Localizing Virtual Cycles for DT4 by Cosections

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2021. 1. 13 IPMU

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arXiv: 2012.13167

§1. Euler class

$X =$ q-proj. scheme or DM stack

$A_*(X) = \mathbb{Z}\{\text{irreducible closed subvarieties in } X\}/_{\text{rat. equiv}}$

- cycle class map $A_k(X) \longrightarrow H_{2k}(X)$
- proper pushforward

  $f: X \to Y$ proper $\Longrightarrow$ $f_*: A_k(X) \longrightarrow A_k(Y)$
- flat pullback

  $f: X \to Y$ flat $\Longrightarrow$ $f^*: A_k(Y) \longrightarrow A_{k+r}(X)$

- Homotopy invariance

$\pi : F \to X$ VB of rank $r \Rightarrow \pi^* : A_k(X) \xrightarrow{\cong} A_{k+r}(F)$ iso

$[Z] \longmapsto [\pi^{-1}(Z)]$

$\Rightarrow \quad 0_F^! = (\pi^*)^{-1} : A_k(F) \longrightarrow A_{k-r}(X)$

$\Rightarrow \quad e(F) : A_k(X) \xrightarrow{0_*} A_k(F) \xrightarrow{0_F^!} A_{k-r}(X)$ bivariant

$\Rightarrow \quad e(F) \in A^r(X)$

- Localization sequence

$Z \overset{j}{\hookrightarrow} X \overset{\iota}{\hookleftarrow} X - Z$

closed open

$A_k(Z) \xrightarrow{j_*} A_k(X) \xrightarrow{\iota^*} A_k(X-Z) \longrightarrow 0$

exact

- More nice properties.

Bivariant classes $A^k(X \xrightarrow{f} Y)$

- $\forall\, Y' \xrightarrow{g} Y$

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$\exists\ A_*(Y') \to A_{*-k}(X')$

- Compatible with proper pushforward, flat pullback and intersection products

<u>Ex</u> $e(F) = c_r(F) \in A^r(X)$, Chern classes $c_i(F) \in A^i(X)$

- $A^k(X) := A^k(X \xrightarrow{id} X)$

if $Y \overset{i}{\hookrightarrow} X$ closed, $A^k_Y(X) := A^k(Y \overset{i}{\hookrightarrow} X)$

§2. Square root Euler class

F = $SO(2n)$-bundle over X

[vector bundle of rank $2n$

$F \otimes_{\mathcal{O}_X} F \xrightarrow{q} \mathcal{O}_X$ symmetric bilinear nondegenerate form

or: $\mathcal{O}_X \to \det F$ (square root of $\det F \underset{\det q}{\cong} \det F^\vee$)

orientation]

$F' \leq F$ is isotropic if $F' \otimes_{\mathcal{O}_X} F' \xrightarrow{q} \mathcal{O}_X$ is 0.

An isotropic F' is maximal if $\operatorname{rank} F' = n$.

A max iso. F' is positive if $\prod_{i=1}^{n} e_i \wedge f_i = \sqrt{-1}^{\,n}$ or.

local basis of F' — dual basis of (e_i)

Edidin-Graham (1995): $\exists$ square root of $e(F)$

$\sqrt{e}(F) : A_k(X) \to A_{k-n}(X)$ bivariant class

$\sqrt{e}(F) \in A^n(X)$, $\sqrt{e}(F)^2 = (-1)^n e(F)$

If $\exists\ \Lambda \leq F$ pos. max. isotropic subbundle, then $\sqrt{e}(F) = e(\Lambda)$. [Fulton's conj.]

$$\begin{tikzcd}
\cdots \arrow[dr, "\pi_3"] & \vdots \arrow[d] & & \\
& Q_2 \arrow[r, hook] \arrow[drr, "\pi_2"'] & \mathbb{P}\left(\mathcal{O}_{Q_1}(-1)^{\perp} / \mathcal{O}_{Q_1}(-1)\right) \arrow[d] & \\
& & Q_1 \arrow[r, hook] \arrow[dr, "\pi_1"'] & \mathbb{P}F \arrow[d] \\
& & & X
\end{tikzcd}$$

