3d mirror symmetry, vertex function and elliptic stable envelope

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Outline

- Physics motivation: 3d mirror symmetry
- Enumerative geometry: quantum K-theory, vertex function, q-difference equations
- Monodromy: elliptic stable envelope
- Conjectures and results.

Physics motivation: 3d mirror symmetry

Physics	Math
3d $\mathcal{N} = 4$ SUSY gauge theory	symp. representation (G, M, ω)
Higgs branch	holo. symp. quotient $\mu^{-1}(0)//_{\theta} G$
Coulomb branch	BFN construction
FI parameter	stab. cond. θ / Kähler parameter
mass parameter	equivariant parameter
R-symmetry $SU(2)_H \times SU(2)_C$	?

Physics background: 3d mirror symmetry

- Intriligator–Seiberg '96, Hanany–Witten '96, Boer–Hori–Ooguri–Oz–Yin '96, ...
- Mirror pair of 3d $\mathcal{N} = 4$ theories, with
  1. Higgs branch $\leftrightarrow$ Coulomb branch
  2. FI parameters $\leftrightarrow$ mass parameters
  3. $SU(2)_H \leftrightarrow SU(2)_C$

Mathematics inspired from 3d $\mathcal{N} = 4$ mirror symmetry:

- Representation theory: Braden–Licata–Proudfoot–Webster, Braverman–Finkelberg–Nakajima, Hilburn–Kamnitzer–Weekes, ...
- Enumerative geometry: Aganagic–Okounkov, Koroteev–Zeitlin, Kamnitzer–McBreen–Proudfoot, McBreen–Sheshmani–Yau, Rimányi–Smirnov–Varchenko–Z, ...

In this talk, a *3d mirror pair* is a pair $(X, X^!)$, where X and $X^!$ are the Higgs branches of a pair of 3d mirror theories.

Assume:

- X and $X^!$ are smooth;
- X and $X^!$ have large enough torus actions, i.e. with isolated fixed points.
- There is a natural bijection between the fixed points.

Example: abelian theory

$G = (\mathbb{C}^*)^k$, $M = T^*\mathbb{C}^n$, $d := n - k$. Action given by

$$0 \longrightarrow \mathbb{Z}^k \xrightarrow{\iota} \mathbb{Z}^n \xrightarrow{\beta} \mathbb{Z}^d \longrightarrow 0.$$

Higgs branch: hypertoric variety.

Mirror theory: $G^! = (\mathbb{C}^*)^d$, $M^! = T^*\mathbb{C}^n$.

$$0 \longrightarrow \mathbb{Z}^d \xrightarrow{\iota^! = \beta^T} \mathbb{Z}^n \xrightarrow{\beta^! = \iota^T} \mathbb{Z}^k \longrightarrow 0.$$

Example: $T^*\mathbb{P}^n$ vs $\mathcal{A}_n$

$G = \mathbb{C}^*$, $M = \mathbb{C}^{n+1}$,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{Z}^{n+1} \xrightarrow{\beta} \mathbb{Z}^n \longrightarrow 0$$

where

$$\iota = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & & & -1 \\ & \ddots & & \vdots \\ & & 1 & -1 \end{pmatrix}$$

Higgs branch $X = T^*\mathbb{P}^n$ (generic stability condition).

Example: $T^*\mathbb{P}^n$ vs $\mathcal{A}_n$

Take the dual of the exact sequence. We get

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{\iota^!} \mathbb{Z}^{n+1} \xrightarrow{\beta^!} \mathbb{Z} \longrightarrow 0$$

where

$$\iota^! = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ -1 & \cdots & -1 \end{pmatrix}, \qquad \beta^! = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}$$

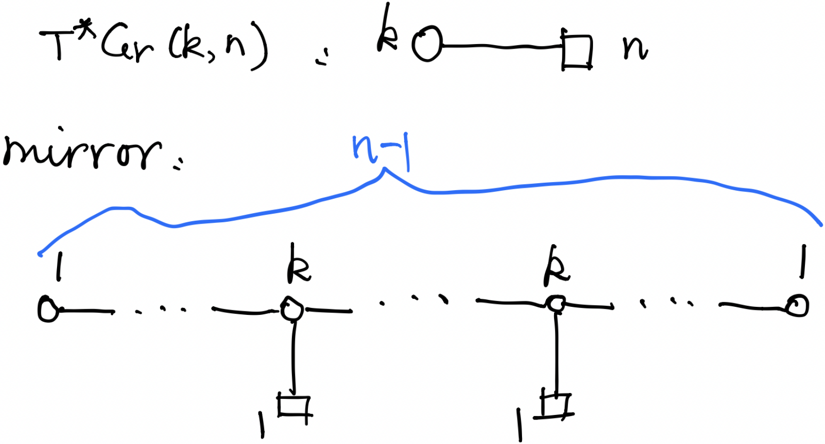
Higgs branch: $\mathcal{A}_n$ surface (generic stability condition), minimal resolution of $\mathbb{C}^2/\mathbb{Z}_{n+1}$.

