Pre-Calabi-Yau algebras

Wai-kit Yeung

Kavli IPMU

wai-kit.yeung@ipmu.jp

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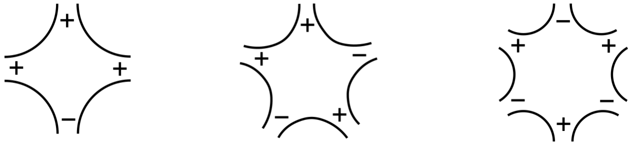
Disk

A disk means:

- The 2-dimensional disk with punctures on boundary
- Each punctured is assigned either:
  "$+$" ("input") or "$-$" ("output").



We only remember the diffeomorphism type of a disk.

D-shaped maps

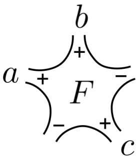
Fix $A \in \mathrm{grVect}_k$.

Given a disk D, a D-**shaped map on** A is a k-linear map

$$A^{\otimes \Sigma^+} \to A^{\otimes \Sigma^-}$$

where $\Sigma^+ = \{"+"\text{ punctures}\}$ and $\Sigma^- = \{"-"\text{ punctures}\}$.

Example.



$$F(a, b, c) \in A \otimes A$$

The ordering is important.

Pre-Calabi-Yau algebras

We will consider a **collection of maps** π which to each disk D assigns a D-shaped map on A

$$\pi(D) : A^{\otimes \Sigma^+} \to A^{\otimes \Sigma^-}$$

Definition

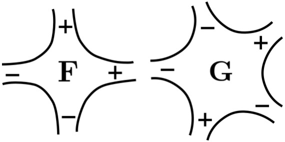
The graded vector space A, together with this collection of maps π is said to be a pre-Calabi-Yau algebra if it satisfies

$$\pi \circ \pi = 0$$

Gluing of disks

We can glue disks along punctures with opposite polarity.

We can also compose the maps as we glue:



This gives a D-shaped map $A^{\otimes 3} \to A^{\otimes 4}$.

The condition $\pi \circ \pi = 0$

Suppose we are given a collection π of maps.

For any given disk D,

$$\begin{aligned} &\text{Each way of writing } D \text{ as a gluing } D = D_1 \# D_2 \\ \Longrightarrow\, &\text{a D-shaped map } \pi(D_1) \circ \pi(D_2) \end{aligned}$$

The condition $\pi \circ \pi = 0$ then says:

For each disk D, we require

$$\sum_{D = D_1 \# D_2} \pm \pi(D_1) \circ \pi(D_2) = 0$$

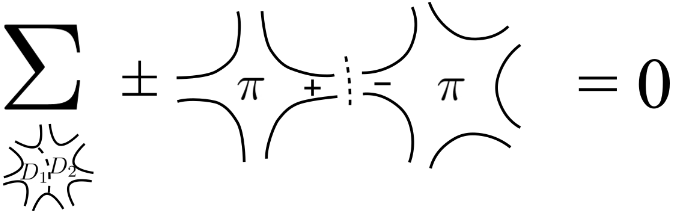
Notice:

1) Each map is a k-linear map $A^{\otimes \Sigma^+} \to A^{\otimes \Sigma^-}$, so we can sum them.

2) There are Koszul signs involved.

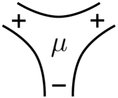
The condition $\pi \circ \pi = 0$

i.e., we require that, for each disk D, we have



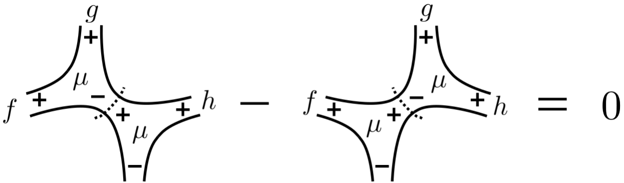
Example 1

Consider the collection that is nonzero only on the shape



$$\begin{array}{c} A \otimes A \\ \downarrow \mu \\ A \end{array}$$

There are two ways to combine two of this disk:



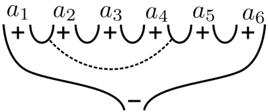
$$\mu(\mu(f, g), h) - \mu(f, \mu(g, h)) = 0$$

Thus (A, μ) is a (non-unital) associative algebra.

Example 2

Consider the collection that is nonzero only on the shapes

The requirement $\pi \circ \pi = 0$ gives one relation for each $n \geq 1$:



$$\sum_{r+s+t=n} \pm\mu_{r+t+1}(a_1, \ldots, a_r, \mu_s(a_{r+1}, \ldots, a_{r+s}), a_{r+s+1}, \ldots, a_n) = 0$$

i.e., $(A, \mu_1, \mu_2, \ldots)$ is an A_∞-algebra.

