

Localizing Virtual Cycles for DT4 by Cosections

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§ I. Euler class

$X =$ g -proj. scheme or DM stack

$$A_*(X) = \mathbb{Z} \{ \text{irreducible closed subvarieties in } X \} / \underline{\text{rat. equiv}}$$

- cycle class map $A_k(X) \rightarrow H_{2k}(X)$

- proper pushforward

$$f: X \rightarrow Y \text{ proper} \implies f_*: A_k(X) \rightarrow A_k(Y)$$

- flat pullback

$$f: X \rightarrow Y \text{ flat} \implies f^*: A_k(Y) \rightarrow A_{k+r}(X)$$

- Homotopy invariance

$$\pi: F \rightarrow X \quad \text{VB of rank } r \Rightarrow \pi^*: A_k(X) \xrightarrow{\cong} A_{k+r}(F) \text{ isom}$$

$$[Z] \mapsto [\pi^{-1}(Z)]$$

$$\Rightarrow \underline{0_F^!} = (\pi^*)^{-1}: A_k(F) \rightarrow A_{k-r}(X)$$

$$\Rightarrow e(F): A_k(X) \xrightarrow{0_*} A_k(F) \xrightarrow{0_F^!} A_{k-r}(X) \text{ bivariant}$$

$$\Rightarrow e(F) \in \underline{A^r(X)}$$

- Localization sequence



$$A_k(Z) \xrightarrow{1_*} A_k(X) \xrightarrow{1^*} A_k(X-Z) \rightarrow 0$$

exact

- More nice properties.

Bivariant classes

$$A^k(X \xrightarrow{f} Y)$$

• $\forall Y' \xrightarrow{g} Y$

$$\begin{array}{ccc}
 X' & \xrightarrow{f'} & Y' \\
 g' \downarrow & \square & \downarrow g \\
 X & \xrightarrow{f} & Y
 \end{array}$$

$\exists A_{X'}(Y') \rightarrow A_{X'-1}(X')$

• Compatible with proper pushforward, flat pullback and intersection products

Ex. $e(F) = c_r(F) \in A^r(X)$, Chern classes $c_i(F) \in A^i(X)$

• $A^k(X) := A^k(X \xrightarrow{id} X)$

if $Y \xrightarrow{c} X$ closed, $A^k_Y(X) := A^k(Y \xrightarrow{c} X)$

§2. Square root Euler class

$F = SO(2n)$ -bundle over X

vector bundle of rank $2n$
 $F \otimes_{\mathbb{Q}} F \xrightarrow{q} \mathcal{O}_X$ symmetric bilinear nondegenerate form
 or: $\mathcal{O}_X \rightarrow \det F$ (square root of $\det F \cong \det F^{\vee}$)
orientation

$F' \leq F$ is isotropic if $F' \otimes_{\mathbb{Q}} F' \xrightarrow{q} \mathcal{O}_X$ is 0.

An isotropic F' is maximal if $\text{rank } F' = n$.

A max iso. F' is positive if $\prod_{i=1}^n e_i \wedge f_i = \sqrt{1}^n$ or.

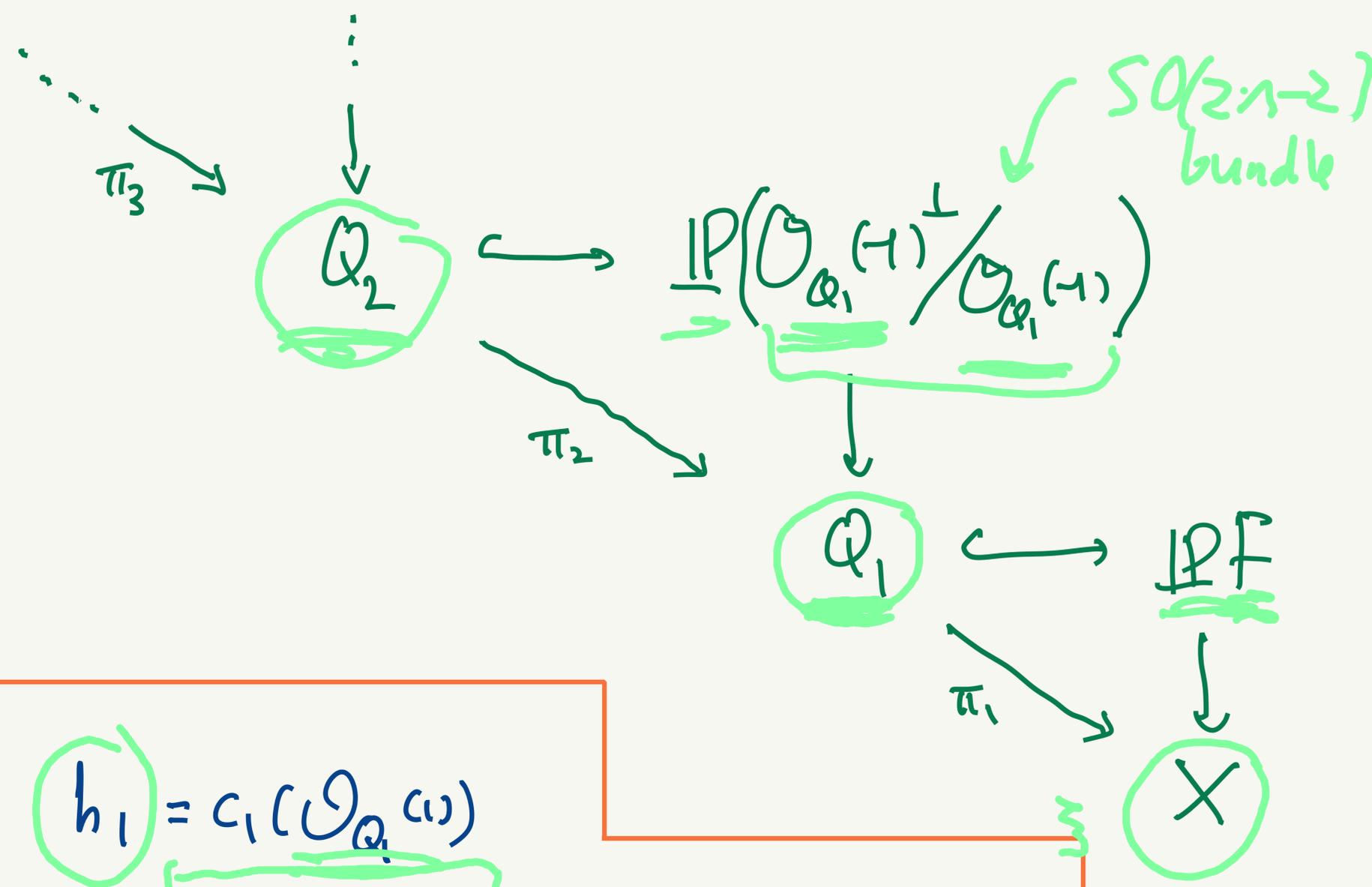
local basis of F' dual basis of (e_i)

Edidin-Graham (1995): \exists square root of $e(F)$

$\sqrt{e}(F) : A_k(X) \rightarrow A_{k-n}(X)$ bivariate class

$\sqrt{e}(F) \in A^n(X)$, $\sqrt{e}(F)^2 = (-1)^n e(F)$

if $\exists \Lambda \leq F$ max. isotropic subbundle, then $\sqrt{e}(F) = e(\Lambda)$. [Fulton's conj.]

