Dimensional reduction in cohomological DT theory

Motivation

DT theory ⋯ counting coh sheaves on CY 3-folds, or rep of quivers w/ potentials. (QP)

categorify $\Big\downarrow$

CoDT theory $\left(\begin{array}{l}\text{Kontsevich–Soibelman : for QP} \\ \text{Joyce and his collaborators. : for CY3}\end{array}\right)$

	DT	CoDT
QP (local)	rich (Szendroi, Nagao, ...)	rich (Kontsevich-Soibelman, Davison - Meinhardt ....)
CY3 (global)	rich (Thomas, MNOP, Joyce-Song, Toda, Bridgeland, ...)	? ?

$\Downarrow$ Q1

Q2 $\Rightarrow$

<u>Q1</u>

Can we "globalize" CoDT for QP ?

(eg Cohomological Hall algebra (CoHA), $\boxed{\text{dimensional reduction}}$, wall-crossing, ...)

3dim $\to$ 2dim

<u>Q2</u>

Can we categorify DT theory for CY3 ?

(eg. DT/PT correspondence, X-indep conj, $\boxed{\text{integrality}}$, ...)

for local CY2.

Today.

What is difficult w/ CoDT for CY3?

① Algebra structure on CoDT (CoHA)

for CY3-fold is yet to be def'd.

(CoHA $\overset{\text{expected}}{\Longrightarrow}$ wall-crossing.)

② Orientation.

(Joyce – Upmeier made a great progress,
but not enough to construct CoHA.)

<u>Today</u> Only consider local surfaces

$X = \mathrm{Tot}_S(K_S)$ (S: sm surface)

(in this case, $\exists$ easy orientation.)

Goal Prove dimensional reduction in CoDT theory, and discuss some applications.

$3\text{-dim} \to 2\text{dim}$

§1 Shifted symplectic geometry and CoDT theory

Derived algebraic geometry $/\mathbb{C}$

"$\doteqdot$" Algebraic geometry over comm. diff. graded algebras $/\mathbb{C}$ (cdga)

Derived scheme "$\doteqdot$" underlying scheme + derived structure e.g. U : sm scheme

$E \to U$: vector bundle

$s \in \Gamma(U, E)$: section.

$$\mathcal{Z}(s) := \mathrm{Spec}_U \left(\to \Lambda^2 E^\vee \xrightarrow{s} E^\vee \xrightarrow{s} \mathcal{O}_U \right)$$

derived enhancement of $s^{-1}(0)$.

$$t_0(\mathcal{Z}(s)) = s^{-1}(0).$$

$\mathcal{Z}(s)$ remembers many information that is forgotten if s is not transverse

e.g.

- $\dim U - \mathrm{rk}\, E = \mathrm{vdim}\, \mathcal{Z}(s)$.

- $e_{loc}(E, s) \in A_{\mathrm{vdim}\, \mathcal{Z}(s)}(s^{-1}(0))$.

: localized euler class.

A derived scheme is <u>quasi-smooth</u>

$\overset{\text{def}}{\iff}$ locally isom to $\mathfrak{X}(s)$.

Many moduli spaces,

(e.g GW moduli, fine moduli of coh sheaves on CY3, ...)

are known to be quasi-smooth.

For a quasi-sm derived sch $\mathfrak{X}$

$\exists\ [\mathfrak{X}]_{vir} \in A_{vdim\,\mathfrak{X}}(\mathfrak{X})$: virtual fundamental class (VFC)

Behrend-Fantech
Li-Tian.

$$[\mathfrak{X}]_{vir}\big|_{\mathfrak{X}(s)} = e_{loc}(E, s).$$

X : proj CY 3-fold.

H : ample div on X, $\gamma \in H^*(X, \mathbb{Q})$.

$M_\gamma^{H\text{-ss}}$: moduli space of H-semistable sheaves on X.

Assume : H-stable $\Leftrightarrow$ H-semistable for E w/ $ch(E) = \gamma$.

<u>Thomas</u>

$$DT_\gamma^H(X) := \int_{[M_\gamma^{H\text{-ss}}]_{vir.}} 1$$

DT inv.