$SO(2n-2)$ bundle

$\mathcal{O}_{Q_1}(-1) \hookrightarrow F|_{Q_1}$ tautological

$\mathcal{O}_{Q_1}(-1)^{\perp}$ orthogonal complement w.r.t. q

$Q_1 \hookrightarrow \mathbb{P}F$ isotropic locus

quadric hypersurface

$h_1 = c_1(\mathcal{O}_{Q_1}(1))$

$$\pi_{1*}\left(\boxed{\frac{h_1^{2n-2}}{2} \cap \pi_1^*(\xi)}\right) = \xi \quad \forall \xi \in A_*(X)$$

$$\pi_{2*}\left(\boxed{\frac{h_2^{2n-4}}{2} \cap \pi_2^*(\xi_1)}\right) = \xi_1 \quad \forall \xi_1 \in A_*(Q_1), \quad h_2 = c_1(\mathcal{O}_{Q_2}(-1))$$

$$
\begin{tikzcd}
\underbrace{\mathbb{P}\left(\mathcal{O}_{Q_{n-1}}(-1)^{\perp}/\mathcal{O}_{Q_{n-1}}(-1)\right)}_{SO(2)} \arrow[d] & & & \\
Q_{n-1} \arrow[u, bend left, "\sigma"] \arrow[r] \arrow[dr, "\pi_{n-1}"'] & \underbrace{\mathbb{P}\left(\mathcal{O}_{Q_{n-2}}(-1)^{\perp}/\mathcal{O}_{Q_{n-2}}(-1)\right)}_{SO(4)} \arrow[d] & & \\
& Q_{n-2} \arrow[r] \arrow[dr, "\pi_{n-2}"'] & \underbrace{\mathbb{P}\left(\mathcal{O}_{Q_{n-3}}(-1)^{\perp}/\mathcal{O}_{Q_{n-3}}(-1)\right)}_{SO(6)} \arrow[d] & \\
& & Q_{n-3} \arrow[r] \arrow[dr, "\pi_{n-3}"'] & \cdots \\
& & & \cdots
\end{tikzcd}
$$

$$0 \to \overset{\exists}{\Lambda} \to F|_{Q_{n-1}} \to \Lambda^{\vee} \to 0$$

max. pos. isotropic subbundle of rank n on Q_{n-1}

$$\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{n-1} : Q_n \to X$$

$$h = \frac{h_1^{2n-2}}{2} \cdot \frac{h_2^{2n-4}}{2} \cdot \cdots \cdot \frac{h_{n-1}^{2}}{2}$$

$$\pi_*(h \cap \pi^*\xi) = \xi \quad \forall\, \xi \in A_*(X)$$

$$\sqrt{e}(F) \cap \xi = \pi_*\left(e(\Lambda)\, h \cap \pi^*(\xi)\right) \quad \forall\, \xi \in A_*(X)$$

$$\sqrt{e}(F) : A_k(X) \to A_{k-n}(X)$$

$\sqrt{e}(F) \in A^n(X)$ bivariant class

§3. Localized Euler class

$$\begin{tikzcd} & F \arrow[d] & \text{VB of rank } r \\ X = s^{-1}(0) \arrow[r, hook, "j"] & Y \arrow[u, bend right, "s"'] & \end{tikzcd}$$

$$e(F,s): A_k(Y) \to A_{k-r}(X)$$

$$j_* \circ e(F,s) = e(F): A_k(Y) \to A_{k-r}(Y)$$

$$Y \overset{\text{deformation}}{\rightsquigarrow} C_{X/Y} \hookrightarrow F|_X \overset{\text{homotopy}}{\rightsquigarrow} X$$

$$\begin{tikzcd} A_k(Y) \arrow[r] \arrow[drr, "sp"'] \arrow[ddrrr, bend right, "e(F,s)"'] & A_{k+1}(Y \times (\mathbb{A}^1 - 0)) \arrow[dr] & A_{k+1}(M_{X/Y}) \arrow[l] \arrow[d, "\text{divisor}"] & \\ & & A_k(C_{X/Y}) \arrow[r] & A_k(F|_X) \arrow[d, "0^!_{F|_X}"] \\ & & & A_{k-r}(X) \end{tikzcd}$$

$$M_{X/Y} = Bl_{X \times \{0\}}(Y \times \mathbb{A}^1) - (\text{proper transf. of } Y \times \{0\})$$

$$\begin{tikzcd} M_{X/Y} \arrow[d, "\text{flat}"'] \\ \mathbb{A}^1 \end{tikzcd}$$

$$\left[\begin{array}{l} M_{X/Y}|_{\mathbb{A}^1 - 0} \cong Y \times (\mathbb{A}^1 - 0) \\ M_{X/Y}|_0 = C_{X/Y} \end{array}\right.$$