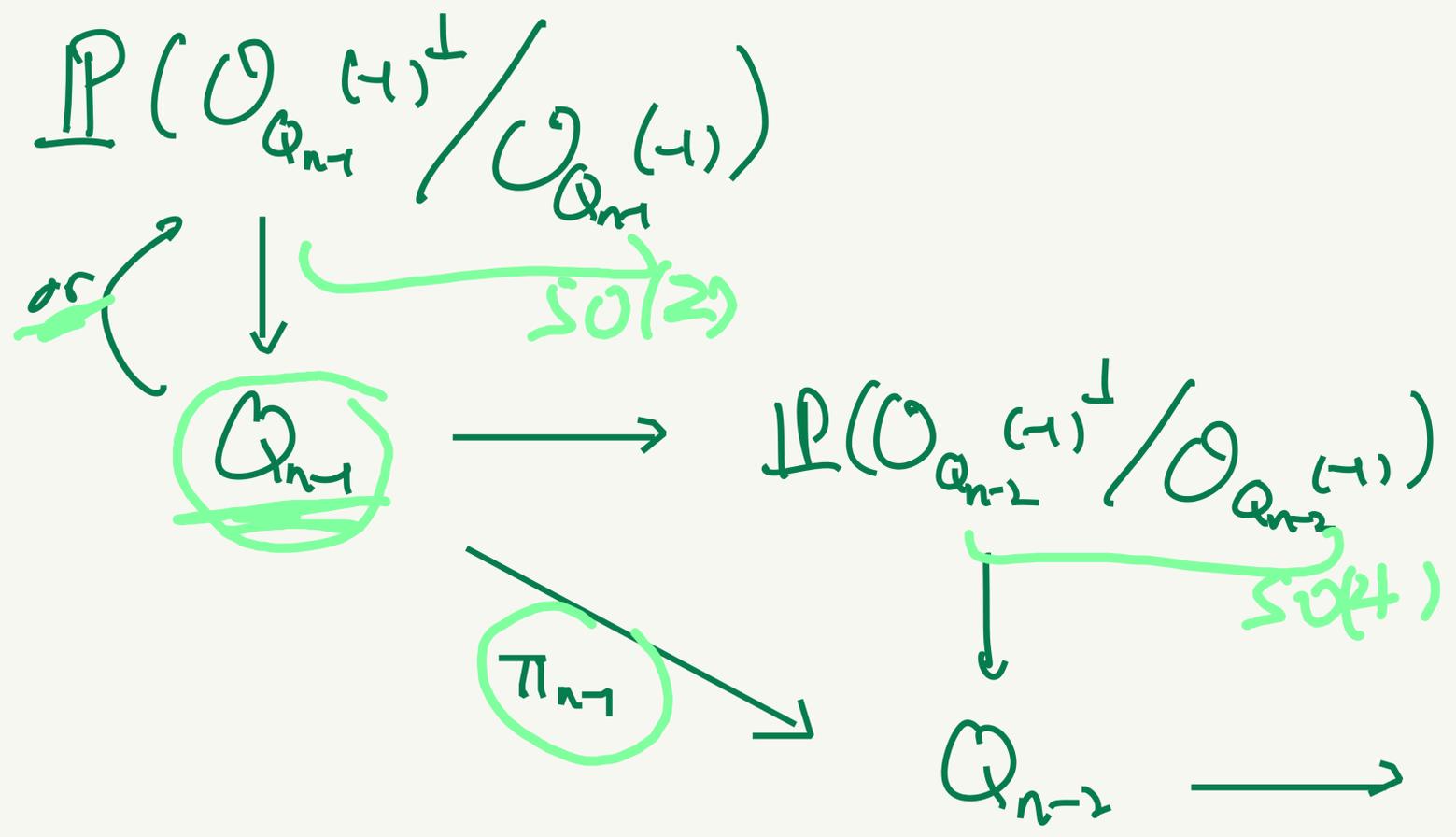


$\mathcal{O}_{Q_1}(-1) \hookrightarrow F|_{Q_1}$ tautological
 $\mathcal{O}_{Q_1}(-1)^\perp$ orthogonal complement w.r.t. g
 $Q_1 \hookrightarrow \mathbb{P}F$ isotropic locus
 quadric hypersurface

$h_1 = c_1(\mathcal{O}_{Q_1}(-1))$

$\pi_{1*} \left(\frac{h_1}{2} \cap \pi_1^*(\zeta) \right) = \zeta \quad \forall \zeta \in A_*(X)$

$\pi_{2*} \left(\frac{h_2}{2} \cap \pi_2^*(\zeta_1) \right) = \zeta_1 \quad \forall \zeta_1 \in A_*(Q_1), \quad h_2 = c_1(\mathcal{O}_{Q_2}(-1))$



$$0 \rightarrow \Lambda \rightarrow Fl_{Q_n} \rightarrow \Lambda^{\vee} \rightarrow 0$$

max. isotropic pos. subbundle of rank n on Q_{n-1}

$$\pi = \pi_1 \circ \pi_2 \circ \dots \circ \pi_{n-1} : Q_n \rightarrow X$$

$$h = \frac{h_{2n-2}}{2} \cdot \frac{h_{2n-4}}{2} \cdot \dots \cdot \frac{h_2}{2}$$