Example 3, 4

Example 3) Consider the collection that is nonzero only on the shape

$$\Delta : A \to A \otimes A$$

Then (A, Δ) is a coassociative coalgebra.

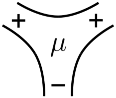
Example 4) Consider the collection that is nonzero only on the shapes

$$\mu : A \otimes A \to A \qquad \text{and} \qquad \Delta : A \to A \otimes A$$

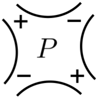
Then (A, μ, Δ) is an infinitesimal bialgebra ($+$ a derivation property).

Example 5

Consider the collection that is nonzero only on the shapes



$$A \otimes A \xrightarrow{\mu} A$$



$$A \otimes A \xrightarrow{P} A \otimes A$$

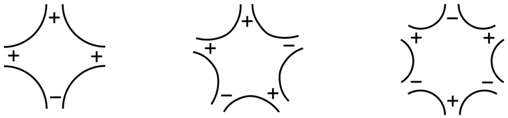
Notice: the disk on the right has an internal C_2 symmetry.

We identify diffeomorphic disks. Accordingly, we require that this map P be C_2-invariant.

We will rewrite $P(a, b) = \{\{a, b\}\} \in A \otimes A$.

Example 5

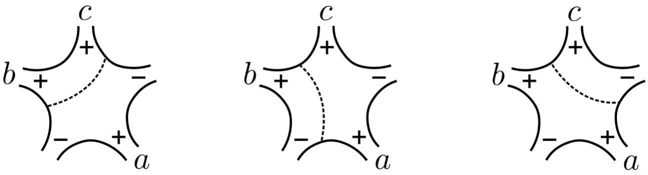
Gluing these disks gives rise to three kinds of disks:



First disk $\Rightarrow (A, \mu)$ is an associative algebra. (We will write $\mu(a, b) = ab$)

Example 5

Second disk



$$\{\{a, bc\}\} = b\{\{a, c\}\} + \{\{a, b\}\}c$$

i.e., $\{\{a, -\}\}$ is a derivation with respect to the outer bimodule structure.

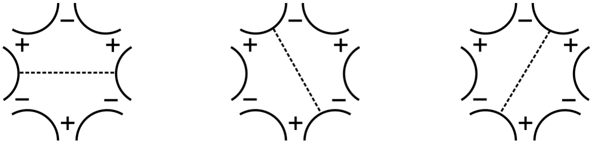
Recall: $A \otimes A$ is a bimodule in two ways.
Outer bimodule: ${}_LA \otimes A_R$ Inner bimodule: $A_R \otimes {}_LA$

By C_2-invariance, $\{\{-, a\}\}$ is a derivation with respect to the inner bimodule structure.

Thus, $\{\{-, -\}\} : A \otimes A \to A \otimes A$ is a double bracket in the sense of Van den Bergh.

Example 5

Third disk



The "double Jacobi identity"

$$\{\{a, \{\{b, c\}\}\}\}_L + \text{ cyclic rotations } = 0$$

Thus, the requirements on $\{\{-, -\}\}$ are:

- C_2 (anti)symmetry
- Derivation on each variable.
- Double Jacobi identity

Thus, $\{\{-, -\}\}$ is a double Poisson structure on the associative algebra (A, μ), as defined by Van den Bergh.

What have we been doing?

Fix $A \in \mathrm{grVect}_k$. Fix $m \in \mathbb{Z}$. Define

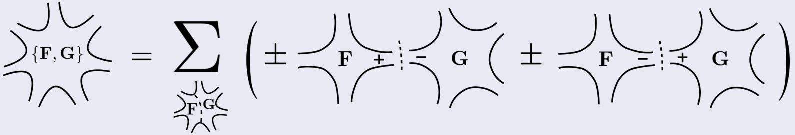
$$\mathfrak{X}^{(p)}(A; m) = \left\{ \begin{array}{l} \text{Collection } F \text{ that assigns a } D\text{-shaped map} \\ F(D) : (A[1])^{\otimes \Sigma^-} \to (A[-m])^{\otimes p} \\ \text{to each disk } D \text{ with } p \text{ outputs} \end{array} \right\}$$

Theorem [Kontsevich-Vlassopoulos]

There is a graded Lie bracket

$$\{-,-\} : \mathfrak{X}^{(p)}(A; m) \otimes \mathfrak{X}^{(q)}(A; m) \to \mathfrak{X}^{(p+q-1)}(A; m)$$

given by the diagram



What have we been doing?