<u>Behrend</u>

$\exists\, \nu$: constructible function

s.t $\displaystyle\int_{M_\gamma^{H\text{-ss}}} \nu = DT_\gamma^H(X)$.

Joyce and his collaborators.

$\exists \varphi \in \mathrm{Perv}(\mathcal{M}_\gamma^{H\text{-}ss})$ s.t

$$\chi(\varphi|_p) = \nu(p).$$

$$\left(\Rightarrow \chi(\varphi) = DT_\gamma^H(x).\right)$$

Main ingrediene : shifted symplectic geometry $\left(\text{by Pantev-Töen -Vaquie-Vezossi}\right)$

$\mathbb{L}_X$: cotangene complex

n- shifted symplectic str on X

$$\text{``}\overset{\varphi}{=}\text{''} \quad \mathbb{L}_X^\vee \overset{\omega}{\simeq} \mathbb{L}_X[n] \quad (\omega \in \wedge^2 \mathbb{L}_X[n])$$

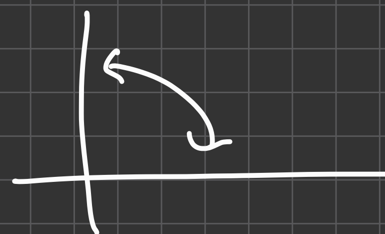
+ closed structure.

e.g. • (PTVV, Calaque)

Y : derived Artin stack.

$$T^*[n]Y := \mathbb{V}(\mathbb{L}_Y[n])$$

is n-shifted symplectic.



• (PTVV, Brav - Dyckerhoff)

X : CY n-fold

$\mathfrak{M}_X$: moduli space of (cptly supp) coh sheaves on X.

$$\mathbb{L}_{\mathfrak{M}_X}|_{[E]} \simeq \mathbb{R}\mathrm{Hom}(E,E)[1].$$

Serre duality $\rightsquigarrow$ $\mathbb{L}^\vee_{\mathfrak{M}_X} \simeq \mathbb{L}_{\mathfrak{M}_X}[2-n]$

PTVV, BD : $\mathfrak{M}_X$ is $(2-n)$-shifted symplectic

<u>Darboux theorem</u> (Ben-Bassat - Brav - Bussi - Joyce)

$\mathfrak{X}$: (-1)-shifted symplectic derived Artin stack.

Then $\mathfrak{X}$ is locally isom to $\mathrm{Crit}(f)$
($f : U \to \mathbb{A}^1$) U sm scheme.

Zariski if $\mathfrak{X}$: schematic
smooth for gen'l $\mathfrak{X}$.

(-1)-shifted symplectic geometry

"=" geometry of the critical loci.

For (-1)-shifted symplectic derived

Artin stack $\mathfrak{X} = \bigcup \mathrm{Crit}(f_i)$

w/ fixed orientation $\sqrt{\det(\mathbb{L}_{\mathfrak{X}})|_{\mathfrak{X}^{red}}}$

[BBBBJ] : $\exists\, \varphi \in \mathrm{Perv}(\mathfrak{X})$,

s.t. $\varphi|_{\mathrm{Crit}(f_i)} \simeq \varphi_{f_i} \otimes_{\mathbb{Z}_2} L_i$

local system depending on the chosen orientation.

§2 dimensional reduction

Thm (K-)

$\mathcal{Y}$: quasi-sm derived Artin stack

$\pi : T^*[-1]\mathcal{Y} \to \mathcal{Y}$

Equip $T^*[-1]\mathcal{Y}$ w/ the canonical orientation.

$\pi^*\mathbb{L}_{\mathcal{Y}} \to \mathbb{L}_{T^*[-1]\mathcal{Y}} \to \pi^*\mathbb{L}_{\mathcal{Y}}^{\vee}[1]$

$\det \downarrow$

$\det(\mathbb{L}_{T^*[-1]\mathcal{Y}}) \simeq \pi^*\det(\mathbb{L}_{\mathcal{Y}})^{\otimes 2}$

Then
$$\begin{cases} \cdot\ \pi_! \varphi \simeq \mathbb{Q}_{\mathcal{Y}}[\operatorname{vdim} \mathcal{Y}]. \\ \cdot\ \pi_* \varphi \simeq \omega_{\mathcal{Y}}[-\operatorname{vdim} \mathcal{Y}]. \end{cases}$$

(idea · $\mathcal{Y} = Z(s)$: due to Davison.