$$\pi_*(h_n \pi^* \xi) = \xi$$

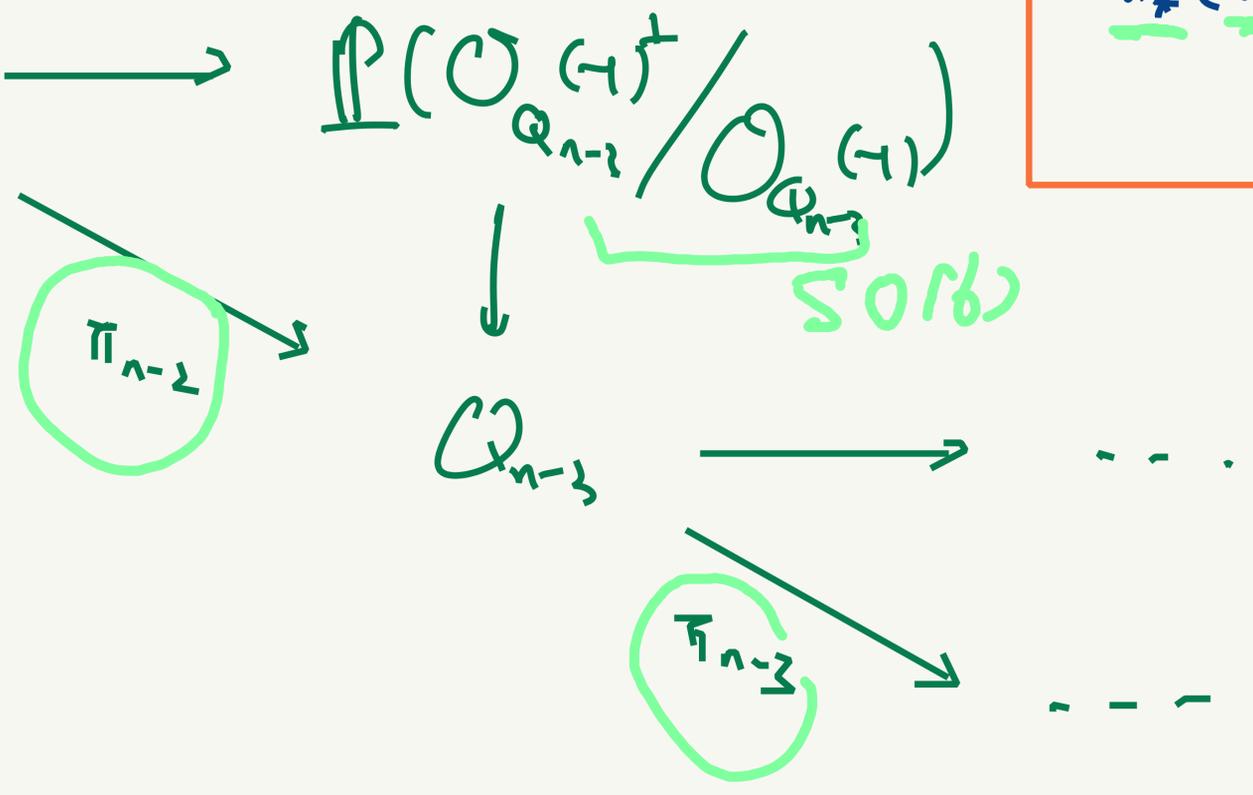
$$\forall \xi \in A_*(X)$$

$$\sqrt{e}(F)_n \xi = \pi_*(e(\Lambda) h_n \pi^*(\xi))$$

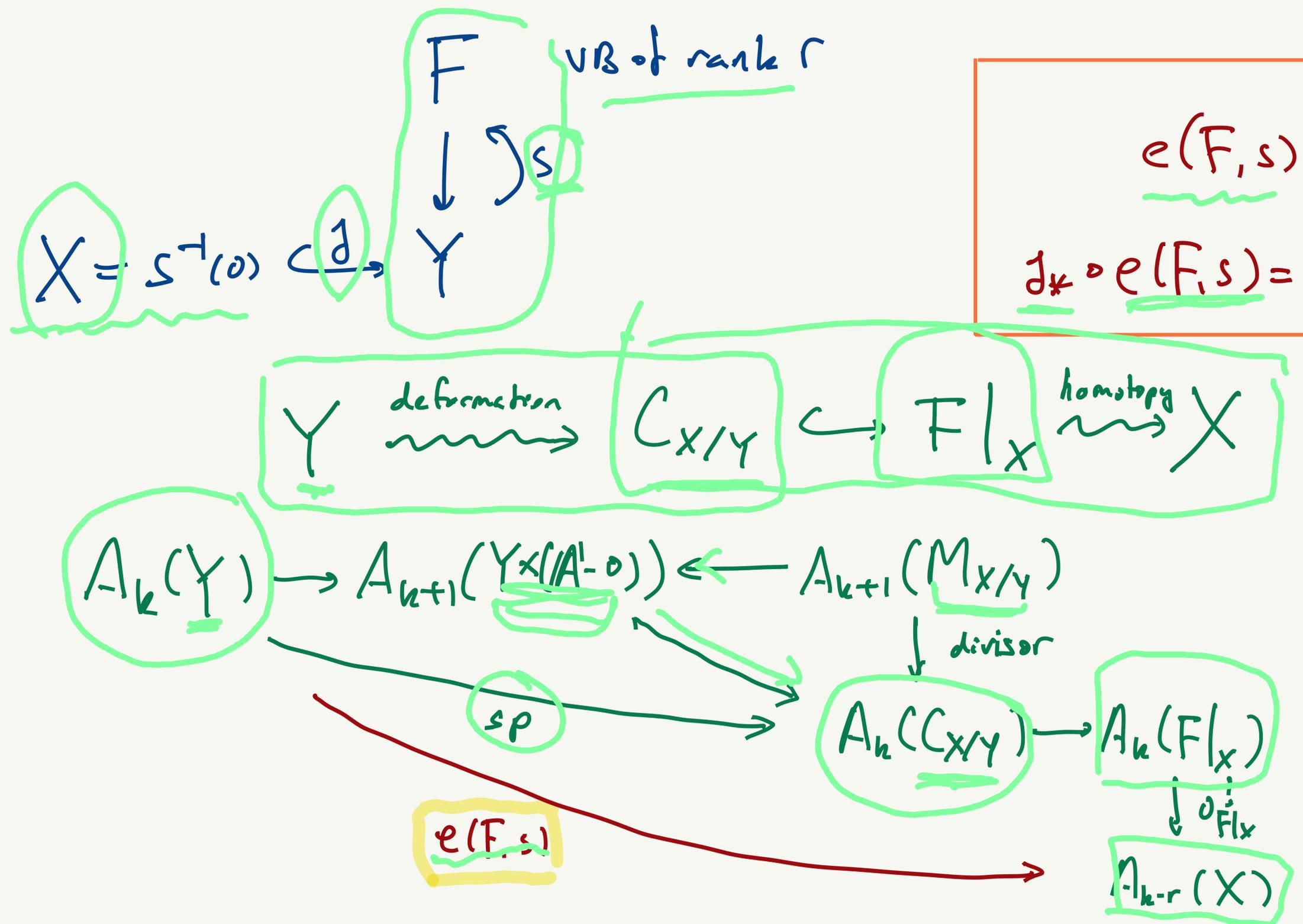
$$\forall \xi \in A_*(X)$$

$$\sqrt{e}(F) : A_k(X) \rightarrow A_{k-n}(X)$$

$$\sqrt{e}(F) \in A^n(X) \quad \text{bivariant class}$$



§3. Localized Euler class



$$e(F, s) : A_k(Y) \rightarrow A_{k-r}(X)$$

$$j_* \circ e(F, s) = e(F) : A_k(Y) \rightarrow A_{k-r}(Y)$$

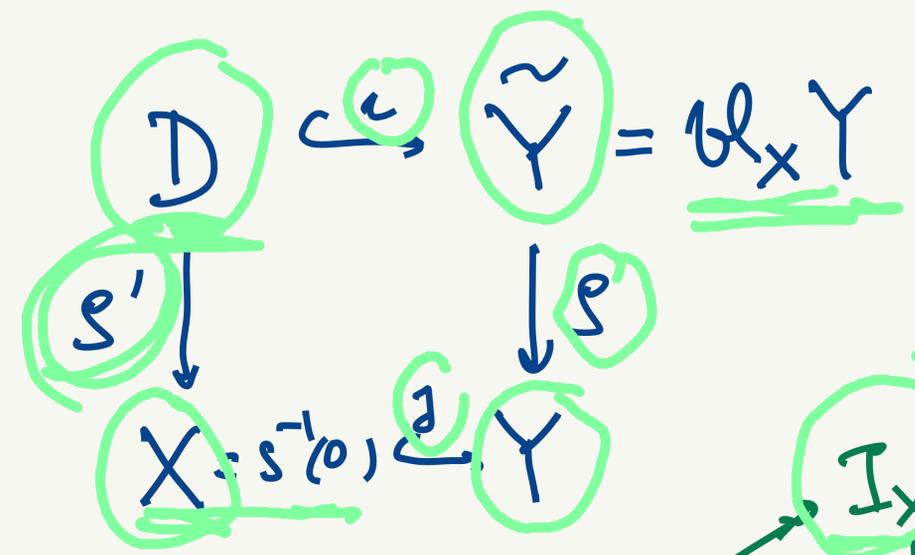
$$M_{X/Y} = \text{bl}_{X \times (0)}(Y \times (A'))$$

— (proper transf. of $Y \times (0)$)

flat \downarrow
 $\leftarrow A'$

$$\left[\begin{array}{l} M_{X/Y} |_{A'=0} \cong Y \times (A'-0) \\ M_{X/Y} |_0 = C_{X/Y} \end{array} \right]$$

Localized Euler class by Blow-up



$$A_*(\tilde{Y}) \oplus A_*(X) \rightarrow A_*(Y)$$

$$(\alpha, \beta) \mapsto \rho_* \alpha + \downarrow_* \beta = \xi$$

$$\mathcal{O}_Y \xrightarrow{s} F \Rightarrow F^\vee \xrightarrow{s^\vee} \mathcal{O}_Y \Rightarrow F^\vee|_{\tilde{Y}} \rightarrow \mathcal{O}_{\tilde{Y}}(-D) \hookrightarrow \mathcal{O}_{\tilde{Y}}$$

$$\Rightarrow 0 \rightarrow \mathcal{O}_{\tilde{Y}}(D) \rightarrow F|_{\tilde{Y}} \rightarrow \bar{F} \rightarrow 0 \quad \text{exact seq. of VB}$$