This bracket gives a graded Lie algebra structure on

$$\hat{\mathfrak{X}}^{\geq 1}(A;m)[m+1] := \prod_{p \geq 1} \mathfrak{X}^{(p)}(A;m)[m+1]$$

Definition

Let $m = 2 - n$. An n-pre-Calabi-Yau algebra is a graded vector space A, together with a Maurer-Cartan element in the graded Lie algebra $\hat{\mathfrak{X}}^{\geq 1}(A;m)[m+1]$.

In other words, we have $\pi = \pi_1 + \pi_2 + \pi_3 + \ldots$ satisfying $\{\pi, \pi\} = 0$.

From now on, we ignore the homological shifts, and so we neglect m.

pre-Calabi-Yau structures

Recall that the bracket $\{-,-\}$ has weight grading -1:

$$\{-,-\} : \mathfrak{X}^{(p)}(A) \otimes \mathfrak{X}^{(q)}(A) \to \mathfrak{X}^{(p+q-1)}(A)$$

In particular, $\mathfrak{X}^{(1)}(A) \subset \mathfrak{X}^{\geq 1}(A)$ is a Lie subalgebra, and $\mathfrak{X}^{\geq 2}(A) \subset \mathfrak{X}^{\geq 1}(A)$ is a Lie ideal.

Write $\pi = \pi_1 + \pi_{\geq 2}$.
Then the condition $\{\pi, \pi\} = 0$ splits into two conditions
1) $\{\pi_1, \pi_1\} = 0$
2) $\{\pi_1, \pi_{\geq 2}\} + \frac{1}{2}\{\pi_{\geq 2}, \pi_{\geq 2}\} = 0$

Thus, a pre-Calabi-Yau algebra is always an A_∞-algebra (A, π_1) with extra structure $\pi_{\geq 2}$.

$$\{-,-\} : \mathfrak{X}^{(p)}(A) \otimes \mathfrak{X}^{(q)}(A) \to \mathfrak{X}^{(p+q-1)}(A)$$

Given an A_∞ structure π_1, then $\{\pi_1, -\}$ preserves each component $\mathfrak{X}^{(q)}(A)$, so that it becomes a chain complex

Definition

The graded vector space $\mathfrak{X}^{(p)}(A)$ together with the differential $d_{\pi_1} = \{\pi_1, -\}$ is called the poly-Hochschild cochains on the A_∞ algebra (A, π_1).
Thus, $(\mathfrak{X}^\bullet(A), d_{\pi_1})$ becomes a DG Lie algebra.

Definition

A pre-Calabi-Yau structure on the A_∞ algebra (A, π_1) is a Maurer-Cartan element in the poly-Hochschild DG Lie algebra $(\hat{\mathfrak{X}}^{\geq 2}(A), d_{\pi_1})$.

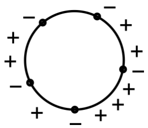
From now on, we assume that A is already given a DG algebra structure.

poly-Hochschild cochains

Recall the definition

$$\mathfrak{X}^{(p)}(A) = \left\{ \begin{array}{l} \text{Collection } F \text{ that assigns a } D\text{-shaped map} \\ F(D) : A^{\otimes \Sigma^-} \to A^{\otimes p} \\ \text{to each disk } D \text{ with } p \text{ outputs} \end{array} \right\}$$

A disk with p outputs is completely determined by the number of inputs between the consecutive outputs:



For example, this disk is specified by the sequence $(3, 2, 0, 2, 1)$.
Any cyclic rotation, e.g., $(2, 0, 2, 1, 3)$, defines the same disk.

poly-Hochschild cochains

Thus we have

$$\mathfrak{X}^{(p)}(A) = \left\{ \begin{array}{l} \text{Collection } F \text{ that assigns a } D\text{-shaped map} \\ F(D) : A^{\otimes \Sigma^-} \to A^{\otimes p} \\ \text{to each disk } D \text{ with } p \text{ outputs} \end{array} \right\}$$

$$= \left[\prod_{(n_1,\ldots,n_p)\in\mathbb{N}^p} \mathrm{Hom}_k(A^{\otimes n_1} \otimes \ldots \otimes A^{\otimes n_p}, A^{\otimes p}) \right]^{C_p}$$

$$= \left[\boldsymbol{R}\mathrm{Hom}_{(A^{\otimes p})^e}(A^{\otimes p}, {}_{\tau}(A^{\otimes p})_{\mathrm{id}}) \right]^{C_p}$$

1) $\mathrm{Hom}_{B^e}(-,-)$ means B-bimodule map.