· For gen'l $\mathcal{Y}$: gluing Davison's isom.)

Cor X : $\mathrm{Tot}(K_S)$ (S : sm surface)

$$H^*(\mathfrak{M}_X; \varphi) \simeq H^{BM}_{-*+\operatorname{vdim}\mathfrak{M}_S}(\mathfrak{M}_S)$$

If K_S is trivial, H : ample div on S,

$$H^*(\mathfrak{M}_X^{H\text{-ss}}; \varphi) \simeq H^{BM}_{-*+\operatorname{vdim}\mathfrak{M}_S}(\mathfrak{M}_S^{H\text{-ss}})$$

☺ We have

$$T^*[-1]\mathfrak{M}_S \simeq \mathfrak{M}_X \text{ (preserving symplectic form)}$$

Ikeda – Qiu

$\mathrm{Perf}(X) \overset{\text{Morita}}{\simeq}$ 3 – CY completion of $\mathrm{Perf}(S)$.

+ Bozec – Calaque – Sherotzke

moduli of n – CY completion

= $(2-n)$ – shifted cotangent.

$$\pi : \mathfrak{M}_X \to \mathfrak{M}_S.$$

K_S : trivial $\Rightarrow$ $\pi^{-1}(\mathfrak{M}_S^{H\text{-ss}}) = \mathfrak{M}_X^{H\text{-ss}}$.

Application

(1) New construction of VFC

$\mathcal{Y}$: quasi-sm derived Artin stack.

$\pi : T^*[-1]\mathcal{Y} \longrightarrow \mathcal{Y}$: projection.

$$\begin{tikzcd}
\pi : \varphi \arrow[r, mapsto] & \pi_* \varphi \\
\mathbb{D}_{\mathcal{Y}}[\mathrm{vdim}\,\mathcal{Y}] \arrow[u, phantom, "\wr|"', "\text{"Thom isom"}"] \arrow[r, dashed, "e(T^*[-1]\mathcal{Y})"'] & \omega_{\mathcal{Y}}[-\mathrm{vdim}\,\mathcal{Y}] \arrow[u, phantom, "\wr|", "\text{"homotopy invariance"}"']
\end{tikzcd}$$

$$e(T^*[-1]\mathcal{Y}) \in H^{BM}_{2\cdot \mathrm{vdim}\,\mathcal{Y}}(\mathcal{Y})$$

Thm (K–, in preparation)

Assume $t_0(\mathcal{Y})$: quasi-proj.

Then $e(T^*[-1]\mathcal{Y}) = (-1)^{\binom{\mathrm{vdim}\,\mathcal{Y}}{2}} [\mathcal{Y}]_{\mathrm{vir}}$.

idea

$$\begin{array}{ccc}
T^*[-1]\mathcal{Y} & \overset{\text{dual}}{\longleftrightarrow} & T[1]\mathcal{Y} \\
 & & \cup \\
\varphi & & C_{\mathrm{int}}
\end{array}$$

C_{int} : intrinsic normal cone

Khan : $FS : D_{\mathbb{G}_m, c}(T^*[-1]\mathcal{Y}) \simeq D_{\mathbb{G}_m, c}(T[1]\mathcal{Y})$

$\mathrm{Supp}(FS(\varphi)) = C_{int}$.

$\rightsquigarrow$ Sheaf theoretic computation leads to the prf.

Thanks to quasi-proj assumption, classical FS transform is enough.

② PBW theorem for 2CY (semi-stable)

Kapravov - Vasseroe CoHA

/ Cohomological integrality for 2CY.

(Work in progress w/ Davison)

S : sm surface

H : ample div on S.