$\bar{F} = F|_{\tilde{Y}} / \mathcal{O}_{\tilde{Y}}(+D)$

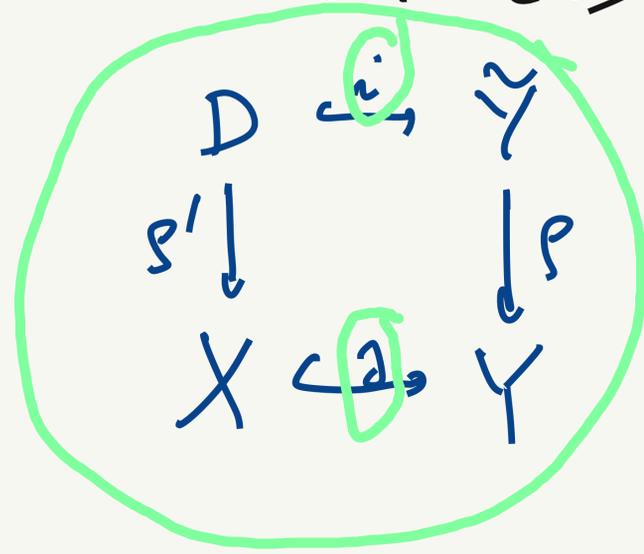
$$e(F, s) \cap \xi = \rho_* (e(\bar{F}) \cap \alpha^* \alpha) + e(F) \cap \beta$$

$$= \rho_* (\iota^* (e(\bar{F}) \cap \alpha)) + e(F) \cap \beta$$

$$e(F, s) : A_k(Y) \rightarrow A_{k-r}(X)$$

Excess intersection formula \Rightarrow

$$\boxed{e(F, s) = \underline{e(F, s)}}$$



- $e(F|_{\tilde{Y}}, s|_{\tilde{Y}}) \cap \alpha = e(\tilde{F}) \cap D \cdot \alpha \quad \forall \alpha \in A_*(\tilde{Y})$
 ($j^! \alpha = e(\tilde{F}) \cap i^! \alpha$ in Fulton's notation)

$$\Rightarrow s'_*(e(\tilde{F}) \cap D \cdot \alpha) = s'_*(e(F|_{\tilde{Y}}, s|_{\tilde{Y}}) \cap \alpha) = \underset{\substack{\uparrow \\ \text{bivariance}}}{e(F, s) \cap s_* \alpha}$$

- $e(F|_X) \cap \beta = \underset{\substack{\uparrow \\ \text{bivariance}}}{e(F, s) \cap s_* \beta}$

$$e(F, s) \cap \xi = e(F, s) (s_* \alpha + j_* \beta) = s'_*(e(\tilde{F}) \cap D \cdot \alpha) + e(F|_X) \cap \beta$$

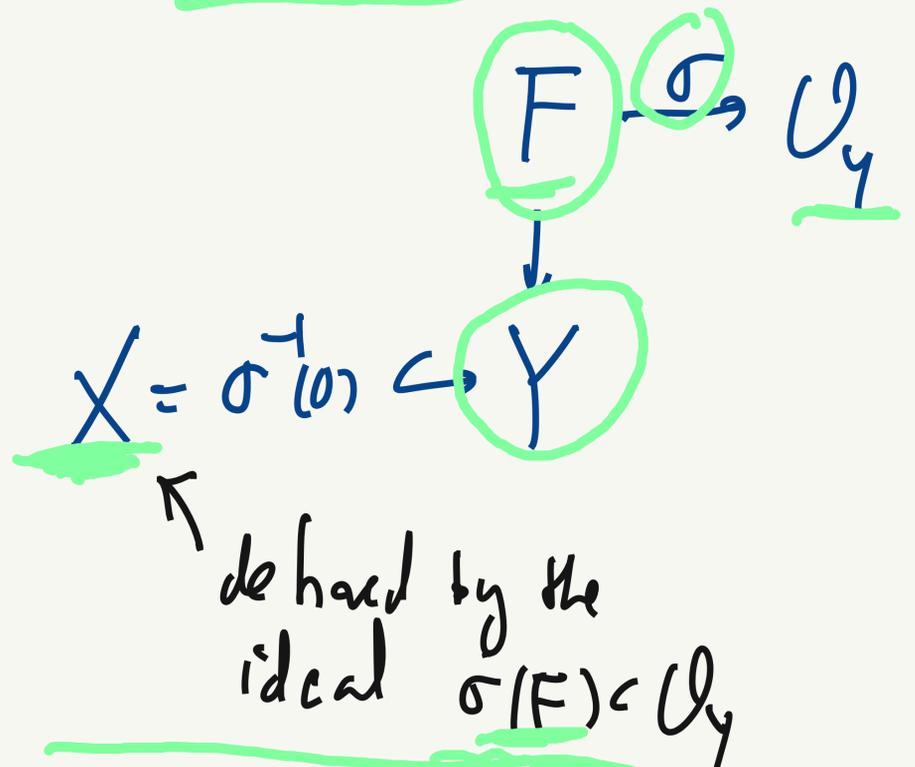
↙ Degeneration to normal cone

$$= e(F, s) (s_* \alpha + j_* \beta) = \underline{e(F, s) \cap \xi}$$

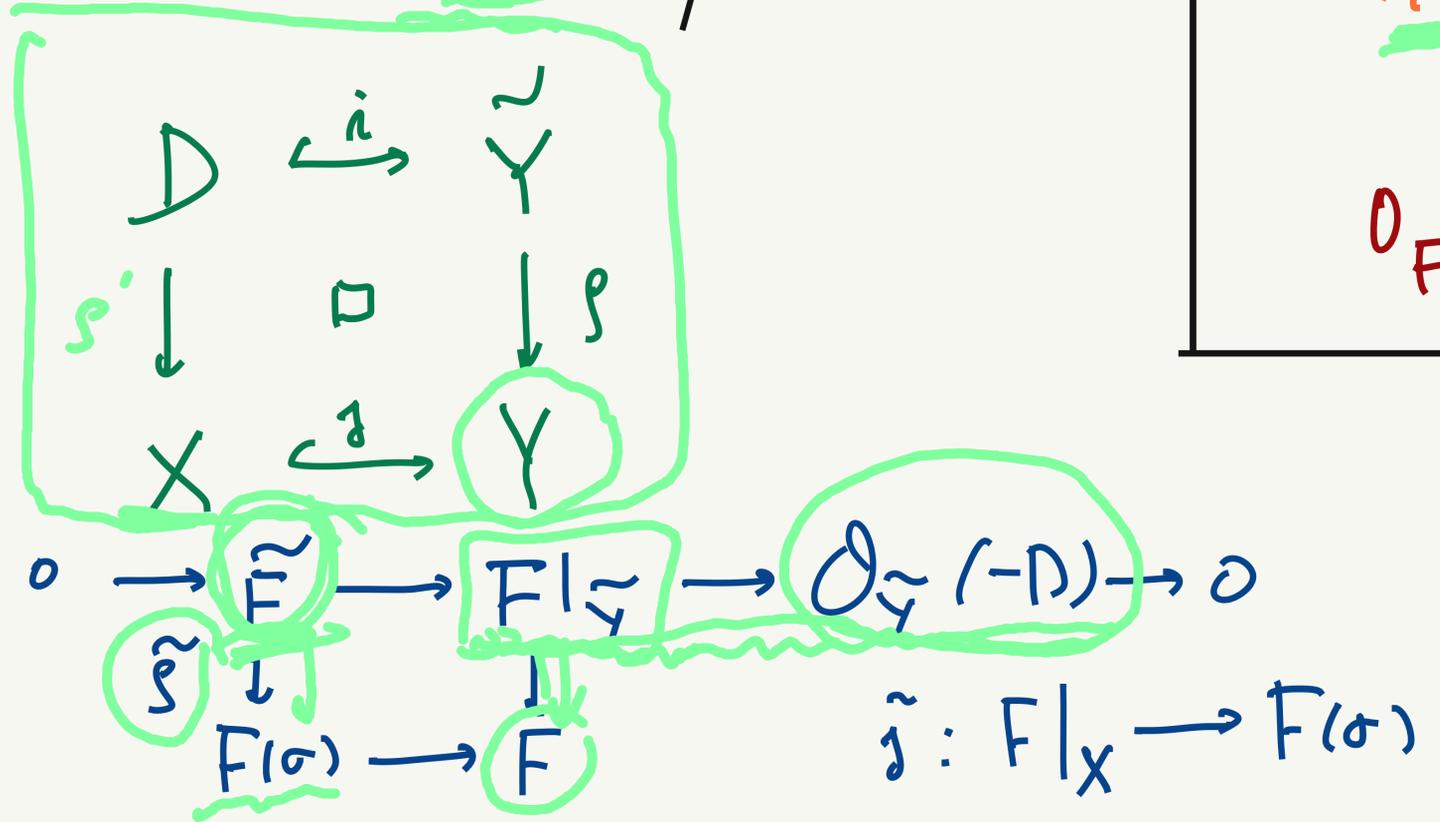
↙ Blowup

Cosection Localized Gysin map (Blowup construction)

$$F(\sigma) := F|_X \cup \ker(\sigma|_{Y-X} : F|_{Y-X} \rightarrow \mathcal{O}_{Y-X})$$