Localized Euler class by Blow-up

$$\begin{tikzcd} D \arrow[r, hook, "\iota"] \arrow[d, "\rho'"] & \tilde{Y} = Bl_X Y \arrow[d, "\rho"] \\ X := s^{-1}(0) \arrow[r, hook, "j"] & Y \end{tikzcd}$$

$$\boxed{\begin{array}{c} A_*(\tilde{Y}) \oplus A_*(X) \twoheadrightarrow A_*(Y) \\ (\alpha, \beta) \longmapsto \rho_* \alpha + j_* \beta = \xi \end{array}}$$

$$\mathcal{O}_Y \xrightarrow{s} F \Rightarrow F^\vee \xrightarrow{s^\vee} \mathcal{O}_Y \Rightarrow F^\vee|_{\tilde{Y}} \twoheadrightarrow \mathcal{O}_{\tilde{Y}}(-D) \hookrightarrow \mathcal{O}_{\tilde{Y}}$$

$$F^\vee \to I_X \hookrightarrow \mathcal{O}_Y$$

$$\Rightarrow 0 \to \mathcal{O}_{\tilde{Y}}(D) \to F|_{\tilde{Y}} \to \bar{F} \to 0 \quad \text{exact seq. of VB}$$

$$\bar{F} = F|_{\tilde{Y}} / \mathcal{O}_{\tilde{Y}}(D)$$

$$e(F, s) \cap \xi = \rho'_*\left(e(\bar{F}) \cap \iota^* \alpha\right) + e(F) \cap \beta$$

$$= \rho'_*\left(\iota^*(e(\bar{F}) \cap \alpha)\right) + e(F) \cap \beta$$

$$e(F, s) : A_k(Y) \to A_{k-r}(X)$$

Excess intersection formula $\Rightarrow$ $\boxed{e(F,s) = e(F,s)}$

$$
\begin{tikzcd}
D \arrow[r, hook, "i"] \arrow[d, "\rho'"'] & \tilde{Y} \arrow[d, "\rho"] \\
X \arrow[r, hook, "j"] & Y
\end{tikzcd}
$$

- $e(F|_{\tilde Y}, s|_{\tilde Y}) \cap \alpha = e(\tilde F) \cap D \cdot \alpha$ $\quad \forall \alpha \in A_*(\tilde Y)$

  ($j^! \alpha = e(\tilde F) \cap i^! \alpha$ in Fulton's notation)

$$\Rightarrow \rho'_*\big(e(\tilde F) \cap D\cdot\alpha\big) = \rho'_*\big(e(F|_{\tilde Y}, s|_{\tilde Y}) \cap \alpha\big) \underset{\text{bivariance}}{=} e(F,s) \cap \rho_* \alpha$$

- $e(F|_X) \cap \beta \underset{\text{bivariance}}{=} e(F,s) \cap j_* \beta$

$$\underbrace{e(F,s) \cap \xi}_{\text{Degeneration to normal cone}} = e(F,s)\,(\rho_*\alpha + j_*\beta) = \rho'_*\big(e(\tilde F) \cap D\cdot\alpha\big) + e(F|_X) \cap \beta$$

$$= e(F,s)\,(\rho_*\alpha + j_*\beta) = \underbrace{e(F,s) \cap \xi}_{\text{Blowup}}$$

Cosection Localized Gysin map (Blowup construction)

$$F \xrightarrow{\sigma} \mathcal{O}_Y, \qquad F \to Y$$

$$X = \sigma^{-1}(0) \hookrightarrow Y$$

defined by the ideal $\sigma(F) \subset \mathcal{O}_Y$

$$F(\sigma) := F|_X \cup \ker\left(\sigma|_{Y-X} : F|_{Y-X} \to \mathcal{O}_{Y-X}\right)$$

$$\begin{tikzcd} D \arrow[r, hook, "i"] \arrow[d, "s'"'] \arrow[dr, phantom, "\square"] & \tilde{Y} \arrow[d, "\rho"] \\ X \arrow[r, hook, "j"] & Y \end{tikzcd}$$

$$0 \to \tilde{F} \to F|_{\tilde{Y}} \to \mathcal{O}_{\tilde{Y}}(-D) \to 0$$

$$\begin{tikzcd} \tilde{F} \arrow[d, "\tilde{s}"'] & F|_{\tilde{Y}} \arrow[d] \\ F(\sigma) \arrow[r] & F \end{tikzcd}$$

$$\tilde{j} : F|_X \to F(\sigma)$$

$$0^!_{F,\sigma} : A_*(F(\sigma)) \longrightarrow A_{*-r}(X)$$

$$\xi \in A_*(F(\sigma)) \xleftarrow{(\tilde{j}_*, \tilde{s}_*)} (\beta, \alpha) \in A_*(F|_X) \oplus A_*(\tilde{F})$$

$$0^!_{F,\sigma}(\tilde{s}_*\alpha + \tilde{j}_*\beta) = -s'_*(D \cdot e(\tilde{F})\alpha) + 0^!_{F|_X}\beta$$

* Well defined
* $j_* \circ 0^!_{F,\sigma} = 0^!_F$
* Works for K-theory, algebraic cobordism, etc

[K.- Park, arXiv: 1908.03440]

§4. Localized square root Euler class by Oh-Thomas

Assume $0 \to \Lambda \to F \to \Lambda^\vee \to 0$ over Y

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] & F \arrow[r] & \Lambda^\vee \arrow[r] & 0 \\
& & \mathcal{O}_Y \arrow[ul, dashed, "s_2 \text{ on } Z(s_1)"] \arrow[u, "s"] \arrow[ur, "s_1"'] & &
\end{tikzcd}$$

Λ: isotropic subbundle of pos. ori.

