Moduli of G-constellations and crepant resolutions for abelian groups

§1 Introduction

Let $G \subset SL_n(\mathbb{C})$ be a finite subgp.

A basic question is :

<u>Q</u> How to construct a <u>crepant</u> resolution $\pi : X \to \mathbb{C}^n/G$ of the singular variety $\mathbb{C}^n/G$ when it exists ?

$K_X = \pi^* K_{\mathbb{C}^n/G}$

In dimension 2, $\mathbb{C}^2/G$ are the ADE-singularities and their crepant resols are the same as the minimal resols., which can be obtained by repeating blowing up the singular points.

There is another construction of crep. resols of $\mathbb{C}^2/G$ which is called the "G-Hilbert scheme."

This is defined as

$$G\text{-Hilb} = \left\{ Z \subset \mathbb{C}^2 \;\middle|\; \begin{array}{l} Z : G\text{-invariant closed subscheme} \\ \text{s.t. } H^0(\mathcal{O}_Z) \cong \mathbb{C}G \text{ as } G\text{-mod} \end{array} \right\}$$

Thm. (Ito-Nakamura)

The Hilbert-Chow morphism

$$\begin{array}{rcl} \pi : G\text{-Hilb} & \to & \mathbb{C}^2/G \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ Z & \mapsto & \mathrm{Supp}(Z) \end{array}$$

gives a crep. resol of $\mathbb{C}^2/G$. ⌟

Rem. π is an isomorphism over the set of free G-orbits

$U = \{ G\cdot x \mid |G\cdot x| = |G| \} \subset \mathbb{C}^2/G$. ⌟

In higher dimensions, G-Hilb is defined in the same way, and it is still a candidate for a crep. resol of $\mathbb{C}^n/G$.

Thm. ($n=3$ case. Nakamura (G: abelian). Bridgeland-King-Reid (G: general))

The Hilbert-Chow morphism $\pi : G\text{-Hilb} \to \mathbb{C}^3/G$ gives a crep. resol of $\mathbb{C}^3/G$. ⌟

However, when $n \geq 4$, the G-Hilbert scheme does not give a crepant resol in general (even if such a resol exists).

Moreover, when $n \geq 3$, crep. resols are not unique in general.

G-Hilb can be considered as a moduli space $\mathcal{M}_{\theta_0}$ of quiver reps with a certain stability condition θ_0 and other candidates $\mathcal{M}_\theta$ of crep. resols of $\mathbb{C}^n/G$ are obtained by varying stability conditions θ.

The moduli space $\mathcal{M}_\theta$

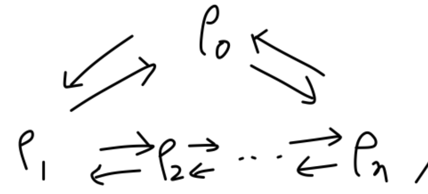
For finite $G \subset SL_n(\mathbb{C})$, we associate to it a quiver Q_G (the McKay quiver) as follows:

Let $\mathrm{Irr}(G) = \{\rho_0, \cdots, \rho_k\}$ be the set of (isom. classes of) irr. reps of G, and let ρ_{st} be the standard rep coming from the inclusion $G \hookrightarrow SL_n(\mathbb{C})$.

The quiver Q_G consists of the vertices v_i indexed by $\mathrm{Irr}(G)$ and $a_{i,i'}$ arrows $v_i \to v_{i'}$ where $a_{i,i'} \in \mathbb{Z}_{\geq 0}$ are defined by $\rho_i \otimes \rho_{st} \cong \bigoplus_{i'} \rho_{i'}^{\oplus a_{i,i'}}$.

Rem. When $n=2$, Q_G are the affine Dynkin quivers of type ADE.

(e.g. type A_n : $G \cong \mathbb{Z}/(n+1)\mathbb{Z}$



)

Let $R = \bigoplus_{\rho \in \mathrm{Irr}(G)} R_\rho \otimes \rho$ be the regular rep. of G and put $V = \mathbb{C}^n$.

