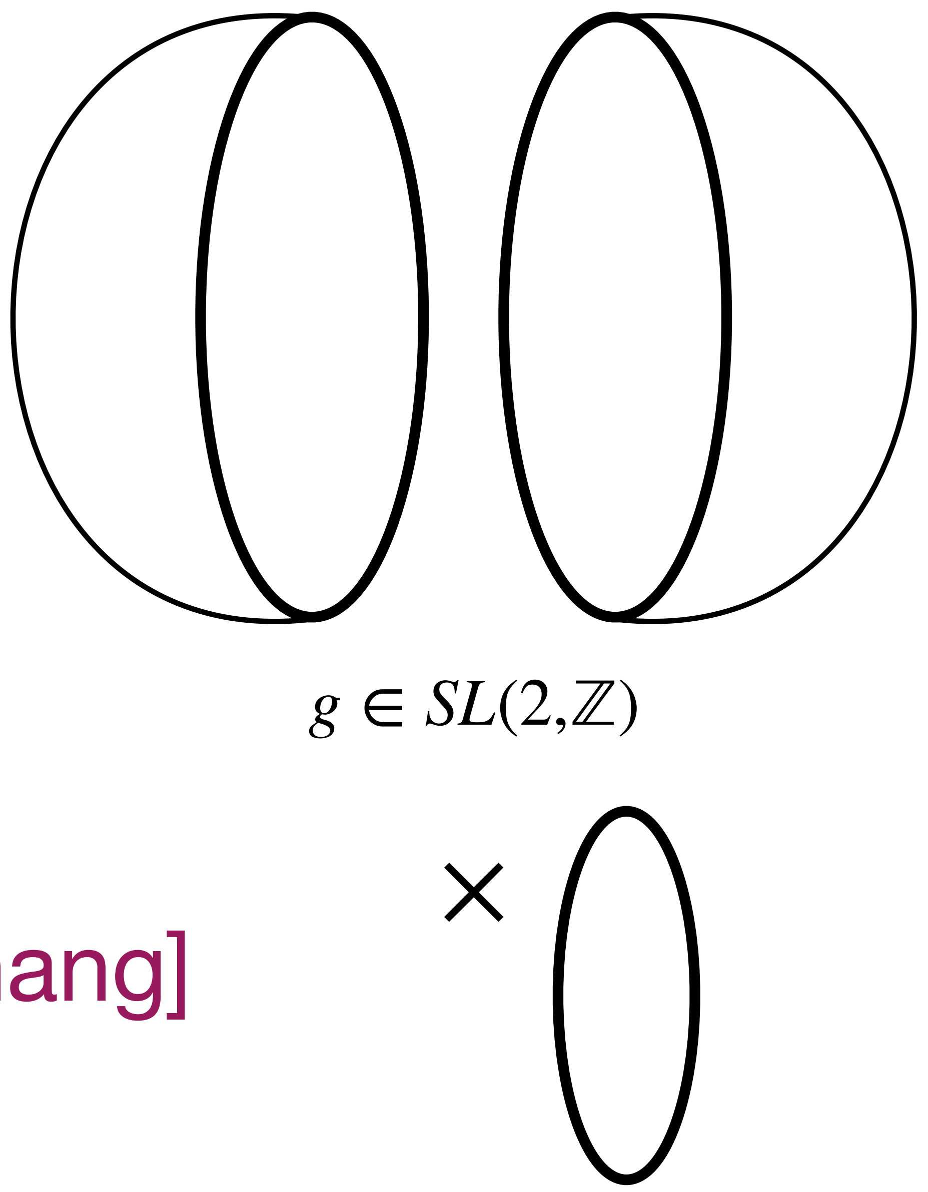
$$\mathcal{Z}_{\mathcal{M}_3} = \sum_\alpha H_\alpha \tilde{H}_\alpha$$

- H_α holomorphic block [Beem, Dimofte and Pasquetti] — 3d analogue of tt^* setup
- Gluing corresponds to Heegaard decomposition of $\mathscr{M}_3$
- Demonstrated in various cases by Coulomb branch localisation
- Examples include $\mathscr{M}_3 = S^2 \times_{A,B} S^1\,, \quad S^3_b\,, \quad S^2 \times S^1$
- Factorisation into hemisphere is *exact*

Factorise

$$\mathcal{Z}_{\mathcal{M}_3} = \sum_\alpha \mathcal{Z}_{\text{Pert.}} \mathcal{Z}_{\text{Vor.}} \mathcal{Z}_{\text{Anti vor.}}$$

- Demonstrated for SQED$[N]$ and ADHM [SC, Bullimore, Zhang] and [SC, Dorey, Zhang]



Application: IR formulae

$$\mathcal{Z}_{\mathcal{M}_3} = \sum_\alpha \mathcal{Z}^\alpha_{S^1\times H^2}\tilde{\mathcal{Z}}^\alpha_{S^1\times H^2}$$

$$\lim_{t\to q^{\pm\frac{1}{2}}} \mathcal{Z}_\alpha(x,\xi,q,t) = \chi^{H,C}_\alpha(x \,\mathrm{or}\, \xi)$$

In the specialised $\mathcal{N} = 4$ limits we find e.g.

$$\mathcal{Z}^B_{\mathrm{SC}} = \sum_\alpha \mathcal{X}^H_\alpha(x)\mathcal{X}^H_\alpha(x^{-1})\,, \qquad \mathcal{Z}^A_{\mathrm{SC}} = \sum_\alpha \mathcal{X}^C_\alpha(\xi)\mathcal{X}^C_\alpha(\xi^{-1})\,,$$

$$\mathcal{Z}^B_{\mathrm{tw}} = \sum_\alpha \mathcal{X}^H_\alpha(x)\mathcal{X}^H_\alpha(x)\,, \qquad \mathcal{Z}^A_{\mathrm{tw}} = \sum_\alpha \mathcal{X}^C_\alpha(\xi)\mathcal{X}^C_\alpha(\xi)$$

$$\mathcal{Z}_{S^3} = \sum_\alpha \hat{\mathcal{X}}^H_\alpha(x)\hat{\mathcal{X}}^C_\alpha(\xi)\,,$$