Example: quiver gauge theories of affine type A

Among this type, they can be obtained from Hanany–Witten's construction.

- $T^*Fl(1, 2, \cdots, n)$ vs $T^*Fl(1, 2, \cdots, n)$;
- $\mathrm{Hilb}^n(\mathbb{C}^2)$ vs $\mathrm{Hilb}(\mathbb{C}^2)$;
- $\mathcal{M}(\mathcal{A}_k, r + 1)$ vs $\mathcal{M}(\mathcal{A}_r, k + 1)$, moduli of instantons.

Example: $T^*Gr(k,n)$, $(n \geq 2k)$ vs mirror



There are other examples, e.g.

- with more general gauge groups, other than type A
- bow varieties [Nakajima–Takayama]

Conjecture (rough idea)

How to formulate enumerative geometric conjectures from 3d mirror symmetry?

- Extract equivariant enumerative invariants from 3d $\mathcal{N} = 4$ theories, depending on Kähler and equivariant parameters. (counting curves/quasimaps on Higgs branch $\rightsquigarrow$ vertex function)
- Relate vertex functions of a mirror pair.

Background: curve counting

X: smooth quasi-projective variety. $\beta \in H_2(X, \mathbb{Z})$.

In Gromov–Witten theory:

- Moduli space: $\overline{\mathcal{M}}_{g,n}(X, \beta)$, consisting of stable maps
$$(C, p_1, \cdots, p_n) \qquad \to \qquad X.$$
- $\overline{\mathcal{M}}_{g,n}(X, \beta)$ admits a perfect obstruction theory, which allows to define a virtual fundamental class $[\mathcal{M}]^{\mathrm{vir}} \in H_*(X)$.
- Invariants:
$$\langle \gamma_1 \psi^{a_1}, \cdots, \gamma_n \psi^{a_n} \rangle_{g,n,\beta} = \int_{[\mathcal{M}]^{\mathrm{vir}}} \prod \psi_i^{a_i} \, \mathrm{ev}_i^* \, \gamma_i,$$
where $\int_{[\mathcal{M}]^{\mathrm{vir}}}$ means the push-forward to a point.
- For $\int_{[\mathcal{M}]^{\mathrm{vir}}}$ to be well-defined, need either:
(i) X is projective;
(ii) X admits T-action with proper T-fixed loci.

$g = 0$: Givental's J-function

One can collecting enumerative invariants into generating functions.

Consider $\mathrm{ev}_1 : \overline{\mathcal{M}}_{0,1}(X, \beta) \to X$.

$$J(z) = \sum_{\beta} \mathrm{ev}_{1*} \frac{1}{-c - \psi} z^{\beta}$$

encodes the information of $g = 0$ GW invariants.

z: Kähler parameter.

I-function

There is another function $I(z)$.

- It is the solution of Picard-Fuchs equation (quantum differential equation w.r.t. Kähler parameters), coming from VHS on the B-model.
- If X is a GIT quotient, $I(z)$ can be interpreted as counting *stable quasimaps* [Ciocan-Fontanine–Kim–Maulik].
- In many cases, easy to compute via equivariant localization.

2d mirror symmetry: $I \approx J$, up to mirror map.

$$2d \rightsquigarrow 3d, \qquad \text{cohomology} \ \rightsquigarrow \ K\text{-theory}$$

In $g = 0$ quantum K-theory, for a GIT quotient X, we also expect an I-function/vertex function which

- is defined by counting quasimaps;
- can be computed explicitly via localization, if X admits a good torus action;
- satisfies a q-difference equation (instead of differential equation) w.r.t. Kähler parameters.