Then the space of repsentations of Q_G can be identified with $\mathrm{Hom}_G(R, V \otimes R)$. If $B \in \mathrm{Hom}_G(R, V\otimes R)$ satisfies $B \wedge B = 0$ in $\mathrm{Hom}_G(R, \wedge^2 V \otimes R)$, then B endows $\underline{R \text{ with a structure of a } G\text{-equivariant } \mathbb{C}[V]\text{-module}}$.

called a G-constellation

$\mathcal{M}_\theta$ is obtained as a GIT quotient of $\mathcal{R} := \{ B \in \mathrm{Hom}_G(R, V \otimes R) \mid B \wedge B = 0 \}$ by the action of $\mathrm{Aut}_G(R) = \prod_{\rho \in \mathrm{Irr}(G)} GL(R_\rho)$.

The scalar subgp $\mathbb{C}^* \subset \mathrm{Aut}_G(R)$ acts trivially, and the character group of $\mathrm{Aut}_G(R)/\mathbb{C}^*$ is naturally identified with $\Theta := \{ \theta \in \mathrm{Hom}_{\mathbb{Z}}(\underline{R(G)}, \mathbb{Z}) \mid \theta(R) = 0 \}$.

the rep. ring of G

<u>Thm.</u> (King)

Take $\theta \in \Theta$ and let χ_θ be the corresponding character of $\mathrm{Aut}_G(R)/\mathbb{C}^*$. Then a G-constellation B is χ_θ-semistable (resp. χ_θ-stable) in the sense of GIT if and only if for any nontrivial subrep $B' \subsetneq B$ (i.e. G-equiv. sub $\mathbb{C}[V]$-mod), we have $\theta(B') \geq 0$.
(resp. $\theta(B') > 0$)

$\square$

In this case B is called θ-semistable.
(resp. θ-stable)

For generic θ (i.e. $R^{\theta\text{-ss}} = R^{\theta\text{-s}}$), we define

$\mathcal{M}_\theta := R /\!\!/_{\chi_\theta} (\mathrm{Aut}_G R / \mathbb{C}^*)$: the moduli space of θ-stable G-constellations

Rem. • There is $\theta_0 \in \Theta$ such that $G\text{-Hilb} \cong \mathcal{M}_{\theta_0}$.

• $\mathcal{M}_\theta$ admists a morphism to $\mathbb{C}^n/G$, which is always projective.

Thm. (Craw-Ishii)

Let $G \subset SL_3(\mathbb{C})$ be a finite abelian subgp.
Then every projective crep. resol X of $\mathbb{C}^3/G$ is isomorphic to $\mathcal{M}_\theta$ for some $\theta \in \Theta$.

Conj (Derived McKay correspondence)

Let $G \subset SL_n(\mathbb{C})$ be a finite subgp and $X \to \mathbb{C}^n/G$ a (not-necessarily projective) crepant resolution.
Then we have a derived equivalence

$$D^b(X) \cong D^b(\mathbb{C}^n)^G$$

where $D^b(\mathbb{C}^n)$ is the derived cat of G-eq. sheaves on $\mathbb{C}^n$.

Rem. This conjecture is affirmative in the following cases:

1. $n=2$ (Kapranov-Vasserot)
2. $n=3$. $X = G\text{-Hilb}$ (BKR)
3. $n=3$. G: abelian, X: any proj. crep. resol (Craw-Ishii)
4. $G \subset Sp_{2n}(\mathbb{C})$ (Bezrukavnikov-Kaledin)
5. G: abelian, X: any proj. crep. resol. (Kawamata)

by using the univ. family of $\mathcal{M}_\theta$

In this talk we will consider when a (not necessarily projective) crep. resol X of $\mathbb{C}^n/G$ for abelian G is obtained as a fine moduli space of G-constellations.

In particular, I will show that X is isomorphic to such a moduli space if X admits a "natural family" of G-constellations which parametrizes indecomposable ones.

We will also see that, if $n \leq 3$, such a family induces a derived equivalence $D^b(X) \cong D^b(\mathbb{C}^n)^G$.

§2 Crepant resolutions as toric varieties

From now on we assume that $G \subset SL_n(\mathbb{C})$ is a finite abelian subgp. We may assume G consists of diagonal matrices.

Let $N = \mathbb{Z}^n$ with the basis $e_1, \cdots, e_n$.

$M := N^\vee = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ with the dual basis $x_1, \cdots, x_n$ which are considered as the coordinates of $\mathbb{C}^n$.