[Gaiotto and Okazaki]

A and B twisted indices

Choose $R_A = 2U(1)_H$ or $R_B = 2U(1)_C$ and place theory on $S^2 \times_{A,B} S^1$ with background R-symmetry flux

↓ E.g. Coulomb branch localisation [Benini and Zaffaroni]

$$\mathcal{Z}_{S^2 \times_{A,B} S^1} = \sum_{\alpha} H_\alpha(x,\xi;q,t) H_\alpha(x,\xi;q^{-1},t)$$

Geometry [SC, Dorey and Zhang]

$$\sum_{\boldsymbol{d}} \xi^{\boldsymbol{d}} \chi_t(\mathcal{Q}^{\boldsymbol{d}}) = \mathcal{Z}_{S^2 \times_A S^1} = \sum_{\alpha} H_\alpha \tilde{H}_\alpha = \sum_{\alpha} || \sum_{\boldsymbol{d}} \xi^{\boldsymbol{d}} \chi_t(\mathfrak{Q}^{\boldsymbol{d}}_\alpha) ||^2$$

Global Laumon space

Angular momentum q is "Q-exact" ⟷ χ_t genus of compact space

Poincaré polynomial limit

$$\mathcal{Z}_{S^1\times H^2} = \mathcal{Z}_{\text{Classical}}\mathcal{Z}_{\text{1-loop}}\mathcal{Z}_{\text{Vortex}}$$

$$\mathcal{Z}_{\text{Vortex}}(x,\xi;q,t)$$

$q \to 0$

$qt \to 0$

$$\mathcal{Z}_{\text{1-loop}}(t,x) = \text{Generating function}$$

$$\sum_{k\geq 0}\xi^k \sum_{\text{f.p.}} t^{\text{Dim. attracting set}}$$

[Bialynicki-Birula]

$$\lim_{q\to 0}\chi_t(\mathfrak{Q}) = P_t(\pi^{-1}(0))$$

Poincaré polynomial

- Only vortices contribute in this limit — R-charge graded Verma characters
- Mirror limit only 1-loop contributions — generating function

3d mirror symmetry $X \leftrightarrow Y$

- IR duality of 3d gauge theories [Intriligator and Seiberg]

$$\mathscr{M}_H \leftrightarrow \mathscr{M}_C \qquad m \leftrightarrow \xi \qquad t \leftrightarrow t^{-1}$$

Neumann

Exceptional Dirichlet ?

Dirichlet

- Boundary conditions transform non-trivially

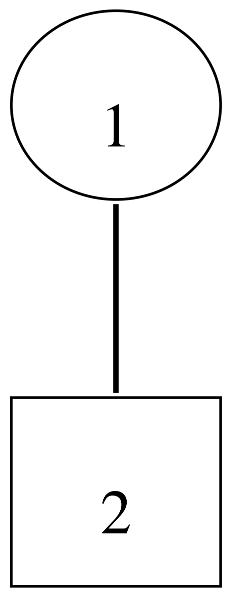
$$\mathcal{Z}_\alpha \to U_{\alpha\beta} \mathcal{Z}_\beta$$

Elliptic stable envelopes

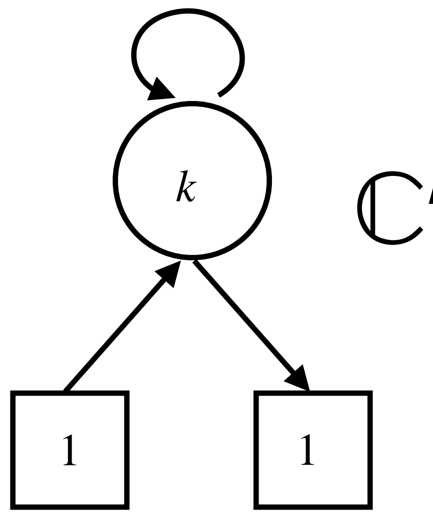
- Two simplifying limits $\lim_{qt^{\pm\frac{1}{2}} \to 0} \mathcal{I}_\alpha^X = \lim_{qt^{\mp\frac{1}{2}} \to 0} \mathcal{I}_\alpha^Y \qquad \lim_{q \to t^{\pm\frac{1}{2}}} \mathcal{I}_\alpha^X = \lim_{q \to t^{\mp\frac{1}{2}}} \mathcal{I}_\alpha^Y$

Proof?

- Exchanges perturbative and non-perturbative contributions



$T^*\mathbb{P}^1$



$\mathbb{C}^k$

$T[SU(2)]$ Example

Self-mirror dual

$$\mathcal{I}^Y = \mathcal{I}_{\text{1-loop}}\mathcal{I}_{\text{Vortex}} = \frac{(qx;q)_\infty}{(tqx;q)_\infty}\sum_{k\geq 0}\xi^k\frac{(tqx;q)_k}{(qx;q)_k}$$

$$\mathcal{I}^X = \mathcal{I}_{\text{1-loop}}\mathcal{I}_{\text{Vortex}} = \frac{(qx;q)_\infty}{(x;q)_\infty}\sum_{k\geq 0}\left(t^{-1}q\xi\right)^k\frac{(tqx;q)_k}{(qx;q)_k}$$