Example: maps and quasimaps to $\mathbb{P}^n$

A map $f : \mathbb{P}^1 \to \mathbb{P}^n$ is equivalent to the data $(L, s_1, \cdots, s_{n+1})$, where

- L is a line bundle on $\mathbb{P}^1$ (actually $L = f^*\mathcal{O}_{\mathbb{P}^n}(1)$);
- $s_1, \cdots, s_{n+1}$ are sections of L, such that for any $x \in \mathbb{P}^1$, $s_i(x)$ does not vanish simultaneously.

Example: maps and quasimaps to $\mathbb{P}^n$

For a *quasimap*, we drop the "base point free" condition.

A quasimap $f : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is defined as $(L, s_1, \cdots, s_{n+1})$, where

- L is a line bundle on $\mathbb{P}^1$;
- $s_1, \cdots, s_{n+1}$ are sections of L, such that for *generic* $x \in \mathbb{P}^1$, $s_i(x)$ does not vanish simultaneously.

$\deg L$ is defined as the *degree* of the quasimap.

The moduli space of quasimaps from $\mathbb{P}^1$ to $\mathbb{P}^n$ of degree d is $\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))^{\oplus(n+1)})$.

Idea:

- bundles and sections on $\mathbb{P}^1$, constructed by data from the target;
- generically maps into stable locus.

Definition (quasimap)

Given G, acting on $T^*M = M \oplus M^*$, moment map $\mu : T^*M \to \mathfrak{g}^*$.

Let X be the GIT quotient $\mu^{-1}(0)/\!/_\theta G = \mu^{-1}(0)^s/G$ (Higgs branch),

where $\theta : G \to \mathbb{C}^*$ is a chosen character, or a stability condition, and $\mu^{-1}(0)^s$ is the stable locus.

(always assume θ to be generically chosen; X is smooth.)

We will consider quasimaps from $\mathbb{P}^1$ to X.

Definition (quasimap)

Definition

- A quasimap from $\mathbb{P}^1$ to the GIT quotient $X = \mu^{-1}(0)//_\theta G$ is a map to the stacky quotient

$$f : \mathbb{P}^1 \to \mathfrak{X} = [\mu^{-1}(0)/G]$$

which maps generically into the stable locus X.

- Alternatively, a principal G-bundle $\mathcal{P}$ over $\mathbb{P}^1$, together with a G-equivariant morphism $\mathcal{P} \to \mu^{-1}(0) \subset T^*M$, which maps generically into the stable locus $\mu^{-1}(0)^s$.

Example: $T^*\mathbb{P}^n$

$G = \mathbb{C}^*$, $T^*M = T^*\mathbb{C}^{n+1}$, action

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(1,\cdots,1)^T} \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z}^n \longrightarrow 0$$

$t \cdot (z_1, \cdots, z_{n+1}, w_1, \cdots, w_{n+1}) =$
$(tz_1, \cdots, tz_{n+1}, t^{-1}w_1, \cdots, t^{-1}w_{n+1})$

Moment map $\mu : T^*\mathbb{C}^{n+1} \to \mathbb{C}$

$$\mu(\vec{z}, \vec{w}) = \sum_{i=1}^{n+1} z_i w_i$$

Example: $T^*\mathbb{P}^n$

$$\mu^{-1}(0) = \{(\vec{z}, \vec{w}) \in T^*\mathbb{C}^{n+1} \mid \vec{z} \cdot \vec{w} = 0\}$$

Two choices of θ

- $\theta > 0$: $\mu^{-1}(0)^s = \{\vec{z} \neq 0\}$, $X = T^*\mathbb{P}^n$
- $\theta < 0$: $\mu^{-1}(0)^s = \{\vec{w} \neq 0\}$, $X = T^*\mathbb{P}^n$

The two $T^*\mathbb{P}^n$'s are symplectic flops of each other.

Example: $T^*\mathbb{P}^n$

A quasimap to $T^*\mathbb{P}^n$ consists of

- a line bundle L on $\mathbb{P}^1$
- sections $s_1, \cdots, s_{n+1}$ of L, $t_1, \cdots, t_{n+1}$ of L^{-1}, such that $\mu(\vec{s}, \vec{t}) = \vec{s} \cdot \vec{t} = 0$, and $\vec{s} \neq 0$ generically

Points on $\mathbb{P}^1$ where $\vec{s} = 0$ are called *base points*.

$\vec{s} \neq 0$ generically $\Rightarrow \deg L \geq 0$, $\vec{t} = 0$.

$X = \mu^{-1}(0)/\!/_\theta G$, $\mathfrak{X} = [\mu^{-1}(0)/G]$.