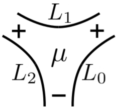
2) Recall that A has a free A-bimodule resolution

$$\ldots \to A \otimes A^{\otimes 2} \otimes A \to A \otimes A^{\otimes 1} \otimes A \to A \otimes A^{\otimes 0} \otimes A \to A \to 0$$

Accordingly, $\mathrm{Bar}(A)^{\otimes p}$ is a $A^{\otimes p}$-bimodule resolution of itself.

Remarks

1) If you require the connected components of the boundaries of disks to be colored by a set of objects, then you get the notion of pre-Calabi-Yau categories. Example:



$$\begin{array}{c} \mathcal{A}(L_1, L_2) \otimes \mathcal{A}(L_0, L_1) \\ \downarrow \mu \\ \mathcal{A}(L_0, L_2) \end{array}$$

2) The Fukaya category is expected to have a pre-Calabi-Yau structure obtained by counting disks with more than one output.
However, there is a technical problem with imposing C_p-invariance.

Noncommutative analogue of Poisson structure

X is a smooth manifold (or variety).
Then a Poisson structure is a bivector field $\pi_2 \in \mathfrak{X}^2(X)$ satisfying $\{\pi_2, \pi_2\} = 0$.

In either deformation quantization or in derived algebraic geometry, we are forced to consider generalized Poisson structures

$$\pi_{\geq 2} = \pi_2 + \pi_3 + \dots$$

satisfying the Maurer-Cartan equation.

This formal parallelism gives a superficial justification that pre-Calabi-Yau structures is a noncommutative analogue of Poisson structures.

Noncommutative calculus

Commutative	**Noncommutative**
Commutative algebra	Associative algebra
Derived stacks	DG categories
Modules or sheaves	Bimodules
Differential forms	Hochschild homology
Closed forms	Negative cyclic homology
de Rham cohomology	Periodic cyclic homology
Symplectic structure	Calabi-Yau structure
Vector fields	Hochschild cohomology
Polyvector fields	Poly-Hochschild cohomology
Poisson structure	pre-Calabi-Yau structure

Symplectic structure vs Calabi-Yau structures

Definition

A **symplectic structure** on X is a **closed** 2-**form** whose **underlying** 2-**form** determines an isomorphism

$$\Omega^1(X)^\vee \xrightarrow{\cong} \Omega^1(X)$$

of sheaves.

Definition [Ginzburg, Kontsevich-Vlassopoulos, Brav-Dyckerhoff]

An n-**Calabi-Yau structure** on A is a **negative cyclic homology class** $\tilde{\eta} \in HC_n^-(A)$ whose **underlying Hochschild homology class** $\eta \in HH_n(A)$ determines an isomorphism

$$A^\vee[n] \xrightarrow{\cong} A$$

in the derived category of DG bimodules.

Non-degenerate pre-Calabi-Yau structures

On a smooth manifold, a symplectic structure is the same as a nondegenerate Poisson structure.

Noncommutative analogue:
Given an n-pre-Calabi-Yau structure $\pi = \pi_2 + \pi_3 + \ldots$, the lowest order term is

$$\pi_2 \in \mathrm{Hom}_{(A^{\otimes 2})^e}(\mathrm{Bar}(A) \otimes \mathrm{Bar}(A), {}_{\tau}(A \otimes A)_{\mathrm{id}})$$

Equivalently, it gives a map of DG bimodules

$$\pi_2^{\#} : \mathrm{Bar}(A) \to \mathrm{Bar}(A)^{\vee}[n]$$

i.e., a map $A \to A^{\vee}[n]$ in the derived category of bimodules.

Definition

The pre-Calabi-Yau structure π is said to be non-degenerate if this map is a quasi-isomorphism.

Non-degenerate pre-Calabi-Yau structures

Theorem [Y.]

An n-Calabi-Yau structure is equivalent to a non-degenerate n-pre-Calabi-Yau structure.

Remark

- More precisely, the Theorem asserts that there is a zig-zag of homotopy equivalences between the space of n-Calabi-Yau structures and the space of non-degenerate n-pre-Calabi-Yau structures.
- Pridham and Calaque-Pantev-Toën-Vaquié-Vezzosi proved a similar theorem for derived stacks. My proof is a direct translation of Pridham's proof to the noncommutative setting.