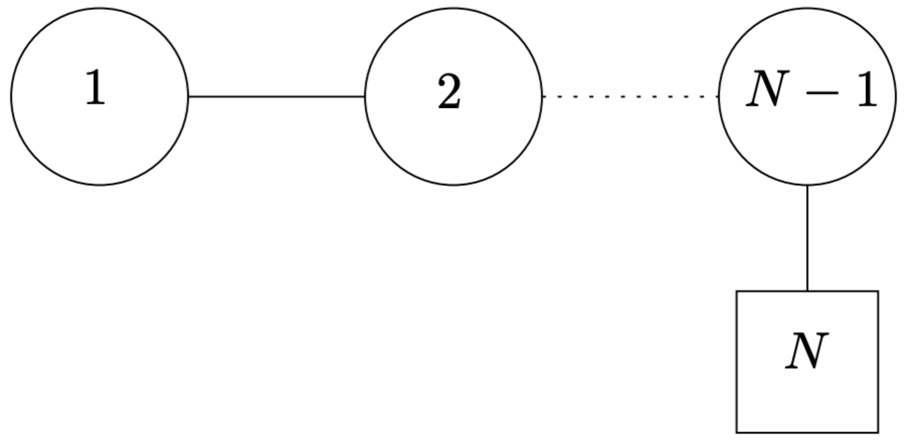
$$\lim_{\text{Poincaré}}\mathcal{I}^Y = \mathcal{I}_{\text{Vortex}} = \sum_{k\geq 0}\xi^k \quad = \quad \lim_{\text{Poincaré}}\mathcal{I}^X = \mathcal{I}_{\text{1-loop}} = \frac{1}{1-x}$$

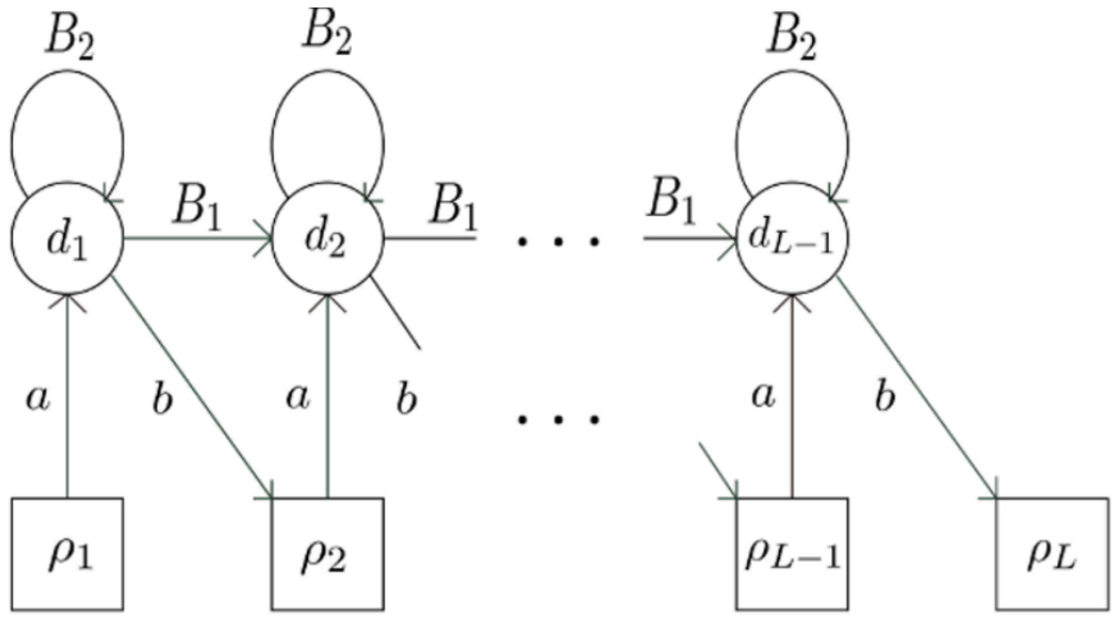
Generating function of handsaw quiver Poincaré polynomials [Nakajima]

Twisted index and Hilbert series

Twisted index on $\Sigma = S^2$ of 3d $\mathcal{N}=4$ theories computes Hilbert series

<u>**Example**</u> $T[SU(N)]$





Twisted index/Hilbert series factorises:

$$\mathcal{Z}_{S^2\times_A S^1} = \sum_{\alpha\in S_N} \mathcal{Z}_{S^1\times H^2}(q,t)\mathcal{Z}_{S^1\times H^2}(q^{-1},t)$$

1-loop contributions only I.e. Poincaré polynomials

$$\sum_{\vec{d}} x^{\vec{d}} P_t(\mathfrak{Q}_\alpha^{\vec{d}})$$

Free to send $q\to 0$

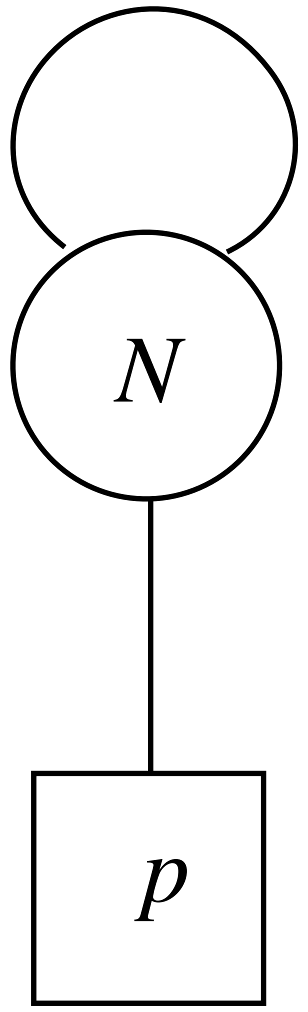
$$\mathcal{Z}_{S^1\times_A S^1} = \sum_{\alpha\in S_N}\prod_{i<j}\frac{1}{1-tx_i/x_j}\prod_{i>j}\frac{1}{1-tx_i/x_j}$$

$$= \prod_{i,j=1}^{N}\frac{1}{1-tx_i/x_j} = \text{H.S.}\,(T[SU(N)])$$

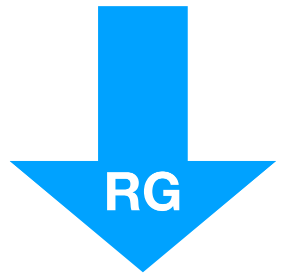
Recap so far

- Twisted index computes Hilbert series
- Factorising twisted index gives formula for Hilbert series in terms of Poincaré polynomial of quasi-map space
- 3d mirror symmetry provides generating function of Poincaré polynomials
- Concrete for $T_\rho[SU(N)]$ theories — Handsaw quivers
- Will now discuss more complicated, “speculative” example