$\mathrm{QM}_d(X)$: moduli space of quasimaps from $\mathbb{P}^1$ to X, of degree d

Obstruction theory/virtual tangent bundle $T_{\mathrm{vir}} = R\pi_* f^* T_{\mathfrak{X}}$

$\rightsquigarrow$ virtual structure sheaf $\mathcal{O}_{\mathrm{vir}} \in K(\mathrm{QM}_d(X))$

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathrm{QM}_d(X) & \xrightarrow{f} & \mathfrak{X} \\ \downarrow \pi & & \\ \mathrm{QM}_d(X) & & \end{array}$$

$\widehat{\mathcal{O}}_{\mathrm{vir}} := \mathcal{O}_{\mathrm{vir}} \otimes (\det T_{\mathrm{vir}}^\vee)^{1/2} \cdot q^{-\deg \mathcal{T}^{1/2}/2}$

$\mathrm{QM}_d^\circ(X)$: open substack where $\infty \in \mathbb{P}^1$ is not a base point.

$\mathrm{ev}_\infty : \mathrm{QM}_d^\circ(X) \to X$ is well-defined, but not proper. Need to work equivariantly.

Assume: M admits a T-action (flavor symmetry), commuting with G and descending to X, such that X^T is proper.

Also $\mathbb{C}_\hbar^*$ scales the cotangent fiber M^* in T^*M

Let $\mathbb{C}_q^*$ scales $\mathbb{P}^1$, $q := T_0\mathbb{P}^1 \in K_{\mathbb{C}_q^*}(\mathrm{pt})$. z: Kähler parameter.

Definition (A. Okounkov)

Vertex function/I-function

$$V(q,a,z) := \sum_\beta z^\beta \, \mathrm{ev}_{\infty *} \, \widehat{\mathcal{O}}_{\mathrm{vir}} \in K_{T\times\mathbb{C}_\hbar^*\times\mathbb{C}_q^*}(X)_{loc},$$

where "loc" means to pass to fraction field of $K_{T\times\mathbb{C}_\hbar^*\times\mathbb{C}_q^*}(\mathrm{pt})$.

Physics motivation

G: reductive group, M: representation.

$T^*M = M \oplus M^*$: quaternionic/symplectic representation.

Physics: 3d $\mathcal{N} = 4$ SUSY gauge theory (G, T^*M)

Moduli space of vacua:

- Higgs branch $\rightsquigarrow$
  hyperkähler/holomorphic symplectic quotient
  $X = \mu^{-1}(0)//_\theta G$
- Coulomb branch

partition functions on $S^1 \times_q D^2 \rightsquigarrow K$-theoretic I-function/vertex function, counting quasimaps to X

IR limit: nonlinear sigma model to $X \rightsquigarrow K$-theoretic GW theory of X

Vertex function for $T^*\mathbb{P}^n$

$T = (\mathbb{C}^*)^{n+1}$, $K_T(\text{pt}) = \mathbb{Q}[a_1^{\pm 1}, \cdots, a_{n+1}^{\pm 1}]$.

A $T \times \mathbb{C}^*_\hbar \times \mathbb{C}^*_q$-fixed quasimap is one that generically maps to a fixed point of $T^*\mathbb{P}^n$

$$p_k = \mu^{-1}(0) \cap \{z_i = w_i = 0 \mid i \neq k\} //_\theta \mathbb{C}^*.$$

Precisely, it means that f maps to the torus-fixed substack $[\mu^{-1}(0) \cap T^*\mathbb{C}/\mathbb{C}^*]$, where $\mathbb{C}$ is the k-th piece in $\mathbb{C}^{n+1}$.

It consists of line bundle $L = \mathcal{O}(d[0])$, sections $s_i = 0$ for $i \neq k$, $s_k = y^d$, $\vec{t} = 0$, where $[x : y]$ is the homogeneous coordinate on $\mathbb{P}^1$, $[0, 1] = \infty \in \mathbb{P}^1$.