F: $SO(2n)$ bundle

$s: \mathcal{O}_Y \to F$ isotropic section

$s^2 = s \cdot s = 0$

$X = s^{-1}(0)$
$= Z(s_1) \cap Z(s_2)$

$$\boxed{\text{Edidin-Graham} \quad \sqrt{e}(F) = e(\Lambda)}$$

If $s \in H^0(\Lambda)$, $\sqrt{e}(F,s) := e(\Lambda, s)$.

Oh-Thomas

$$Y \overset{\text{deformation}}{\rightsquigarrow} C_{Z(s_1)/Y} \hookrightarrow \Lambda^\vee|_{Z(s_1)} \xrightarrow{s_2^\vee} \mathcal{O}_{Z(s_1)}.$$

$$e^{\frac{1}{2}}(F,s,\Lambda): A_k(Y) \xrightarrow{sp} A_k(C_{Z(s_1)/Y}) \longrightarrow A_k(\Lambda^\vee|_{Z(s_1)}) \xrightarrow{(-1)^n 0^!_{\Lambda^\vee|_{Z(s_1)}, s_2^\vee}} A_{k-n}(X)$$

cosection localized Gysin map

- $\iota_* \circ e^{\frac{1}{2}}(F,s,\Lambda) = e(\Lambda) = \sqrt{e}(F)$

For a general $SO(2n)$-bundle F and an isotropic section s, use the isotropic flag variety $\pi : Q_{n-1} \to Y$ such that $\exists$ exact seq

$$0 \to \Lambda \to F|_{Q_{n-1}} \to \Lambda^\vee \to 0 ,$$

Λ = positive max isotropic subbundle of $F|_{Q_{n-1}}$

$$e^{\frac{1}{2}}(F, s) \cap \xi = \pi'_* \left(e^{\frac{1}{2}}(F|_{Q_{n-1}}, s|_{Q_{n-1}}, \Lambda)\, h \cap \pi^* \xi \right).$$

$$\begin{tikzcd} s|_{Q_{n-1}}^{-1}(0) \arrow[r, hook] \arrow[d, "\pi'"'] & Q_{n-1} \arrow[d, "\pi"] \\ X = s^{-1}(0) \arrow[r, hook, "j"] & Y \end{tikzcd}$$

$$\xi = \pi_* (\underbrace{h \cap \pi^* \xi}_{\text{lift of } \xi \text{ to } Q_{n-1}})$$

- $j_* \circ e^{\frac{1}{2}}(F, s) = \sqrt{e}(F)$
- $\sqrt{0_F^!} := e^{\frac{1}{2}}(F|_{Q_1}, \underset{\text{tautological section}}{\tau}) : A_*(\underset{\text{quadric hypersurface of isotropic vectors}}{Q_1}) \to A_{*-n}(Y)$, $\quad Q_1 \subset F$

※ This construction is not suitable for a further localization! ← need it for cosection localization

Localized square root Euler class by Blowup

$$\begin{tikzcd} D \arrow[r, hook, "\iota"] \arrow[d, "\rho'"'] & \tilde{Y} = bl_X Y \arrow[d, "\rho"] & \\ X = s^{-1}(0) \arrow[r, hook, "j"] & Y & F \arrow[l] \arrow[l, bend left, "s"'] \end{tikzcd}$$

$$\begin{tikzcd} A_k(Y) & A_k(\tilde{Y}) \oplus A_k(X) \arrow[l, two heads] \\ \xi = \rho_*\alpha + j_*\beta & (\alpha, \beta) \arrow[l, mapsto] \end{tikzcd}$$

$s : \mathcal{O}_Y \to F$ isotropic $\Rightarrow$ $L := \underbrace{\mathcal{O}_{\tilde{Y}}(D)}_{\text{isotropic subbundle}} \hookrightarrow F|_{\tilde{Y}}$

$F^\vee \twoheadrightarrow \mathcal{O}_{\tilde{Y}}(-D)$

$0 \to L \to L^\perp \to \tilde{F} \to 0$ $\quad SO(2n-2)$

$F|_{\tilde{Y}} \to L^\vee \to 0$

Let $\tilde{F} = L^\perp / L$ $SO(2n-2)$-bundle.