3d ADHM Example



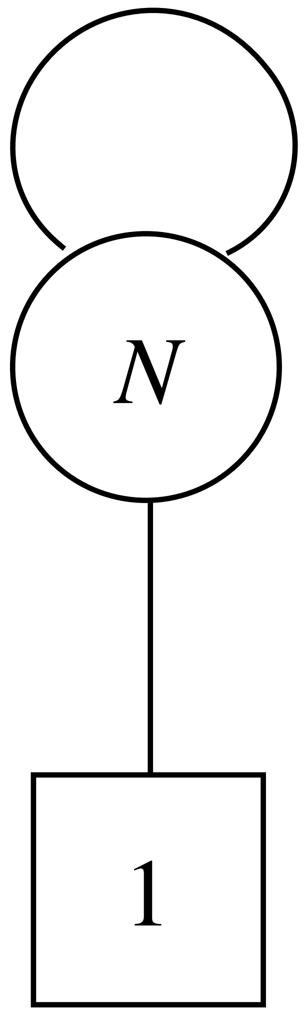
Worldvolume theory $N\;D2$ branes on single $D6$ in type IIA $\quad p = 1$



$N\;M2$ branes $\longleftrightarrow$ SUGRA AdS$_4 \times S^7$

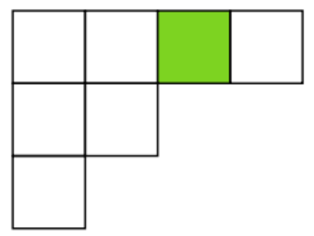
BH entropy in Cardy limit [Choi and Hwang] from vortex partition functions

$$\lim_{q\to 1} \oint e^{\frac{1}{\log(q)}\mathscr{W}} = H^* \sim e^{\frac{1}{\log(q)} N^{\frac{3}{2}}\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}}$$



Torus action on $\mathrm{Hilb}^N(\mathbb{C}^2)$

- Mirror self dual
- $\mathscr{M}_H = \mathrm{Hilb}^N(\mathbb{C}^2)$ with $G_H = U(1)$ and $R_H - R_C = U(1)$
- Vacua labelled by partitions $|\lambda| = N$



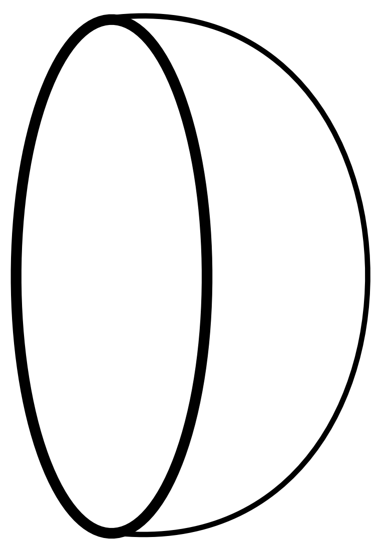
- $\hat{\mathbb{C}}[\mathscr{M}_C]$ described by [Nakajima and Kodera]
    - Cyclotomic rational Cherednik algebra
- $\mathbb{C}[\mathscr{M}_H]$ gauge invariant polynomials in (A, B, I, J)

$$E_{k,t} = \sum_{a=1}^{N_k} \frac{\prod_b w_{k,a} - w_{k-1,b} - \epsilon_2}{\prod_{b\neq a}(w_{k,a} - w_{k,b})} w_{k,a}^t v_{k,a}$$

$$F_{k,t} = \sum_{a=1}^{N} \frac{\prod_b w_{k,a} - w_{k+1,b} + \epsilon_2}{\prod_{b\neq a}(w_{k,a} - w_{k,b})} v_{k,a}^{-1} w_{k,a}^{t+\delta_{0,k}}$$

$$\mathrm{QM}^{\boldsymbol{d}}_{\lambda}(\mathbb{P}^1 \to \mathrm{Hilb}^N(\mathbb{C}^2)) \qquad \mathcal{Z}_{\mathrm{Vortex}} = \sum_{\boldsymbol{d}} \zeta^d \chi(\hat{\mathcal{O}}_{\mathrm{Vir.}})$$

ADHM hemisphere partition function

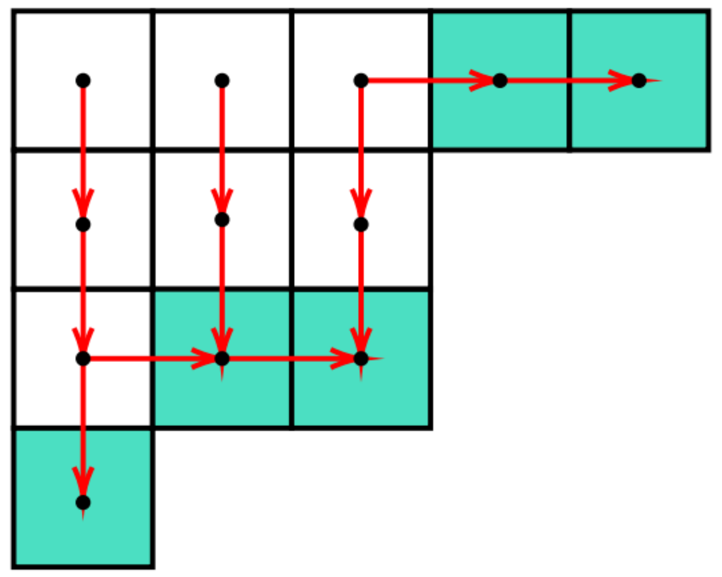


$$\mathcal{Z}^{\lambda}_{S^1\times D} = \mathcal{Z}^{\lambda}_{\text{Classical}}\mathcal{Z}^{\lambda}_{\text{1-loop}}\mathcal{Z}^{\lambda}_{\text{Vortex}}$$