Two flavors of noncommutative algebraic geometry

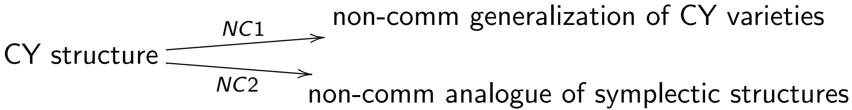
NC1

For any notion P on varieties/derived stacks, etc, its **noncommutative generalization** should be a notion $\widetilde{P}$ on associative algebras, so that it reduces to P for (smooth) commutative algebras.

NC2 [The Kontsevich-Rosenberg principle]

For any structure P on varieties/derived stacks, etc, its **noncommutative analogue** should be a structure P_{nc} on an associative algebra A which induces the structure P on the moduli space of representations of A.

Example:



The Kontsevich-Rosenberg principle

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de Rham cohomology	Periodic cyclic homology
Symplectic structure	Calabi-Yau structure
Vector fields	Hochschild cohomology
Polyvector fields	Poly-Hochschild cohomology
Poisson structure	pre-Calabi-Yau structure

All these can be justified by the Kontsevich-Rosenberg principle.

Two sides of NC2

Actually, the way I see it, there are two sides of NC2:

- Phenomenological side:
  The Kontsevich-Rosenberg principle
- Ontological side:
  Use the analogy

Commutative	**Noncommutative**
Commutative algebra	Associative algebra
Derived stacks	DG categories
Modules or sheaves	Bimodules

together with some aesthetic principles, guided by some basic examples, to develop noncommutative geometry.

Keypoint: These two sides end up doing the same thing!

Symplectic and Poisson structure on moduli spaces

Theorem [Pantev-Toën-Vaquié-Vezzosi, Brav-Dyckerhoff, Y.]

Any n-Calabi-Yau structure on A induces a $(2-n)$-shifted symplectic structure on the derived moduli stack of representations of A.

Theorem [Y.]

Any n-pre-Calabi-Yau structure on A induces a $(2-n)$-shifted Poisson structure on the derived moduli stack of representations of A.

Caveat

We have used an a priori different definition of shifted symplectic/Poisson structure, which is conjecturally equivalent to the ones by PTVV and CPTVV. Accordingly our proof is quite different from the ones in the literature.

Proof (Sketch)

A is a smooth DGA. X is the moduli space of representations of A.

Induce symplectic structure

$\Upsilon^{(p)}(A) \simeq (A \otimes^{\mathbf{L}}_A \dots \otimes^{\mathbf{L}}_A A)_{\natural, C_p}$

$\exists$ explicit model with mixed structure $B : \Upsilon^{(p)}(A) \to \Upsilon^{(p+1)}(A)$

$\exists$ map $\Upsilon^{(p)}(A) \to \mathcal{A}^p(X)$ respecting the mixed structure.

Induce Poisson structure

$\mathfrak{X}^{(p)}(A) \simeq (A^\vee \otimes^{\mathbf{L}}_A \dots \otimes^{\mathbf{L}}_A A^\vee)_{\natural, C_p}$

$\exists$ explicit model with a Lie bracket $\{-,-\}$.

$\exists$ map $\mathfrak{X}^{(p)}(A) \to \mathfrak{X}^{(p)}_{\mathrm{com}}(X)$ of DGLAs.

In fact, for the right hand side, we use a smaller model $\mathfrak{X}_{\mathrm{res}}(A)$, called the extended necklace DGLA. It fits into

$$\mathfrak{X}_{\mathrm{res,fr}}(A) \hookrightarrow \mathfrak{X}_{\mathrm{res}}(A) \xrightarrow{\sim} \mathfrak{X}(A)$$

Here, $\mathfrak{X}_{\mathrm{res,fr}}(A)$ is the necklace Lie algebra, whose weight 2 Maurer-Cartan elements are precisely double Poisson brackets.

Natural examples

Symplectic topology

Fukaya categories are expected to have natural pre-CY structure.

Algebraic geometry

X (proper) n-dimensional variety with at most Gorenstein singularities. Then any $s \in H^0(X, \omega_X^\vee)$ gives an n-pre-CY structure on $\mathcal{A} = \mathcal{D}_{\mathrm{perf}}(X)$.

Topology

M compact connected n (real) dimensional oriented manifold, possibly with boundary. Then $A = C_*(\Omega M)$ has an n-pre-CY structure.

Representation theory

2-CY: (derived) deformed preprojective algebra

3-CY: Ginzburg DG algebra

Their relative version.