Def. (K.-Park) $\sqrt{e}(F,s) \cap \xi := \rho'_*(\sqrt{e}(\tilde{F}) \cap \iota^*\alpha) + \sqrt{e}(F) \cap \beta$

for $\xi = \rho_*\alpha + j_*\beta \in A_k(Y)$, $\quad \beta \in A_*(X)$

Thm 1. (K.-Park) (1) $\sqrt{e}(F,s)$ is independent of α, β.

$\sqrt{e}(F,s)\xi = \rho'_*(\sqrt{e}(\tilde{F}) \cap \underbrace{\iota^*\alpha}_{D\cdot\alpha}) + \sqrt{e}(F) \cap \beta \quad \forall\ \xi = \rho_*\alpha + g_*\beta.$

(3) $\sqrt{e}(F,s) = e^{\frac{1}{2}}(F,s)$.

(2) $\sqrt{e}(F,s) \in A^n_X(Y)$ is a bivariant class.

(4) If t is another isotropic section of F with $s.t = 0$,

orth.

then $\sqrt{e}(F,s)$ is further localized to

$$\sqrt{e}(F,s;t) : A_*(Y) \longrightarrow A_{*-n}(X \cap Z)$$

$Y-Z$

where $\underset{t^{-1}(0)}{Z} \subset Y$ is closed and s,t are independent away from Z.

locally on $Y-Z$, $F = F' \oplus F''$

$s \in H^0(F')$, $t \in H^0(F'')$

$$\sqrt{e}(F,s;t)\xi = \rho''_*(\sqrt{e}(\tilde{F},\tilde{t}) \cap \iota^*\alpha) + \sqrt{e}(F,t) \cap \beta$$

$\xi = \rho_*\alpha + g_*\beta$, $\quad \tilde{F} = L^\perp/L \quad (L = \mathcal{O}_{\tilde{Y}}(D) \leq F|_{\tilde{Y}})$, $\quad t|_{\tilde{Y}} \in H^0(L^\perp) \to \tilde{t} \in H^0(\tilde{F})$

$\tilde{Y}$

§5. Virtual cycles for DT4 invariants

X = quasi-projective moduli space of stable sheaves or simple perfect complexes $\mathcal{E}$ over a Calabi-Yau 4-fold W

$$\mathcal{E} \downarrow X \times W \xrightarrow{p} X$$

$\mathrm{Ext}^\bullet_0(\mathcal{E},\mathcal{E})$, $\exists$ 3-term symmetric obst. th on X

$\phi : \mathbb{E}^\bullet \to \mathbb{L}_X$. $\mathbb{E}^\bullet$ = perfect of amplitude $[-2,0]$

$\mathbb{E}^{\bullet\vee} = Rp_* R\mathcal{H}om(\mathcal{E},\mathcal{E})_0[1]$

$h^0(\phi)$ isom, $h^{-1}(\phi)$ surj. ✓

$\mathbb{E}^{\bullet\vee}[2] \underset{\theta}{\xrightarrow{\simeq}} \mathbb{E}^\bullet$, $\theta^\vee[2] = \theta$ ✓

Thm (Oh-Thomas) $\exists$ 3-term cx of loc. free sheaves $[B \xrightarrow{d} F \xrightarrow{d^\vee} B^\vee]$ on X

F: $SO(2n)$-bundle, $F \cong F^\vee$

$$\mathbb{E}^\bullet \cong [B \xrightarrow{d} F \cong \boxed{F^\vee \xrightarrow{d^\vee} B^\vee]}$$

pert. obs. th. on X

Behrend-Fantechi : $\exists$ intrinsic normal cone $\mathfrak{C}_X \hookrightarrow [F/B] \leftarrow F$

(if $\begin{array}{c} U \hookrightarrow V^{sm} \\ \downarrow \text{étale} \\ X \end{array}$, $\mathfrak{C}_X|_U \cong [C_{U/V}/TV|_U]$)

$$Y := C_{red} \subset C := \mathfrak{C}_X \times_{F/B} F \hookrightarrow \begin{array}{c} F \\ \downarrow \\ X \end{array}$$

$$F|_Y = F \times_X Y$$

$$\begin{tikzcd}
 & F|_Y \arrow[dl] \arrow[dr] & \\
F \arrow[dr] & Y \arrow[u, "\tau"] \arrow[l] \arrow[r, "id"] & Y \arrow[dl] \\
 & X &
\end{tikzcd}$$

tautological section $\tau : Y \longrightarrow F|_Y \in H^0(F|_Y)$ isotropic section [BBBJ, Darboux thm]

$\tau^{-1}(0) = X$

Def (Oh-Thomas) $A_*(X) \ni [X]^{vir} := \sqrt{e}(F|_Y, \tau) \cap [C] \in A_*(Y)$

Thm (Oh-Thomas) Same as Borisov-Joyce; Deformation inv. Torus localization, ...