New

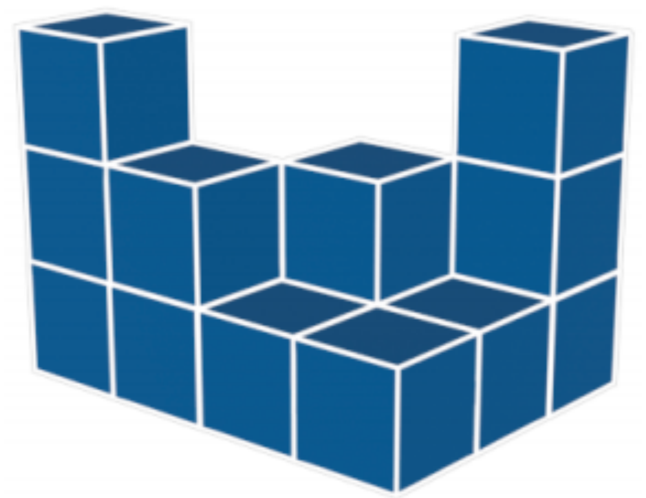
$$\mathcal{Z}^{\lambda}_{\text{Classical}} = e^{-\left[\sum_{s\in\lambda} c(s)\right]\frac{\log\zeta\log z}{\log q}} e^{\left[\sum_{s\in\lambda} h(s)\right]\frac{\log v\log z}{\log q}} e^{\left[\sum_{s\in\lambda} h(s)\right]\frac{\log u\log \zeta}{\log q}} e^{-\left[\sum_{s\in\lambda} c(s)\right]\frac{\log u\log v}{\log q}}$$

$$\mathcal{Z}^{\lambda}_{\text{1-loop}} = \prod_{s\in\lambda}\frac{(qz^{a_\lambda(s)+l_\lambda(s)+1}u^{-a_\lambda(s)+l_\lambda(s)-1};q)_\infty}{(z^{a_\lambda(s)+l_\lambda(s)+1}u^{-a_\lambda(s)+l_\lambda(s)+1};q)_\infty}$$



Familiar-ish

$$\mathcal{Z}^{\lambda}_{\text{Vortex}} = \sum_{\pi\in\text{RPP}(\lambda)}\left(\zeta t^{\frac{1}{2}}q^{-\frac{1}{4}}\right)^{|\pi|}\prod_{s\in\lambda}\frac{(u^2v_s^{-1};q)_{-\pi_s}}{(qv_s^{-1};q)_{-\pi_s}}\prod_{\substack{s,t\in\lambda\\ s\neq t}}\frac{\left(qu^{-2}\frac{v_t}{v_s};q\right)_{\pi_t-\pi_s}}{\left(\frac{v_t}{v_s};q\right)_{\pi_t-\pi_s}}\frac{\left(zu\frac{v_t}{v_s};q\right)_{\pi_t-\pi_s}}{\left(qzu^{-1}\frac{v_t}{v_s};q\right)_{\pi_t-\pi_s}}$$



$u = t^{\frac{1}{2}}q^{\frac{1}{4}},\ v = t^{-\frac{1}{2}}q^{\frac{1}{4}}$

Fuses exactly!

Verma character limits

$$\mathcal{Z}^{\lambda}_{\text{Vortex}} = \sum_{\pi \in \text{RPP}(\lambda)} \left(\zeta t^{\frac{1}{2}} q^{-\frac{1}{4}}\right)^{|\pi|} \prod_{s\in\lambda} \frac{(u^2 v_s^{-1};q)_{-\pi_s}}{(q v_s^{-1};q)_{-\pi_s}} \prod_{\substack{s,t\in\lambda\\ s\neq t}} \frac{\left(q u^{-2}\frac{v_t}{v_s};q\right)_{\pi_t-\pi_s}}{\left(\frac{v_t}{v_s};q\right)_{\pi_t-\pi_s}} \frac{\left(z u \frac{v_t}{v_s};q\right)_{\pi_t-\pi_s}}{\left(q z u^{-1}\frac{v_t}{v_s};q\right)_{\pi_t-\pi_s}}$$

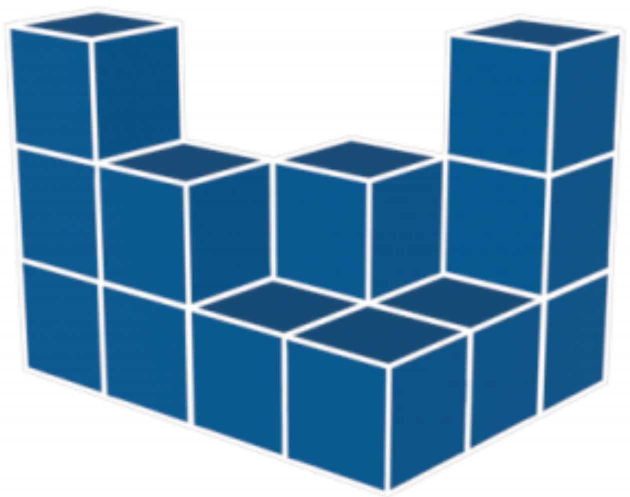
$$\mathcal{Z}^{\lambda}_{\text{1-loop}} = \prod_{s\in\lambda} \frac{(q z^{a_\lambda(s)+l_\lambda(s)+1} u^{-a_\lambda(s)+l_\lambda(s)-1};q)_\infty}{(z^{a_\lambda(s)+l_\lambda(s)+1} u^{-a_\lambda(s)+l_\lambda(s)+1};q)_\infty}$$

Verma limit

Mirror Verma limit

$$\sum_{\pi\in\text{RPP}(\lambda)} \zeta^{|\pi|} = \prod_{s\in\lambda} \frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1}}$$

Fixed points on QM space



Generating function

Representation theory

- Verma modules of $\hat{\mathbb{C}}[\mathscr{M}_C]$?
- π boundary operators $\longleftrightarrow$ Fixed points on $\mathrm{QM}^{\boldsymbol{d}}_{\lambda}(\mathrm{Hilb}^N(\mathbb{C}^2))$
- Action of $\hat{\mathbb{C}}[\mathscr{M}_C]$ on RPPs?