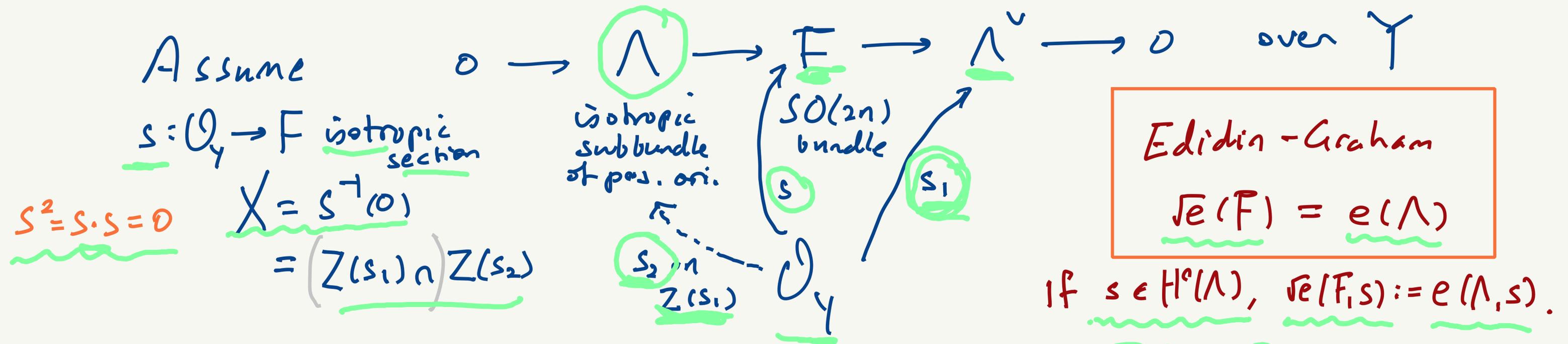


$$\begin{aligned}
 & \circ_{F, \sigma}^! : A_* (F(\sigma)) \longrightarrow A_{*-r} (X) \\
 & \quad \quad \quad \uparrow (\tilde{\mathcal{F}}_*, \tilde{\mathcal{S}}_*) \\
 & (\beta, \alpha) \in A_* (F|_X) \oplus A_* (\tilde{F}) \\
 & \circ_{F, \sigma}^! (\tilde{\mathcal{S}}_* \alpha + \tilde{\mathcal{J}}_* \beta) = -\tilde{\mathcal{S}}_* (D \cdot e(\tilde{F}) \alpha) + \circ_{F|_X}^! \beta
 \end{aligned}$$

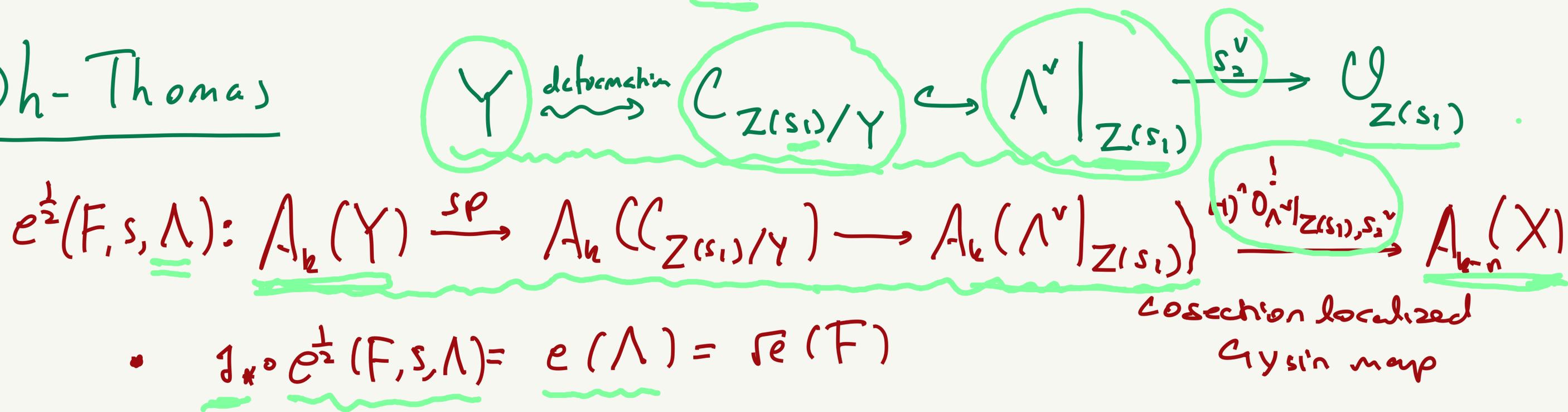


- * Well defined
- * $\tilde{\mathcal{J}}_* \circ \circ_{F, \sigma}^! = \circ_{F|_X}^!$
- * Works for K-theory, algebraic cobordism, etc
 [K.-Park, arXiv: 1908.03440]

§4. Localized square root Euler class by Oh-Thomas



Oh-Thomas

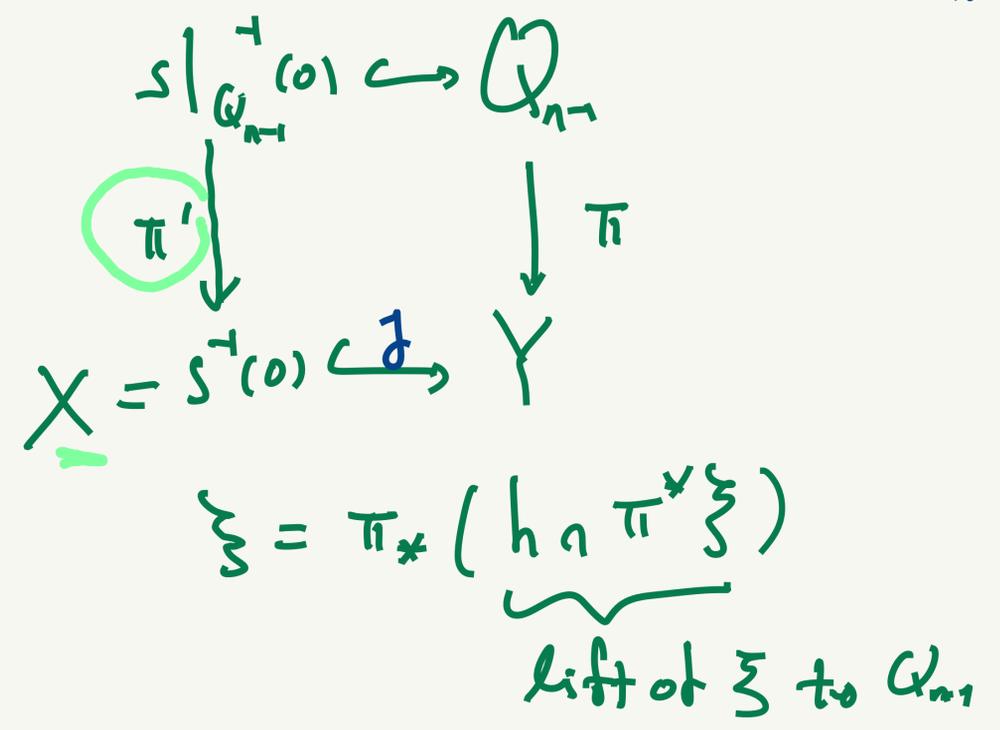


For a general SO(2n)-bundle F and an isotropic section S,

use the isotropic flag variety $\pi: Q_n \rightarrow Y$ such that \exists exact seq

$$0 \rightarrow \Lambda \rightarrow F|_{Q_n} \rightarrow \Lambda^\vee \rightarrow 0, \quad \Lambda = \text{positive max isotropic subbundle of } F|_{Q_n}$$