DT4 invariant = counting sheaves on CY4.

$$DT4 = \int_{[X]^{vir}} \eta \,, \qquad \eta \in A^*(X)$$

X = q-proj. moduli sp of simple complexes on CY4.

History

1) 1990s Donaldson - Thomas

$$Ext^1(E,E)_0 \longrightarrow Ext^2(E,E)_0^+ \oplus Ext^2(E,E)_0^-$$

2) Early 2010s
Cao - Leung constructed vir. cycles for some cases like moduli of vector bundles.

3) Mid 2010s
Borisov - Joyce constructed (real) vir. cycles using derived differential geometry.

4) 2020
Oh - Thomas constructed algebraic vir. cycle by $e^{\frac{1}{2}}(F, s)$.

5) Late 2010s
Cao - Maulik - Toda conjectured connections with GW. (GV inv.)

Cosection Localization of virtual cycles : 2-term case.

- X = DM stack, $\mathbb{E}^\bullet = [F^\vee \xrightarrow{d^\vee} B^\vee] \xrightarrow{\phi} \mathbb{L}_X^{\geq -1}$ obstruction theory
  $h^0(\phi)$ iso. $h^{-1}(\phi)$ surj.
  loc. free

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[d] \\ \text{intrinsic normal cone} = \mathfrak{C}_X \arrow[r, hook] & [F/B] \end{tikzcd} \Rightarrow [X]^{vir} := 0_F^![C] \in A_*(X).$$

- $\sigma : Ob_X = h^1(\mathbb{E}_\bullet) \to \mathcal{O}_X$ cosection $(\mathbb{E}_\bullet = (\mathbb{E}^\bullet)^\vee)$

$\Rightarrow \tilde{\sigma} : F \twoheadrightarrow \operatorname{coker} d = Ob_X \xrightarrow{\sigma} \mathcal{O}_X$

1) cone reduction $C_{red} \subset F(\tilde{\sigma}) \Rightarrow [C] \in A_*(F(\tilde{\sigma}))$

2) cosection localized Gysin map $[X]^{vir}_{loc} := 0^!_{F,\tilde{\sigma}}[C] \in A_*(X(\sigma))$

$[X]^{vir}_{loc}$ turned out to be quite useful. (e.g. FJRW, MSP, ...)

Question. Cosection localization for DT4 vir. cycles?

Cosection of Obstruction Theory

- 2-term $\phi: \mathbb{E}^\bullet \to \mathbb{L}_X^{\geq -1}$ perf. ob. th., $\mathbb{E}_\bullet = (\mathbb{E}^\bullet)^\vee$

locally isom. to a 2-term cx $[B_1 \to B_0]$ of loc. free sh.

✓ $\sigma: Ob_X = h^1(\mathbb{E}_\bullet) = \tau^{\geq 1}\tau^{\leq 1}\mathbb{E}_\bullet \longrightarrow \mathcal{O}_X$ cosection of the obstruction sheaf Ob_X.

$\Rightarrow \sigma_\bullet: \mathbb{E}_\bullet \xleftarrow{\cong} \tau^{\leq 1}\mathbb{E}_\bullet \longrightarrow \tau^{\geq 1}\tau^{\leq 1}\mathbb{E}_\bullet = Ob_X[-1] \xrightarrow{\sigma} \mathcal{O}_X[-1]$

perf of amplitude $[0,1]$

✓ Conversely, a morphism $\mathbb{E}_\bullet \xrightarrow{\sigma_\bullet} \mathcal{O}_X[-1]$ induces

$\Rightarrow \sigma = h^1(\sigma_\bullet): Ob_X = h^1(\mathbb{E}_\bullet) \longrightarrow h^1(\mathcal{O}_X[-1]) = \mathcal{O}_X$.

Def. Let $\phi: \mathbb{E}^\bullet \to \mathbb{L}_X^{\geq -1}$ be an obstruction theory ($h^0(\phi)$ isom, $h^{-1}(\phi)$ surj.)

A cosection of the obstr. th. is a morphism $\boxed{\sigma_\bullet: \mathbb{E}_\bullet = (\mathbb{E}^\bullet)^\vee \to \mathcal{O}_X[-1]}$ ✓

A cosection of an obstruction theory induces a cosection of the obstruction sheaf.

[$\sigma_\cdot : \mathbb{E}_\cdot \to \mathcal{O}_X[-1] \Rightarrow \sigma = h^1(\sigma_\cdot) : Ob_X \to \mathcal{O}_X$.]