$\mathcal{Z}_{\text{Classical}}$ $\qquad$ $\mathcal{Z}_{\text{Vortex}}$

$$\chi_\lambda(\hat{\mathbb{C}}[\mathcal{M}_C]) = e^{\frac{\xi m}{2\beta}\sum_{s\in\lambda} c_\lambda(s) + \frac{\xi}{2}\sum_{s\in\lambda} h_\lambda(s)} \sum_{\pi\in\mathrm{RPP}(\lambda)} \zeta^{|\pi|}$$

Poincaré polynomial limit

$$\mathcal{Z}^{\lambda}_{\text{Vortex}} = \sum_{\pi \in \text{RPP}(\lambda)} \left(\zeta t^{\frac{1}{2}} q^{-\frac{1}{4}}\right)^{|\pi|} \prod_{s\in\lambda} \frac{(u^2 v_s^{-1};q)_{-\pi_s}}{(q v_s^{-1};q)_{-\pi_s}} \prod_{\substack{s,t\in\lambda \\ s\neq t}} \frac{\left(qu^{-2}\frac{v_t}{v_s};q\right)_{\pi_t-\pi_s}}{\left(\frac{v_t}{v_s};q\right)_{\pi_t-\pi_s}} \frac{\left(zu\frac{v_t}{v_s};q\right)_{\pi_t-\pi_s}}{\left(qzu^{-1}\frac{v_t}{v_s};q\right)_{\pi_t-\pi_s}}$$

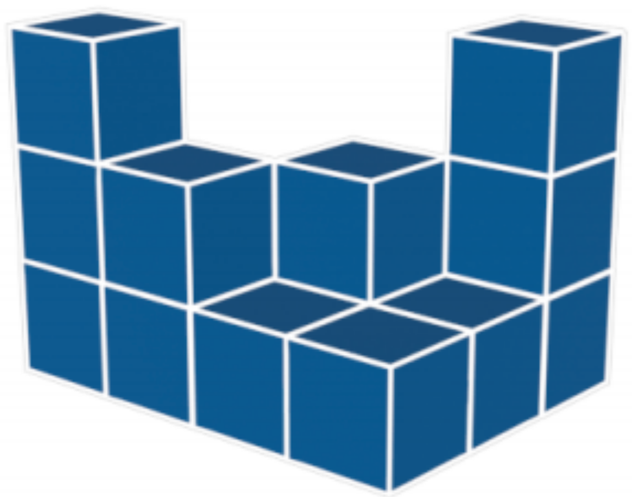
$$\mathcal{Z}^{\lambda}_{\text{1-loop}} = \prod_{s\in\lambda} \frac{(qz^{a_\lambda(s)+l_\lambda(s)+1}u^{-a_\lambda(s)+l_\lambda(s)-1};q)_\infty}{(z^{a_\lambda(s)+l_\lambda(s)+1}u^{-a_\lambda(s)+l_\lambda(s)+1};q)_\infty}$$

“Poincaré limit $q \to 0$”

Mirror “Poincaré limit”

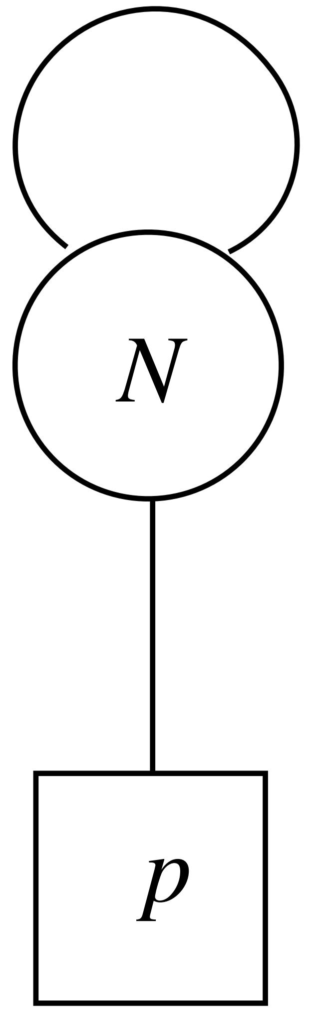
$$\sum_{\pi\in\text{RPP}(\lambda)} t^{\frac{1}{2}(\text{ht}'(\pi)-\text{ht}(\pi)+b(\pi))} z^{|\pi|} = \prod_{s\in\lambda} \frac{1}{1-z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(-a_\lambda(s)+l_\lambda(s)+1)}}$$

Refined generating function of RPPs



Neumann boundary condition

- Throughout we have been using particular Dirichlet boundary conditions
- Neumann boundary condition is expected to yield simple modules [Bullimore, Dimofte, Gaiotto and Hilburn]
- Consider $p = N_f > 1$. I.e. general Jordan quiver
- $\mathscr{M}_H = \mathscr{M}_{N,p}$ instanton moduli space $G_H = SU(p)$