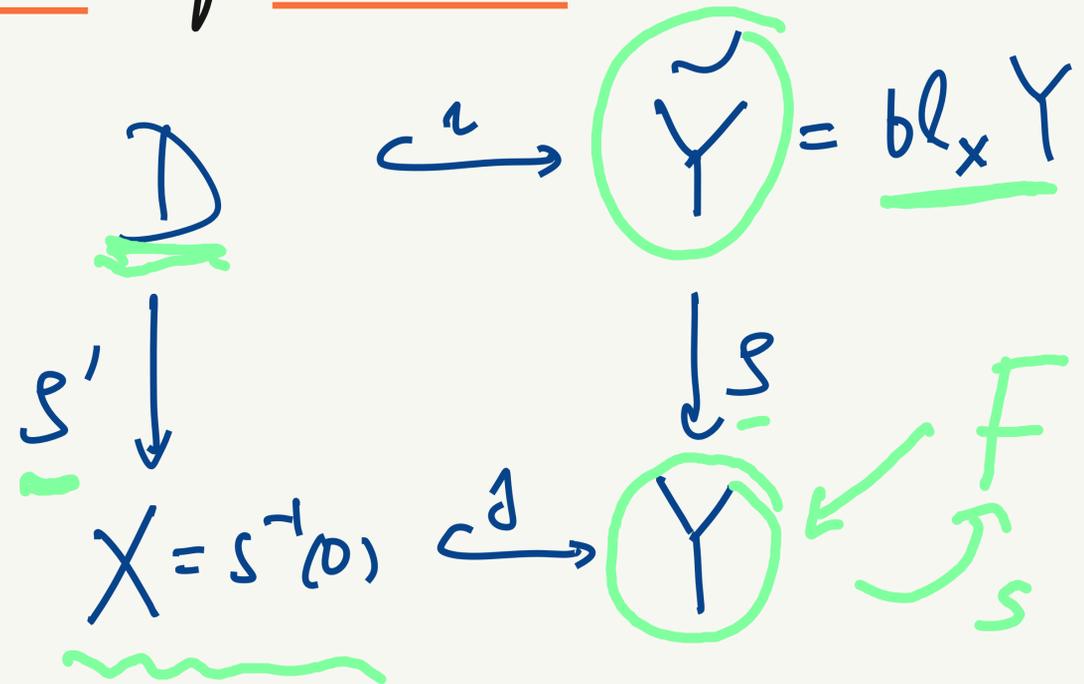
$$e^{\frac{1}{2}}(F, s)_n \xi = \pi'_* \left(e^{\frac{1}{2}}(F|_{Q_n}, s|_{Q_n}, \Lambda) \wedge \pi^* \xi \right)$$



- $\int_* \circ e^{\frac{1}{2}}(F, s) = \int(F)$
- $\sqrt{0}_F := e^{\frac{1}{2}}(F|_{\hat{Q}_1}, \tau) : A_*(\hat{Q}_1) \rightarrow A_{*-n}(Y)$
 where τ is the tautological section and \hat{Q}_1 is a quadric hypersurface of isotropic vectors.

~~*~~ This construction is not suitable for a further localization! need it for cosection localization

Localized square root Euler class by Blowup



$$A_k(Y) \leftarrow A_k(\tilde{Y}) \oplus A_k(X)$$

$$\tilde{\zeta} = \rho_* \alpha + j_* \beta$$

$$F^\vee \rightarrow \mathcal{O}_{\tilde{Y}}(-D)$$

$$s: \mathcal{O}_Y \rightarrow F \text{ isotropic} \Rightarrow L := \mathcal{O}_{\tilde{Y}}(D) \hookrightarrow F|_{\tilde{Y}}$$

Isotropic subbundle

Let $\tilde{F} = L^\perp / L$ $SO(2n-2)$ -bundle.

Def. (K. Park) $\sqrt{e}(F, s) \cap \tilde{\zeta} := \rho'_*(\sqrt{e}(\tilde{F}) \cap \alpha^* \alpha) + \sqrt{e}(F) \cap \beta$

for $\tilde{\zeta} = \rho_* \alpha + j_* \beta \in A_k(Y)$ $\overset{\cap}{\longleftarrow} A_*(X)$

Thm¹ (K.-Park) (1) $\sqrt{e}(F, s)$ is independent of α, β .

(3) $\sqrt{e}(F, s) = e^{\frac{1}{2}}(F, s)$. $\sqrt{e}(F, s) \xi = \int_{D \cdot \alpha} (\sqrt{e}(\tilde{F}) \cap z^* \alpha) + \int_{\beta} \sqrt{e}(F) \cap \beta$
 $\xi = \int_{D \cdot \alpha} + \int_{\beta}$

(2) $\sqrt{e}(F, s) \in A_{X-n}^n(Y)$ is a bivariant class.

(4) If t is another isotropic section of F with $s \cdot t = 0$, then $\sqrt{e}(F, s)$ is further localized to $orth.$

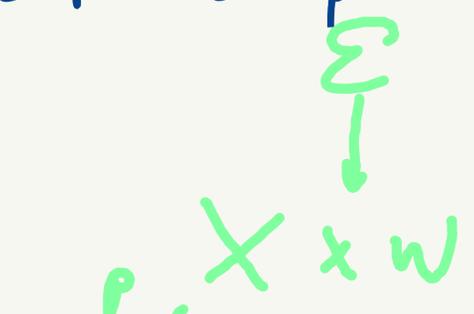
$\sqrt{e}(F, s; t) : A_*(Y) \rightarrow A_{X-n}(X \cap Z)$ $Y \rightarrow Z$

where $Z \subset Y$ is closed and s, t are independent away from Z.
 $t^{-1}(0)$ \uparrow locally on $Y-Z, F = F' \oplus F''$
 $(s) \in H^0(F'), (t) \in H^0(F'')$

$\sqrt{e}(F, s; t) \xi = \int_{D \cdot \alpha} (\sqrt{e}(\tilde{F}, \tilde{t}) \cap z^* \alpha) + \int_{\beta} \sqrt{e}(F, t) \cap \beta$
 $\xi = \int_{D \cdot \alpha} + \int_{\beta}, \tilde{F} = L^\perp / L \ (L = \mathcal{O}_{\tilde{Y}}(D) \leq F|_{\tilde{Y}}), \int_{\tilde{Y}} \tilde{t} \in H^0(L^\perp) \rightarrow \tilde{t} \in H^0(\tilde{F})$

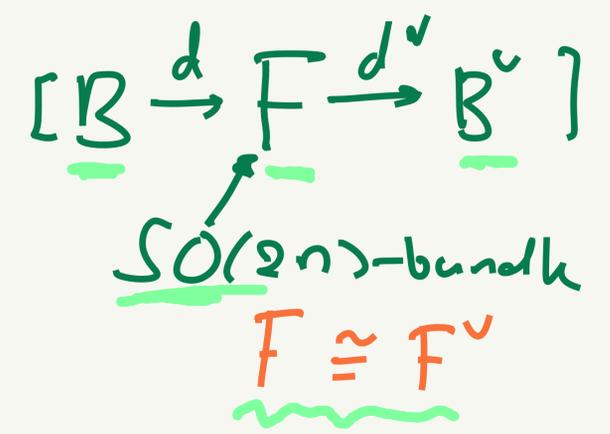
§§. Virtual cycles for DT4 invariants

$X =$ quasi-projective moduli space of stable sheaves or simple perfect complexes over a Calabi-Yau 4-fold W



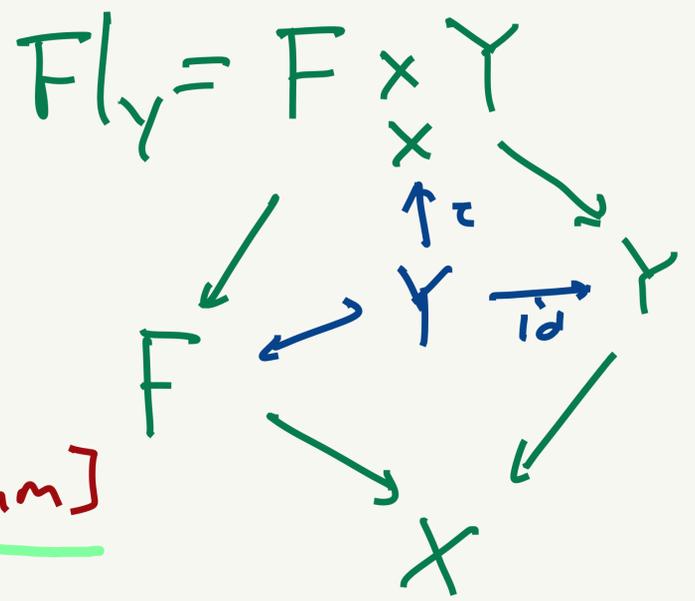
$Ext^0(\mathcal{E}, \mathcal{E}), \exists$ 3-term symmetric obst. th on X
 $\mathbb{E}^\bullet = R\rho_* R\theta_* \mathcal{E}^\bullet[1]$ $\mathbb{E}^\bullet =$ perfect of amplitude $[-2, 0]$
 $\phi: \mathbb{E}^\bullet \rightarrow \mathcal{L}_X$, $h^0(\phi)$ isom, $h^1(\phi)$ surj. ✓
 $\mathbb{E}^\bullet[2] \xrightarrow[\theta]{\cong} \mathbb{E}^\bullet$, $\theta^\bullet[2] = \theta$ ✓