If $\mathbb{E}^\cdot \cong \mathbb{E}_\cdot[2]$ symmetric, a cosection $\sigma_\cdot : \mathbb{E}_\cdot \to \mathcal{O}_X[-1]$ is isotropic if $\mathcal{O}_X[-1] \xrightarrow{\sigma_\cdot^\vee} \mathbb{E}^\cdot[-2] \xrightarrow{\cong} \mathbb{E}_\cdot \xrightarrow{\sigma_\cdot} \mathcal{O}_X[-1]$ is 0

Lemma. Let $\mathbb{E}^\cdot \xrightarrow{\phi} \mathbb{L}_X^{\geq -1}$ be a symmetric obstruction theory of amplitude $[-2,0]$ over q-proj. X.

Let $\sigma_\cdot : \mathbb{E}_\cdot \to \mathcal{O}_X[-1]$ be a cosection.

SO(2n)-bundle

Then $\exists$ global loc. free resolution $\mathbb{E}^\cdot \cong [B \xrightarrow{d} F \cong F^\vee \xrightarrow{d^\vee} B^\vee]$

and a homomorphism $\tilde{\sigma} : F \to \mathcal{O}_X$ such that

$$\begin{tikzcd}
\mathbb{E}_\cdot \arrow[d, "\sigma_\cdot"'] & = & {[B} \arrow[r, "d"] & F \arrow[d, "\tilde{\sigma}"] & \cong F^\vee \arrow[r, "d^\vee"] & B^\vee] \\
\mathcal{O}_X[-1] & = & & \mathcal{O}_X & &
\end{tikzcd}$$

[$\tilde{\sigma}^\vee : \mathcal{O}_X \to F^\vee \cong F$ is isotr
isotropic if $\sigma_\cdot$ is isotr

DT4 Cosection Localization

Thm 2. (K.-Park) If X = q-projective DM stack with $\phi: \mathbb{E}^{\bullet} \to \mathbb{L}_X^{\geq -1}$ sym. ob. th. perf. of amplitude $[-2,0]$

admits an isotropic cosection $\sigma_\bullet : \mathbb{E}_\bullet \longrightarrow \mathcal{O}_X[-1]$, then

($\mathcal{O}_X[-1] \xrightarrow{\sigma^\vee} \mathbb{E}[-2] \cong \mathbb{E}^\vee \xrightarrow{\sigma} \mathcal{O}_X[-1]$ is zero)

$\exists \quad [X]^{vir}_{loc} \in A_*(X(\sigma))$ where $X(\sigma) = \mathrm{zero}(\sigma) \xhookrightarrow{\iota} X$

$\sigma = h^1(\sigma_\bullet) : Ob_X \to \mathcal{O}_X$

s.t.

- $\iota_* [X]^{vir}_{loc} = [X]^{vir}$
- deformation invariant.

Recall $\boxed{[X]^{vir} = \sqrt{e}(F|_Y, \tau)\,[C]}$ where
$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[d] \\ \mathcal{C}_X \arrow[r, hook] & {[F/B]} \end{tikzcd} \qquad \begin{tikzcd} Y = C_{red} \arrow[d] \\ X \end{tikzcd}$$

$$\sqrt{e}(F,\tau) : A_*(Y) \to A_{*-n}(X).$$

- $\tilde{\sigma} : F \to \mathcal{O}_X$ induces $\tilde{\sigma}|_Y : F|_Y \to \mathcal{O}_Y \Rightarrow \tilde{\sigma}|_Y^\vee \in H^0(F|_Y)$
  isotropic ... isotropic section

- $E_\cdot \simeq [B \xrightarrow{d} F \xrightarrow{d^\vee} B^\vee]$ $\Longrightarrow$ $Y \subset F(\tilde{\sigma}) \Longrightarrow \tau \cdot \tilde{\sigma}|_Y^\vee = 0$
$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "0"'] & F \arrow[d, "\tilde{\sigma}"] \\ & \mathcal{O}_X \end{tikzcd}$$
cone reduction

$\rightsquigarrow$ $[X]^{vir}_{loc} := \sqrt{e}(F|_Y, \tau ; \tilde{\sigma}|_Y^\vee)\,[C] \in A_*(X(\sigma))$

τ and $\tilde{\sigma}|_Y^\vee$ are independent away from $Y \times_X X(\sigma)$.