$T[SU(N)]$ analogy

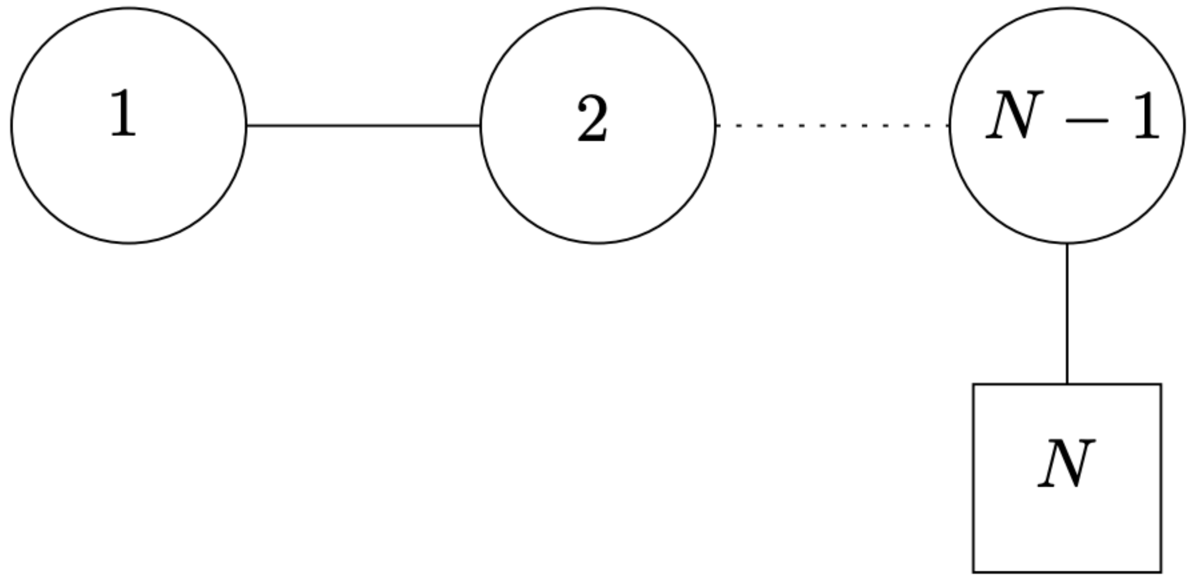
- Neumann boundary condition is computed via contour integral

$$\mathcal{Z}_{S^1\times D}[z_i,\zeta_i;q,t] =$$
$$\oint_\Gamma \prod_{a=1}^{N}\prod_{i=1}^{a}\frac{dx_a^{(i)}}{x_i^{(a)}}e^{\log\left(x_i^{(a)}\right)\log\left(\zeta_i^{(a)}\right)}\prod_{a=1}^{N-1}\frac{\prod_{i\neq j}^{a}\left(x_j^{(a)}/x_i^{(a)};q\right)_\infty}{\prod_{i,j}^{N}\left(tqx_j^{(a)}/x_i^{(a)};q\right)_\infty}\prod_{a=1}^{N-1}\prod_{i=1}^{a}\prod_{j=1}^{a+1}\frac{\left(tqx_j^{(a+1)}/x_i^{(a)};q\right)_\infty}{\left(x_j^{(a+1)}/x_i^{(a)};q\right)_\infty}$$

Integral form of Macdonald polynomial

$\downarrow$ Verma limit

s_λ



Borel-Weil-Bott — compact Lagrangian core is complete flag

Equivalent for ADHM

$$\mathcal{Z}^B_{S^1\times D} = \frac{1}{N!}\oint_{\mathrm{JK}} \prod_{a=1}^{N} \frac{ds_a}{2\pi i s_a} s_a^{-\zeta} \frac{\prod_{a\neq b}^{N}(s_a s_b^{-1};q)_\infty}{\prod_{a,b=1}^{N}(s_a s_b^{-1} t^{-1} q;q)_\infty} \prod_{a,b=1}^{N} \frac{(s_a s_b^{-1} z t^{-\frac{1}{2}} q;q)_\infty}{(s_a s_b^{-1} z t^{\frac{1}{2}};q)_\infty}$$

$$\prod_{a=1}^{N}\prod_{i=1}^{p} \frac{(s_a x_i t^{-\frac{1}{2}} q;q)_\infty}{(s_a x_i t^{\frac{1}{2}};q)_\infty}\,.$$

Choice of $\mathscr{B}$

Verma limit

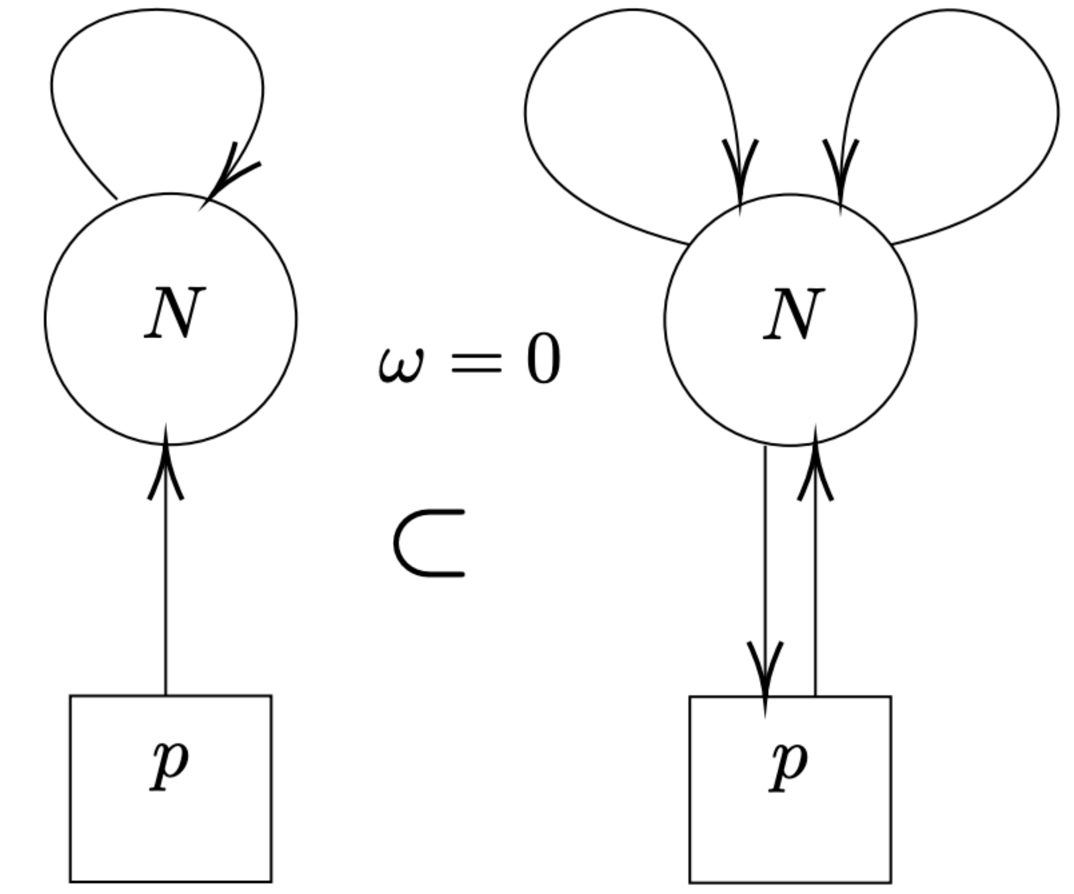
$$\mathcal{Z}_{N,p}(\mathcal{B}) = \frac{1}{N!}\oint_{(S^1)^N} \prod_{a=1}^{N} \frac{ds_a}{2\pi i s_a} s_a^{-\zeta} \frac{\prod_{a\neq b}^{N}\left(1 - s_a s_b^{-1}\right)}{\prod_{a,b=1}^{N}\left(1 - z s_a s_b^{-1}\right)} \prod_{a=1}^{N}\prod_{i=1}^{p} \frac{1}{1 - x_i s_a}$$