For $g = \begin{pmatrix} \zeta^{a_1} & & \\ & \ddots & \\ & & \zeta^{a_n} \end{pmatrix} \in G$, we set $v_g = \frac{1}{r}(a_1, \cdots, a_n)$.

$$\left(\begin{array}{l} \zeta = e^{\frac{2\pi i}{r}} \\ r = |G| \end{array} \right)$$

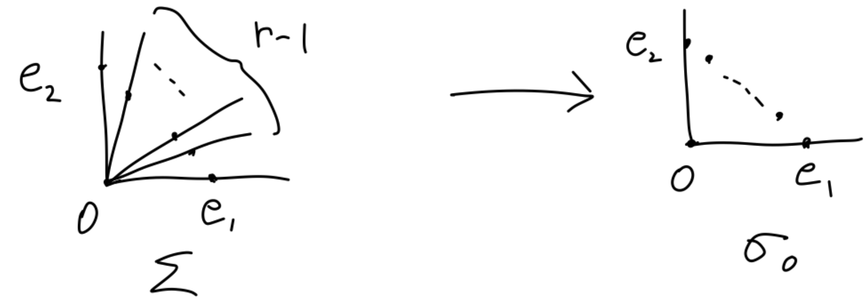
We define $N' = N + \sum_{g \in G} \mathbb{Z} \cdot v_g \subset \frac{1}{r}\mathbb{Z}^n$.

Then the quotient $\mathbb{C}^n/G$ is realized as the toric variety $X(\sigma_0)$ of the cone $\sigma_0 = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} \cdot e_i \subset N' \otimes \mathbb{R}$.

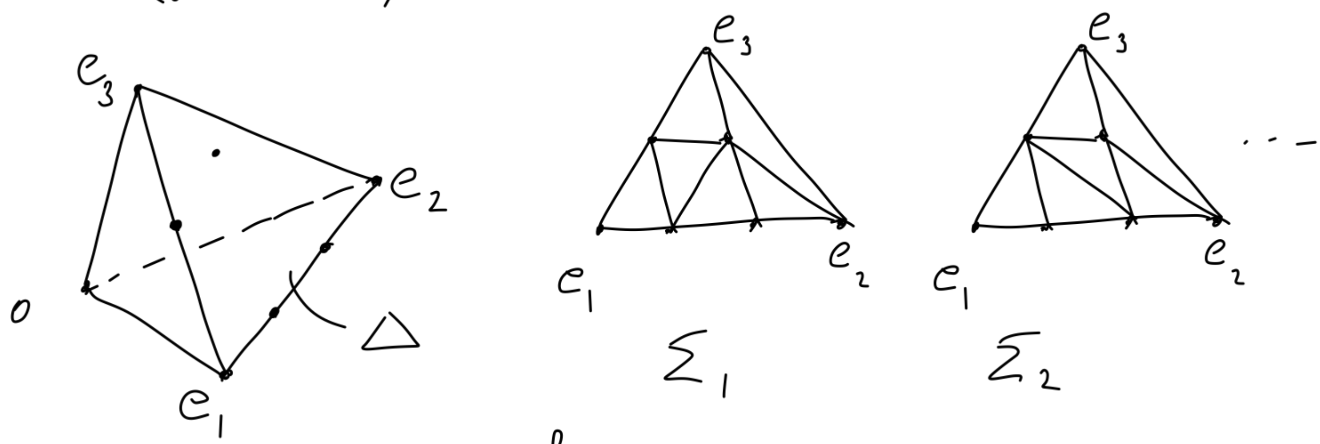
<u>Fact</u> Every crepant resol $X \to \mathbb{C}^n/G$ is realized as the toric variety $X(\Sigma)$ where Σ is a fan obtained as a subdivision of σ_0 into smooth cones whose one-dim'l cones are generated by the lattice points on the junior simplex $\Delta := \{ \sum_{i=1}^{n} a_i e_i \mid \sum a_i = 1 \} \subset \sigma_0$.

Ex. 1. (2-dim'l case)

$G = \langle \frac{1}{r}(1, r-1) \rangle$. $\mathbb{C}^2/G = (A_{r-1}\text{-singularity})$



2. $G = \langle \frac{1}{6}(1, 2, 3) \rangle < SL_3(\mathbb{C})$



In this case crep. resol's are non-unique.

Rem. When $n \geq 4$, crep. resol's do not exist in general.