For $X = T^*\mathbb{P}^n$, equivariantly,

$$T_{\mathfrak{X}} = \sum_{i=1}^{n+1} a_i \mathcal{L} + \sum_{i=1}^{n+1} \hbar^{-1} a_i^{-1} \mathcal{L}^{-1} - (1+\hbar^{-1})\mathcal{O}$$

where $\mathcal{L}$ is the tautological line bundle on $B\mathbb{C}^*$,
$\mu(z,w) = \vec{z} \cdot \vec{w} \in Lie(\mathbb{C}^*)^\vee$.

$$\begin{array}{ccccccc} \mathbb{P}^1 & \xrightarrow{f} & [\mu^{-1}(0)/\mathbb{C}^*] & \longrightarrow & [T^*\mathbb{C}^{n+1}/\mathbb{C}^*] & \longrightarrow & B\mathbb{C}^* \\ \vdots & \nearrow & & & & & \\ T^*\mathbb{P}^n & & & & & & \end{array}$$

The pullback of $\mathcal{L}$ from $B\mathbb{C}^*$ to $T^*\mathbb{P}^n$ is $\mathcal{O}_{\mathbb{P}^n}(1)$, and to $\mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(d)$.

Its restriction to p_k is the 1-dimensional representation a_k^{-1}.

Virtual tangent bundle at the particular quasimap f which maps to p_k is

$$T_{\mathrm{vir}} = H^\bullet\Big(\mathbb{P}^1, \sum_{i=1}^{n+1} a_i a_k^{-1}\mathcal{O}(d) + \sum_{i=1}^{n+1} \hbar^{-1} a_i^{-1} a_k \mathcal{O}(-d) - (1+\hbar^{-1})\mathcal{O}\Big)$$

where $H^\bullet := H^0 - H^1$, $d \geq 0$.

Need to compute $H^\bullet(\mathbb{P}^1, \mathcal{O}(d))$, as a class in $K_{\mathbb{C}_q^*}(\mathrm{pt})$.

For $d \geq 0$, $H^0(\mathbb{P}^1, \mathcal{O}(d))$ is spanned by

$$x^d, x^{d-1}y, \cdots, xy^{d-1}, y^d.$$

In $K_{\mathbb{C}_q^*}(\mathrm{pt})$,

$$H^0(\mathbb{P}^1, \mathcal{O}(d)) = 1 + q + \cdots + q^d.$$

For $d \geq 2$, $H^1(\mathbb{P}^1, \mathcal{O}(-d)) \cong H^0(\mathbb{P}^1, \mathcal{O}(d)) \otimes \omega$ is spanned by (the dual of)

$$x^{d-1}y, \cdots, xy^{d-1}.$$

In $K_{\mathbb{C}_q^*}(\mathrm{pt})$,

$$H^1(\mathbb{P}^1, \mathcal{O}(-d)) = q^{-1} + \cdots + q^{-d+1}.$$

Alternatively, compute by $\mathbb{C}_q^*$-localization. K-theoretic localization

$$\chi(X,\mathcal{E}) = \sum_{F\subset X^T} \frac{\chi(F,\mathcal{E}|_F)}{\Lambda^\bullet N_F^\vee}$$

where

$$\Lambda^\bullet(\sum_i x_i)^\vee = \prod_i (1-x_i^{-1})$$

is the K-theoretic Euler class.

Recall that $q = T_0\mathbb{P}^1$. $\mathcal{O}(d)|_\infty = 1$, $\mathcal{O}(d)|_0 = q^d$.

$$H^\bullet(\mathbb{P}^1,\mathcal{O}(d)) = \frac{q^d}{1-q^{-1}} + \frac{1}{1-q} = \frac{q^{d+1}-1}{q-1}.$$

For a quasimap f to p_k, of degree $d \geq 0$,

$$T_{\mathrm{vir}} = \sum_{i=1}^{n+1} \frac{a_i}{a_k}(1+q+\cdots+q^d) - \sum_{i=1}^{n+1} \hbar^{-1} \frac{a_k}{a_i}(q^{-1}+\cdots+q^{-d+1}) - 1 - \hbar^{-1}.$$

$$T_{\mathrm{vir}} - T_{p_k}X = \sum_{i=1}^{n+1} \frac{a_i}{a_k}(q+\cdots+q^d) - \sum_{i=1}^{n+1} \hbar^{-1} \frac{a_k}{a_i}(1+q^{-1}+\cdots+q^{-d+1})$$

Localization implies that

$$V(q,a,z)|_{p_k} = \sum_{d=0}^{\infty} \frac{q^{-(n+1)d/2}(\det T_{\mathrm{vir}}^{\vee})^{1/2}}{\Lambda^{\bullet}(T_{\mathrm{vir}} - T_{p_k}X)^{\vee}} z_d,$$