$\gamma \in H^*(X, \mathbb{Q})$

<u>KV</u> : $\mathcal{H}_{KV} := \bigoplus_{n \geq 0} H_*^{BM}\left(\mathfrak{M}^{H\text{-ss}}_{S, n\gamma}\right)$

carries a convolution produce.

(CoHA)

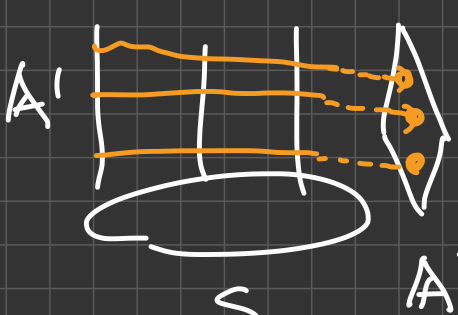
Stack of short exact seq.

$$\begin{array}{ccc} \mathfrak{M}_S^{H\text{-ss},\,ext} & \xrightarrow[\text{proper}]{ev_2} & \mathfrak{M}_S^{H\text{-ss}} \\ {\scriptstyle \text{quasi-sm}}\downarrow{\scriptstyle (ev_1,\, ev_3)} & & \\ \mathfrak{M}_S^{H\text{-ss}} \times \mathfrak{M}_S^{H\text{-ss}} & & \end{array}$$

ev_i : i-th evaluation.

Assume K_S is trivial.

$$X = \mathrm{Tot}(K_S) = S \times \mathbb{A}^1$$

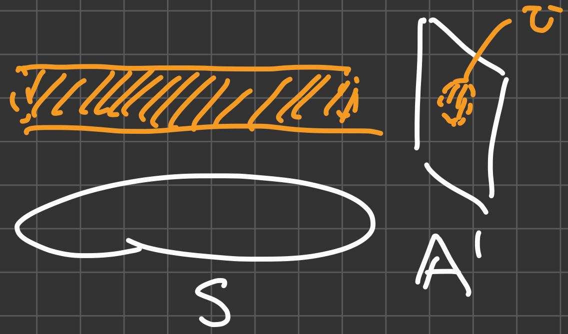


$$\begin{array}{ccc} \mathfrak{M}_X^{H\text{-ss}} & \xrightarrow{\lambda} & \mathrm{Sym}(\mathbb{A}^1) \\ \downarrow & & \\ \mathfrak{M}_S^{H\text{-ss}} & & \end{array}$$

taking supp over $\mathbb{A}^1$.

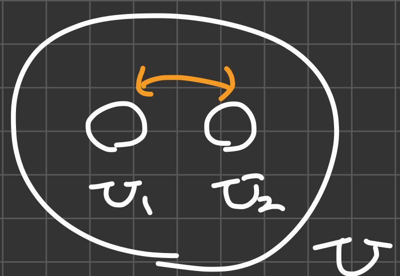
For $U \subset \mathbb{A}^1$: analytic open ball,

$$U \mapsto H^*(\lambda^{-1}(\mathrm{Sym}(U)), \varphi)$$



$$\simeq H_*^{BM}(\mathfrak{M}_S^{H\text{-ss}})$$

dim red.



$$\mathrm{res}^{U}_{U_1 \amalg U_2} : H_*^{BM}(\mathfrak{M}_S^{H\text{-ss}}) \longrightarrow H_*^{BM}(\mathfrak{M}_S^{H\text{-ss}})^{\otimes 2}$$

: cocommutative coproduct on $\mathcal{H}_{KV}$.

Milnor–Moore

equiv parameter

$$\rightsquigarrow \mathcal{H}_{KV} \simeq U(\mathfrak{g}[u])$$

$$\mathfrak{g} = \bigoplus_{i,n} \mathfrak{g}_{i,n}$$: BPS Lie algebra.

$$\sum (-1)^i \dim \mathfrak{g}_{i,n} = \text{BPS number for } nr.$$

$\Uparrow$

cohomological

integrality.

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Possible future direction

Q1 : Can we construct other enumerative inv (e.g DT4 inv) using dimensional reduction or its variant ?

Q2 : Can we study wall-crossing formula of CoDT invs for local K3 surfaces using KV CoHA ?