§3 The quotient construction of X

For $g \in G$ with $v_g = \frac{1}{r}(a_1, \cdots, a_r)$, we define the age of g as $\mathrm{age}(g) = \frac{1}{r}\sum a_i \in \mathbb{Z}_{\geq 0}$.

From the toric descreption of X above, we have a one-to-one correspondence:

$$\{\text{irr. exc. divisor } E \text{ of } X \to \mathbb{C}^n/G\} \overset{1:1}{\longleftrightarrow} \{g \in G \text{ with } \mathrm{age}(g) = 1\}$$

Rem. This correspondence is generalized to non-abelian G (Ito – Reid)

Let $g_1, \cdots, g_m$ be the set of $g \in G$ with $\mathrm{age}(g) = 1$, and let $E_1, \cdots, E_m$ be the corresponding exceptional divisors.

We define a graded ring S as the subring of $\mathbb{C}[x_1, \cdots, x_n][t_1^{\pm 1}, \cdots, t_m^{\pm 1}]$ generated by

$x_i t_1^{a_{i,1}} \cdots t_m^{a_{i,m}}$ $(i = 1, \cdots, n)$ and t_k^{-r} $(k = 1, \cdots, m)$

where $a_{i,k} \in \mathbb{Z}_{\geq 0}$ is defined by $v_{g_k} = \frac{1}{r}(a_{1,k}, \cdots, a_{n,k})$.

S is graded by $\mathrm{Pic}(X)$ (so that t_k^{-r} is in the component of E_k).

The torus $T_X := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(X), \mathbb{C}^*) \cong (\mathbb{C}^*)^m$ naturally acts on $\mathcal{X} = \mathrm{Spec}(S)$.

Thm. (Cox)

X is obtained as the geometric quotient of a certain open subset $U_X \subset \mathcal{X}$ by the T_X-action.

Rem. · The ring S is identified with the total coordinate ring of a toric variety X (introduced by Cox). The notion is generalized to non-toric varieties (and called the Cox ring).

· S is independent of the choice of a crep. resol, but U_X depends on X.

§4 Natural family of G-constellations

Let $\pi : X \to \mathbb{C}^n/G$ be a crepant resol for a finite abelian subgp $G \subset SL_n(\mathbb{C})$, and let $q: \mathbb{C}^n \to \mathbb{C}^n/G$ be the quotient map.

Def. A gnat family (short for G-natural or geometrically natural) $\mathcal{F}$ on X is a flat family of G-constellations on X such that for any ~~x~~ $P \in X$, we have

$$q(\mathrm{Supp}_{\mathbb{C}^n} \mathcal{F}|_P) = \pi(P).$$

Ex. If X is given as a moduli space $\mathcal{M}_\theta$ for some θ (e.g. $X = G\text{-Hilb}$), then its universal family $\mathcal{U}_\theta$ is a gnat family on X.

Recall that $M = N^\vee$ is the free abelian gp of Laurent monomials in $x_1, \ldots, x_n$. We can deompose it as

$M = \bigoplus_{\chi \in G^\vee} M_\chi$ as G-mod. Set $M_\chi^+ := M_\chi \cap \mathbb{C}[x_1, \ldots, x_n]$.

For each $f \in M$ and an exceptional divisor E_k, we can define a rational number $v_{E_k}(f)$ by

$v_{E_k}(f) = f(v_{g_k})$. (Recall that $v_{g_k} \in N$.)

Prop. (Logvinenko)

Every gnat family $\mathcal{F}$ on X is of the form

$$\mathcal{F} = \bigoplus_{\chi \in G^\vee} \mathcal{O}_X\Big(-\sum_{k=1}^{m} b_{\chi,k} E_k\Big) \quad \text{with } b_{\chi,k} \in \mathbb{Q}$$

such that $b_{\chi,k} + v_{E_k}(f) - b_{\chi\chi',k} \in \mathbb{Z}_{\geq 0}$ for any $f \in M^+_{\chi'}$.

Hereafter we assume $\mathcal{F}$ is normalized i.e. $b_{\chi_0,k} = 0$ for the trivial character χ_0.

Fact Among normalized gnat families, there is a unique family $\mathcal{F}_{max}$ (called the maximal shift family) whose coefficients $b_{\chi,k}$ are maximal.