$$\mathcal{Z}_{N,p}(\mathcal{B}) = Q'_{(\zeta)^N}(x_1,\ldots,x_p;z)$$

Milne polynomial $q \to 0$ of Haiman-Garcia Macdonald polynomial

Character of KR module / XXZ spin chain partition function

Geometry



Hanany-Tong Lagrangian $\chi(\mathscr{O}(\zeta))$

Simple module of Nakajima and Kodera $\hat{\mathbb{C}}[\mathscr{M}_C]$?

Refined Topological Vertex

- Quasi-maps to $\mathrm{Hilb}^N(\mathbb{C}^2)$ should be related to vertex
- $q \to 0$ limit coincides with the refined topological vertex

1-loop piece

$$\lim_{q\to 0} \mathcal{Z}^{A,\lambda}_{S^1\times D} = t_1^{\frac{1}{4}||\lambda||^2} t_2^{-\frac{1}{4}||\lambda^\vee||^2} \prod_{s\in\lambda} \frac{1}{1 - t_1^{l_\lambda(s)+1} t_2^{-a_\lambda(s)}}$$

Classical piece = framing factors

$$C^{(\mathrm{IKV})}_{\emptyset,\emptyset,\lambda}(t = t_2^{-1}, q = t_1)$$

Twisted index gluing

$$\sum_{\substack{\lambda\\|\lambda|=N}} \lim_{q\to 0} \mathcal{Z}^{B,\lambda}_{\text{Classical}} \bar{\mathcal{Z}}^{B,\lambda}_{\text{Classical}} \mathcal{Z}^{B,\lambda}_{\text{1-loop}} \bar{\mathcal{Z}}^{B,\lambda}_{\text{1-loop}}$$

- Twisted index = Hilbert series of $\text{Hilb}^N(\mathbb{C}^2)$
- Identify the twisted index gluing with the gluing of vertices.

Poincaré polynomials

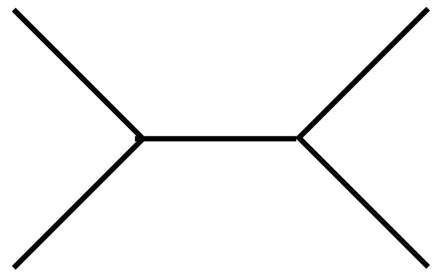
$$\lim_{q\to 0} \mathcal{Z}^{B,\lambda}_{\text{1-loop}}(z,\zeta;q,t) = \prod_{s\in\lambda} \frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(-a_\lambda(s)+l_\lambda(s)+1)}},$$

$$\lim_{q\to 0} \mathcal{Z}^{B,\lambda}_{\text{1-loop}}(z,\zeta;q^{-1},t) = \prod_{s\in\lambda} \frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(-a_\lambda(s)+l_\lambda(s)-1)}},$$

$$\lim_{q\to 0} \mathcal{Z}^{B,\lambda}_{\text{Classical}}(z,\zeta;q,t) = z^{\frac{1}{2}\sum_{s\in\lambda} h_\lambda(s)} t^{-\frac{1}{4}\sum_{s\in\lambda} c_\lambda(s)},$$

$$\lim_{q\to 0} \mathcal{Z}^{B,\lambda}_{\text{Classical}}(z,\zeta;q^{-1},t) = z^{\frac{1}{2}\sum_{s\in\lambda} h_\lambda(s)} t^{-\frac{1}{4}\sum_{s\in\lambda} c_\lambda(s)}.$$

$$\mathcal{Z}^{A,\lambda}_{\text{Vortex}} = V^{\emptyset,\emptyset,\lambda}_{\text{PT}}$$



Interpret as conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ amplitude

Independence of q...

$$= \sum_{\substack{\lambda\\|\lambda|=N}} \prod_{s\in\lambda} \frac{z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(-a_\lambda(s)+l_\lambda(s))}}{\left(1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(-a_\lambda(s)+l_\lambda(s)+1)}\right)\left(1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(-a_\lambda(s)+l_\lambda(s)-1)}\right)}$$

Nekrasov's partition function

Summary

- Hemisphere partition functions realise factorisation
- Verma character formulae for $\mathscr{M}_3$ partition functions
  - Geometric interpretation
- Twisted index, Hilbert series and Poincaré polynomials
- Detailed study of ADHM example
  - Connections to topological vertex

Further directions

- Quasi-map interpretation of 1-loop contributions
- Mirror symmetry of boundary conditions
- Geometric interpretation of Cardy limit $\lim_{q \to 1} \mathcal{Z}^{\lambda}_{S^1 \times D}$
  - Phase with non-trivial scaling of $N = |\lambda|$ i.e. $\sim e^{N^{3/2}}$
  - Relevance of Hanany-Tong lagrangian and simple modules?

Thanks