$$Y|_{X - X(\sigma)} \subset (\tilde{\sigma}^\vee)^\perp / \langle \tilde{\sigma}^\vee \rangle \Big|_{X - X(\sigma)}$$

orthogonal version of cone reduction

Reduced DT4 invariant

Thm 3. (Ki-Park) If $\exists$ isotropic cosection $\sigma_\bullet : \mathbb{E}_\bullet \to \mathcal{O}_X[-1]$ with $\sigma = h^1(\sigma_\bullet) : Ob_X \twoheadrightarrow \mathcal{O}_X$ surjective

so that $[X]^{vir} = 0 \in A_{vd}(X)$ $\left(\because [X]^{vir}_{loc} \in A_*(X(\sigma)) = A_*(\emptyset)\right)$ $\overset{\shortparallel}{0}$

where $vd = \frac{1}{2}\operatorname{rank}\mathbb{E}^\bullet$,

F

then $\exists$ reduced virtual cycle

$$[X]^{vir}_{red} := \sqrt{e}\left(\underbrace{(\tilde{\sigma}^\vee)^\perp / \langle\tilde{\sigma}\rangle}_{SO(2n-2)}\Big|_Y, \tau\right)[C] \in A_{vd+1}(X).$$

- Orthogonal cone reduction $Y \subset \underbrace{(\tilde{\sigma}^\vee)^\perp}_{\text{isotropic}} / \underbrace{\langle\tilde{\sigma}\rangle}_{\text{isotropic}}$: $SO(2n-2)$-bdle

  ($\exists$ local isotropic section $d^\vee(a) = 0$, $\tilde{\sigma}^\vee \cdot a = 0$
  $Y \subset \langle a, \tilde{\sigma}^\vee\rangle^\perp$)

Vanishing of DT4 for non-isotropic cosection.

Thm 4. (K.-Park) $\sigma_\bullet : E_\bullet \to \mathcal{O}_X[-1]$ cosection, $\sigma_\bullet^2 = \text{constant} \neq 0$

$\Rightarrow \quad [X]^{vir} = 0$

$\left(X = \text{proj. moduli} \Rightarrow \sigma_\bullet^2 \in H^0(X, \mathcal{O}_X) \right.$
$\left. \Rightarrow \sigma_\bullet^2|_{X_{red}} = \text{constant.} \right)$

$\left[\because \ s = \sigma^\vee : \mathcal{O}_Y \to F|_Y = \Lambda \oplus \Lambda^\vee \quad (s', s'') \right.$
$0 \neq s^2 = 2s' s''$
$\left. \sqrt{e}(F|_Y) = e(\Lambda) = 0 \ (\because \exists s' \in H^0(\Lambda) \text{ nowhere zero}) \right]$

Everything discussed so far works with K-theory and virtual structure sheaves!

More generally, virtual intersection theories [arXiv: 1908.03440]

Cosections for DT4 moduli

$At(\mathcal{E}) : \mathcal{E} \to \mathcal{E} \otimes \Omega_W[1]$

① [Cao–Maulik–Toda] $(2,0)$-form $\theta \in H^0(\Omega^2_W)$ on CY4 W $\Rightarrow$ cosection

$$\sigma^\theta_\bullet : Rp_* R\mathcal{H}om(\mathcal{E},\mathcal{E})[1] \xrightarrow{At(\mathcal{E})^2} Rp_* R\mathcal{H}om(\mathcal{E}, \mathcal{E}\otimes\Omega^2_W)[3] \xrightarrow{tr} Rp_*\Omega^2_W[3]$$
$$\xrightarrow{\theta} Rp_*\Omega^4_W[3] \xrightarrow{\tau^{\geq 1}} \mathcal{O}_X[-1]$$

② [R. Thomas] $(3,1)$-form $\delta \in H^1(\Omega^3_W)$ on W $\Rightarrow$ cosection

$$\sigma^\delta_\bullet : Rp_* R\mathcal{H}om(\mathcal{E},\mathcal{E})[1] \xrightarrow{At(\mathcal{E})} Rp_* R\mathcal{H}om(\mathcal{E}, \mathcal{E}\otimes\Omega_W)[2] \xrightarrow{tr} Rp_*\Omega_W[2]$$
$$\xrightarrow{\delta} Rp_*\Omega^4_W[3] \xrightarrow{\tau^{\geq 1}} \mathcal{O}_X[-1]$$

③ [R. Thomas] $(0,2)$-form $\gamma \in H^2(\mathcal{O}_W)$ on W $\Rightarrow$ cosection

$$\sigma^\gamma_\bullet : Rp_* R\mathcal{H}om(\mathcal{E},\mathcal{E})[1] \xrightarrow{tr} Rp_*\mathcal{O}[1] \xrightarrow{\gamma} Rp_*\mathcal{O}[3] \xrightarrow{\omega} Rp_*\Omega^4_W[3] \xrightarrow{\tau^{\geq 1}} \mathcal{O}_X[-1].$$

$\rightsquigarrow$ Localization & vanishing results for DT4!

Thank you for your attention!

Happy New Year!