Thm (Oh-Thomas) \exists 3-term ex of loc. free sheaves on X
 $\mathbb{E}^\bullet \cong [B \xrightarrow{d} F \xrightarrow{d^\bullet} B^\bullet]$ per. obs. th. on X



Behrend - Fantechi : \exists intrinsic normal cone $\mathcal{C}_Y \hookrightarrow [F/B] \xleftarrow{F}$
 (if $U \hookrightarrow V^{sm}$, $\mathcal{E}_X|_U \cong [C_{U/V}/TV|_U]$
 $\downarrow \text{étale}$
 X)

$Y := C_{red} \subset C := \mathcal{C}_X \times_{F/B} F \hookrightarrow F$
 \downarrow
 X



tantological section $\tau: Y \rightarrow F|_Y \in H^0(F|_Y)$
 $\tau^{-1}(0) = X$

isotropic section
 [BBB], Darboux thm

Def (Oh-Thomas)

$[X]^{vir} := \sqrt{e}(F|_Y, \tau) \cap [C] \in A_*(Y)$

Thm (Oh-Thomas)

Same as Borisov-Joyce ; Deformation inv. Torus localization, ...

DT4 invariant = counting sheaves on CY4.

$$DT4 = \int_{\mathcal{X}} \eta, \quad \eta \in \underline{A^*(X)}$$

$X = \mathfrak{g}\text{-proj. moduli sp of simple complexes on CY4.}$

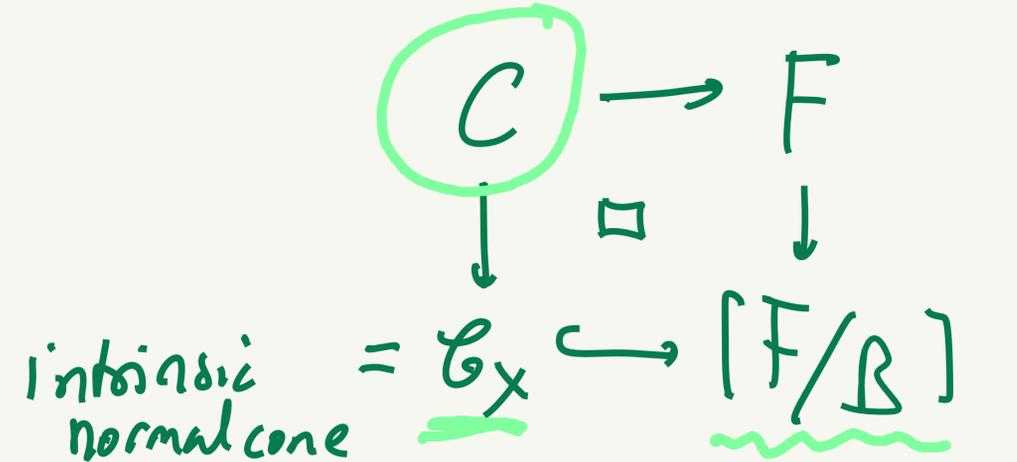
History y 1990s Donaldson - Thomas

$$\underline{Ext^1(E, E)_0} \rightarrow \underline{Ext^2(E, E)_0^+} \oplus \underline{Ext^2(E, E)_0^-}$$

- 2) ^{Early 2010s} Cao - Leung constructed vir. cycles for e
 some cases like moduli of vector bundles.
- 3) ^{Mid 2010s} Borisov - Joyce constructed vir. cycles using ^(real) differential ^{derived} geometry.
- 4) ²⁰²⁰ Oh - Thomas constructed algebraic vir. cycle by $e_2^{\pm}(F, s)$.
- 5) ^{Late 2010s} Cao - Maulik - Toda conjectured connections with GW. (GV inv.)

Cosection Localization of virtual cycles: 2-term case.

• $X = DM$ stack, $\mathbb{E}^\bullet = [F^\vee \xrightarrow{d^\vee} B^\vee] \xrightarrow{\phi} \mathbb{L}_X^{\geq -1}$ obstruction theory
 $h^0(\phi)$ via $h^1(\phi)$ surj.



$\Rightarrow [X]^{vir} := \mathcal{O}_F^! [C] \in A_*(X)$

• $\sigma: \mathcal{O}_X = h^1(\mathbb{E}^\bullet) \rightarrow \mathcal{O}_X$ cosection $(\mathbb{E}^\bullet = (\mathbb{E}^\bullet)^\vee)$

$\Rightarrow \tilde{\sigma}: F \rightarrow \text{coker } d = \mathcal{O}_X \xrightarrow{\sigma} \mathcal{O}_X$

1) cone reduction

2) cosection localized Gysin map

$C_{red} \subset F(\tilde{\sigma}) \Rightarrow [C] \in A_*(F(\tilde{\sigma}))$
 $[X]_{loc}^{vir} := \mathcal{O}_{F, \tilde{\sigma}}^! [C] \in A_*(X(\sigma))$

$[X]_{loc}^{vir}$ turned out to be quite useful. (e.g. FJRW, MSP, ...)

Question. Cosection localization for DT4 vir. cycles?

Cosection of Obstruction Theory

• 2-term

$$\phi: \mathbb{E}^\bullet \rightarrow \mathcal{L}_X^{\geq -1} \text{ perf. ob. th.}, \quad \mathbb{E}_\bullet = (\mathbb{E}^\bullet)^\vee$$

locally isom. to a 2-term ex $[\mathcal{B}_1 \rightarrow \mathcal{B}_0]$ of loc. free sh.

✓ $\sigma: \text{Ob}_X = h^1(\mathbb{E}_\bullet) = \tau^{\geq 1} \tau^{\leq 1} \mathbb{E}_\bullet \rightarrow \mathcal{O}_X$ cosection of the obstruction sheaf Ob_X .

$\Rightarrow \sigma: \mathbb{E}_\bullet \xleftarrow{\cong} \tau^{\leq 1} \mathbb{E}_\bullet \xrightarrow{\tau^{\geq 1} \tau^{\leq 1}} \tau^{\geq 1} \tau^{\leq 1} \mathbb{E}_\bullet = \text{Ob}_X[-1] \xrightarrow{\sigma} \mathcal{O}_X[-1]$
part of amplitude $[0,1]$

✓ Conversely, a morphism $\mathbb{E}_\bullet \xrightarrow{\sigma} \mathcal{O}_X[-1]$ induces

$\Rightarrow \sigma = h^1(\sigma): \text{Ob}_X = h^1(\mathbb{E}_\bullet) \rightarrow h^1(\mathcal{O}_X[-1]) = \mathcal{O}_X$.

Def. Let $\phi: \mathbb{E}^\bullet \rightarrow \mathcal{L}_X^{\geq -1}$ be an obstruction theory ($h^0(\phi)$ isom, $h^1(\phi)$ surj.)