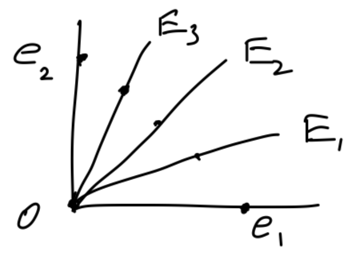
If $X = G\text{-Hilb}$, then its universal family is isom. to $\mathcal{F}_{max}$.

The maximal shift family is explicitly computed by

$$b_{\chi,k} = \min_{f \in M^+_\chi} v_{E_k}(f).$$

Ex. ($n=2$, A_3-singularity)

$$G = \langle \underset{\substack{\shortparallel \\ g_1}}{\tfrac{1}{4}(3,1)} \rangle$$



$E_1 \longleftrightarrow v_{g_1} = \frac{1}{4}(3,1)$, $E_2 \longleftrightarrow v_{g_2} = \frac{1}{4}(2,2)$, $E_3 \longleftrightarrow v_{g_3} = \frac{1}{4}(1,3)$

$G^\vee = \{\chi_0, \chi_1, \chi_2, \chi_3\}$, $\chi_j : g_1 \mapsto e^{\frac{2\pi i}{4} j}$

$M_{\chi_0}^+ = \{1, x^4, xy, y^4, \cdots\}$

$M_{\chi_1}^+ = \{y, x^3, xy^2, \cdots\}$

$M_{\chi_2}^+ = \{x^2, y^2, x^3y, \cdots\}$

$M_{\chi_3}^+ = \{x, y^3, x^2y, \cdots\}$

$\longrightarrow$ the coefficients of $\mathcal{F}_{max}$ are :

$b_{\chi_0,k} = 0$ ($\forall k$), $b_{\chi_1,1} = v_{g_1}(y) = \frac{1}{4}$, $b_{\chi_1,2} = v_{g_2}(y) = \frac{1}{2}$

$b_{\chi_1,3} = v_{g_3}(y) = \frac{3}{4}$, $b_{\chi_2,1} = v_{g_1}(y^2) = \frac{2}{4}$, $b_{\chi_2,2} = v(y^2) = 1$,

$b_{\chi_2,3} = v_{g_3}(x^2) = \frac{1}{2}$, and so on.

In this case another gnat family is given, for example,

by $\mathcal{F}_1 = \mathcal{O}_X \oplus \mathcal{O}_X(-\frac{1}{4}(3E_1 + 2E_2 + E_3))$

$\oplus\, \mathcal{O}_X(-\frac{1}{4}(2E_1 \quad + 2E_3))$

$\oplus\, \mathcal{O}_X(-\frac{1}{4}(E_1 + 2E_2 + 3E_3))$.

Since $\mathcal{F}_{max}$ is the universal family of G-Hilb, all the G-constellations parametrized by $\mathcal{F}_{max}$ is indecomposable (i.e. does not decompose into smaller G-equiv. $\mathbb{C}[V]$-modules).

One can also check that $\mathcal{F}_1$ has the same property.

However, this does not hold in general (even in dim 2).

§5 Morphism from X to a moduli space

Fix a crep. resol X and a gnat family $\mathcal{F}$ on X.

Recall that the moduli space $\mathcal{M}_\theta$ is obtained as a GIT quotient of an affine scheme $\mathcal{R}$ by the group $\mathrm{Aut}_G(R)/\mathbb{C}^*$.

When G is abelian, $\mathcal{R}$ contains an irr. comp. Z which is a possibly non-normal toric variety (Craw-Mclagan-Thomas).

Moreover $T_G := \mathrm{Aut}_G(R)/\mathbb{C}^* \cong (\mathbb{C}^*)^{r-1}$ $(r = |G|)$.

Recall also that X is obtained as a quotient:

$$
\begin{array}{l}
T_X \curvearrowright \mathfrak{X} = \mathrm{Spec}(S) \\
\qquad\quad \cup \\
p : U_X \xrightarrow[/T_X]{} X
\end{array}
$$

<u>Prop.</u> There is a morphism $\varphi_{\mathcal{F}} : \mathfrak{X} = \mathrm{Spec}(S) \to Z$ such that $\varphi_{\mathcal{F}}(x)$ as a rep. of Q_G is isomorphic to $\mathcal{F}_{p(x)}$ for any $x \in U_X$.