A cosection of the obstr. th. is a morphism $\sigma: \mathbb{E}_\bullet = (\mathbb{E}^\bullet)^\vee \rightarrow \mathcal{O}_X[-1]$ ✓

A cosection of an obstruction theory induces a cosection of the obstruction sheaf.

$$[\sigma: \mathbb{E} \rightarrow \mathcal{O}_X[-1] \Rightarrow \sigma = h^*(\sigma) = \text{Ob}_X \rightarrow \mathcal{O}_X.]$$

If $\mathbb{E}' \cong \mathbb{E}[2]$ symmetric,
a cosection $\sigma: \mathbb{E} \rightarrow \mathcal{O}_X[-1]$ is isotropic if $\mathcal{O}_X[-1] \xrightarrow{\sigma^\vee} \mathbb{E}'[-2] \cong \mathbb{E}[-2] \rightarrow \mathcal{O}_X[-1]$ is 0

Lemma. Let $\mathbb{E} \xrightarrow{\phi} \mathbb{L}_X^{\geq -1}$ be a symmetric obstruction theory of amplitude $[-2, 0]$ over g -proj. X .

Let $\sigma: \mathbb{E} \rightarrow \mathcal{O}_X[-1]$ be a cosection. SO(2n)-bundle

Then \exists global resolution loc. free $\mathbb{E} \cong [B \xrightarrow{d} F \cong F^\vee \xrightarrow{d^\vee} B^\vee]$

and a homomorphism $\tilde{\sigma}^\vee: F \rightarrow \mathcal{O}_X$ such that

$$\sigma \downarrow \mathbb{E} = [B \xrightarrow{d} \underbrace{F}_{\cong \mathcal{O}_X} \xrightarrow{\tilde{\sigma}^\vee} F^\vee \xrightarrow{d^\vee} B^\vee]$$

$$\mathcal{O}_X[-1] = \mathcal{O}_X$$

[$\tilde{\sigma}^\vee: \mathcal{O}_X \rightarrow F^\vee \cong F$ is isotropic if σ is isotr.]

DT4 Cosection Localization

Thm 2. (K.-Pack) If $X =$ DM stack with g-projective $\phi: E \rightarrow L_x^{\geq 1}$ sym. ob. H. perf. of amplitude [-2, 0]

admits an isotropic cosection $\sigma: E \rightarrow \mathcal{O}_X[-1]$, then
($\mathcal{O}_X[-1] \xrightarrow{\sigma^v} E[-2] \cong E^v \xrightarrow{\sigma} \mathcal{O}_X[-1]$ is zero)

\exists $[X]_{loc}^{vir}$ $\in A_{*}(X(\sigma))$ where $X(\sigma) = \text{zero}(\sigma) \hookrightarrow X$
 $\sigma = h'(\sigma.): \sigma_b \rightarrow \mathcal{O}_X$

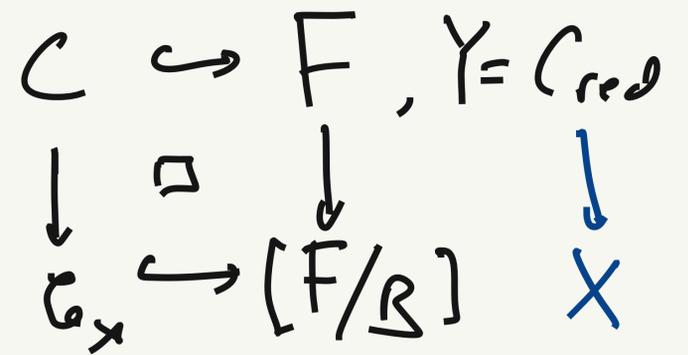
- s.t.
- $\mathcal{L}_* [X]_{loc}^{vir} = [X]^{vir}$
 - deformation invariant.

Recall

$$[X]^{vir} = \sqrt{e}(F|_Y, \tau) [C]$$

where

$$\sqrt{e}(F, \tau) : A_*(Y) \rightarrow A_{*-n}(X).$$



- $\tilde{\sigma} : F \rightarrow \mathcal{O}_X$ induces $\tilde{\sigma}|_Y : F|_Y \rightarrow \mathcal{O}_Y \Rightarrow \tilde{\sigma}|_Y^v \in H^0(F|_Y)$
isotropic isotropic section

- $\mathbb{E} \equiv [B \xrightarrow{d} F \xrightarrow{d'} B'] \Rightarrow Y \subset F(\tilde{\sigma}) \Rightarrow \tau \cdot \tilde{\sigma}|_Y^v = 0$
 $\begin{array}{ccc} & & \mathcal{O}_X \\ & \searrow & \downarrow \tilde{\sigma} \\ & & \mathcal{O}_X \end{array}$
cone reduction

\rightsquigarrow

$$[X]_{loc}^{vir} := \sqrt{e}(F|_Y, \tau; \tilde{\sigma}|_Y^v) [C] \in A_*(X(\sigma))$$

τ and $\tilde{\sigma}|_Y^v$ are independent away from $Y \times X(\sigma)$.

$$Y|_{X-X(\sigma)} \subset \frac{(\tilde{\sigma}^v)^\perp}{\langle \tilde{\sigma}^v \rangle} |_{X-X(\sigma)}$$

orthogonal version of cone reduction

Reduced DT4 invariant

Thm 3. (K. Park) If \exists isotropic cosection

$\sigma : \mathbb{E} \rightarrow \mathcal{O}_X[1]$ with $\sigma = h^1(\sigma) : \mathcal{O}_b \rightarrow \mathcal{O}_X$ surjective

so that $[X]^{vir} = 0 \in A_{vd}(X)$
where $vd = \frac{1}{2} \text{rank } \mathbb{E}$,

($\because [X]^{vir} \in A_{*}(X(\sigma)) = A_{*}(\emptyset)$)

then \exists reduced virtual cycle

(F)

$$[X]_{red}^{vir} := \sqrt{e} \left(\frac{(\tilde{\sigma}^v)^\perp}{\langle \tilde{\sigma} \rangle} \Big|_{Y, z} \right) [C] \in A_{vd+1}(X).$$

• Orthogonal cone reduction

$Y \subset \frac{(\tilde{\sigma}^v)^\perp}{\langle \tilde{\sigma} \rangle} : SO(2n-2)$ -bundle
isotropic isotropic

(\exists local isotropic section
 $d^v(a) = 0, \tilde{\sigma}^v \cdot a = 0$
 $Y \subset \langle a, \tilde{\sigma}^v \rangle^\perp$)

Cosections for DT4 moduli

$$A_{\theta}(E) : E \rightarrow E \otimes \Omega_W[1]$$

① [Cao-Maulik-Toda] $(2,0)$ -form $\theta \in H^0(\Omega_W^2)$ on CY4 $W \Rightarrow$ cosection

$$\sigma^{\theta} : R\rho_* R\mathcal{H}om(E, E)[1] \xrightarrow{A_{\theta}(E)^2} R\rho_* R\mathcal{H}om(E, E \otimes \Omega_W^2)[3] \xrightarrow{tr} R\rho_* \Omega_W^2[3] \\ \xrightarrow{\theta} R\rho_* \Omega_W^4[3] \xrightarrow{\tau^{\geq 1}} \mathcal{O}_X[-1]$$

② [R. Thomas] $(3,1)$ -form $\delta \in H^1(\Omega_W^3)$ on $W \Rightarrow$ cosection

$$\sigma^{\delta} : R\rho_* R\mathcal{H}om(E, E)[1] \xrightarrow{A_{\theta}(E)} R\rho_* R\mathcal{H}om(E, E \otimes \Omega_W)[2] \xrightarrow{tr} R\rho_* \Omega_W[2] \\ \xrightarrow{\delta} R\rho_* \Omega_W^4[3] \xrightarrow{\tau^{\geq 1}} \mathcal{O}_X[-1]$$

③ [R. Thomas] $(0,2)$ -form $\gamma \in H^2(\mathcal{O}_W)$ on $W \Rightarrow$ cosection

$$\sigma^{\gamma} : R\rho_* R\mathcal{H}om(E, E)[1] \xrightarrow{tr} R\rho_* \mathcal{O}[1] \xrightarrow{\gamma} R\rho_* \mathcal{O}[3] \xrightarrow{\omega} R\rho_* \Omega_W^4[3] \xrightarrow{\tau^{\geq 1}} \mathcal{O}_X[-1]$$

\rightsquigarrow Localization & vanishing results for DT4!

Thank you for
your attention!

Happy New Year!