Moreover, there is a group hom. $\psi_{\mathcal{F}} : T_X \to T_G$ s.t. $\varphi_{\mathcal{F}}$ is equivariant w.r.t. $\psi_{\mathcal{F}}$.

Using this proposition, we obtain the following diagram:

$$
\begin{tikzcd}
\Psi_{\mathcal{F}} : T_X \arrow[r] & T_G \\
\circlearrowright & \circlearrowright \\
\varphi_{\mathcal{F}} : \mathcal{X} \arrow[r] & \mathcal{Z} \\
\cup & \cup \\
U_X \arrow[r] \arrow[d, "p"'] & V_{\mathcal{F},X} \arrow[d, "p'"] \\
X \arrow[r] & V_{\mathcal{F},X} /\!/ T_G \\
 & \mathcal{M}_{\mathcal{F},X} \arrow[u, equal]
\end{tikzcd}
$$

Here, $V_{\mathcal{F},X} \subset \mathcal{Z}$ is the smallest T_G-invariant open subset containing $\varphi_{\mathcal{F}}(U_X)$, and $\mathcal{M}_{\mathcal{F},X}$ is the categorical quotient of $V_{\mathcal{F},X}$ by T_G in the category of alg. vars.

<u>Rem.</u> p is always a geometric quotient while p' may not be a geometric quotient in general.

$\mathcal{M}_{\mathcal{F},X}$ is not normal in general, and we take the normalization $\widetilde{\mathcal{M}}_{\mathcal{F},X} \to \mathcal{M}_{\mathcal{F},X}$.

Then the induced map $\tilde{\varphi}_{X,\mathcal{F}} : X \to \widetilde{\mathcal{M}}_{\mathcal{F},X}$ is a proper birational toric morphism.

Note that if X itself is given as (an irr. comp. of) a moduli space $\mathcal{M}_\theta$ and if $\mathcal{F}$ is the universal family $\mathcal{U}_\theta$, then $\tilde{\varphi}_{X,\mathcal{F}}$ is tautologically an isomorphism.

The main result is stated as follows.

Thm. (Y) Keep the notation as above.
The map $\tilde{\varphi}_{X,\mathcal{F}}$ is an isomorphism
$\Longleftrightarrow$ P' is a geometric quotient
$\Longleftrightarrow$ F_x is indecomposable for all $x \in X$.

Moreover, if $n = \dim X \leq 3$, then the above equivalent conditions also imply that $\mathcal{F}$ induces a derived equiv.

$$D^b(X) \xrightarrow{\sim} D^b(\mathbb{C}^n)^G.$$

Rem. 1. There exists a non-proj. crep. resol $X \to \mathbb{C}^3/G$ which admits a gnat family $\mathcal{F}$ satisfying the above conditions. Thus, $\mathcal{M}_{X,\mathcal{F}}$ gives a generalization of moduli spaces $\mathcal{M}_\theta$ to non-proj. ones.

2. It is not known whether or not every crep. resol X admits a gnat family $\mathcal{F}$ satisfying the conditions. By the result of Craw-Ishii, there is always such $\mathcal{F}$ if X is a proj. resol of dim 3.

The proof of the theorem is done by showing that the condition that F_x is indecomposable ($\forall x \in X$) is equivalent to the condition that the fan of the toric variety $V_{X,\mathcal{F}}$ (or its normalization) is isom. to the fan Σ defining X.

(In the proof, crepantness of X plays a crucial role.)

As for the derived equivalence, it suffices to show that $\mathrm{Hom}(\mathcal{F}_x, \mathcal{F}_y) = 0$ for all pairs $(x, y) \in X \times X$ with $x \neq y$ (Bridgeland-King-Reid, Logvinenko).

By using the "(V-2) property" of a toric variety (i.e. any two points of X lies in an affine open subset), $\mathcal{F}_x$ and $\mathcal{F}_y$ are both θ-stable for some θ. In particular this implies $\mathrm{Hom}(\mathcal{F}_x, \mathcal{F}_y) = 0$.

§6 Towards non-abelian cases

Even if G is not abelian, the similar results are expected to hold. The main difficulty is to determine the structure of a gnat family since it is no longer a direct sum of line bundles.

However, the Cox ring of a crep. resol X can still be defined and is finitely generated. This particularly implies that X has the (V-2) property.

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