Example: $T^*\mathbb{P}^n$

We have $K_T(T^*\mathbb{P}^n) = \mathbb{C}[\hbar^{\pm 1}, a_1^{\pm 1}, \cdots, a_{n+1}^{\pm 1}, s^{\pm 1}] / \prod_{i=1}^{n+1}(1 - a_i s)$, where $s = [\mathcal{O}(-1)]$. For the fixed point p_k, $s|_{p_k} = a_k^{-1}$

$$\boxed{V(q,a,z) = \sum_{d=0}^{\infty} (-q\hbar^{-1/2})^{(n+1)d} \prod_{i=1}^{n+1} \frac{(\hbar a_i s)_d}{(q a_i s)_d} \cdot z^d}$$

$$\boxed{V(q,a,z)|_{p_k} = \sum_{d=0}^{\infty} (-q\hbar^{-1/2})^{(n+1)d} \prod_{i=1}^{n+1} \frac{(\hbar a_i / a_k)_d}{(q a_i / a_k)_d} \cdot z^d}$$

which lies in $K_{T \times \mathbb{C}_\hbar^* \times \mathbb{C}_q^*}(T^*\mathbb{P}^n)_{loc}[[z]]$, where $q \in \mathbb{C}$, $|q| < 1$,

$$(y)_d := \frac{(y)_\infty}{(q^d y)_\infty} = (1-y)\cdots(1-q^{d-1}y), \qquad (y)_\infty := \prod_{l=0}^{\infty}(1-q^l y).$$

Properties of vertex functions: q-difference equations

Consider a normalized version of vertex function

$$\widetilde{V}(q, a, z) := (prefactor) \cdot V(q, a, z) \in K_{T \times \mathbb{C}^*_\hbar \times \mathbb{C}^*_q}(X^T)_{loc}[[z]].$$

Theorem (Okounkov)

The $K(X^T)$-valued function $\widetilde{V}(q, a, z)$ generates a holonomic module under the action of q-difference operators in both Kähler and equivariant parameters.

The q-difference equations are obtained geometrically, also by counting some kind of quasimaps. In general, they are not explicitly presented.

Properties of vertex functions

The q-difference module generated by $\widetilde{V}$ is the quantum D_q-module of the quantum K-theory, defined by quasimaps.

Vertex function lies in Givental's lagrangian cone for a modified version of permutation-equivariant quantum K-theory.

From the vertex function one can also deduce the *Bethe algebra*/ "quantum K-theory ring" / "chiral ring".

Example: $T^*\mathbb{P}^n$

Introduce "redundant" Kähler parameters $z_1, \cdots, z_{n+1}$, such that $z := \prod_{i=1}^{n+1} z_i$. $z_{\sharp,i} := (-\hbar^{1/2})z_{\sharp,i}$.

- $\widetilde{V}|_{p_k} := \prod_{i\neq k} \frac{(q\hbar^{-1}a_k a_i^{-1})_\infty}{(a_k a_i^{-1})_\infty} \cdot e^{\sum_{i=1}^{n+1} \frac{\ln(a_i/a_k)\ln z_{i,\sharp}}{\ln q}} \cdot V(q,z)|_{p_k}$, is annihilated by

$$\boxed{\prod_{i=1}^{n+1}(1 - q^{-z_i\partial_{z_i}}) - z_{1,\sharp}\cdots z_{n+1,\sharp}\prod_{i=1}^{n+1}(1 - \hbar^{-1}q^{-z_i\partial_{z_i}}).}$$

- $\widetilde{V}|_{p_k} \cdot e^{-\sum_{i=1}^{n+1} \frac{\ln a_i \ln z_{i,\sharp}}{\ln q}}$ is annihilated by

$$\boxed{(1 - q^{-a_i\partial_{a_i}})(1 - q\hbar^{-1}q^{-a_j\partial_{a_j}}) - \frac{a_j}{a_i}(1 - q\hbar^{-1}q^{-a_i\partial_{a_i}})(1 - q^{-a_j\partial_{a_j}})}$$

for any $i \neq j$.

Example: $\mathcal{A}_n$

Consider $G = (\mathbb{C}^*)^n$ acting on $T^*\mathbb{C}^{n+1}$ as

$$\begin{aligned} & (\lambda_1, \cdots, \lambda_n) \cdot (z_1, \cdots, z_{n+1}, w_1, \cdots, w_{n+1}) \\ = \; & (\lambda_1 z_1, \lambda_1^{-1}\lambda_2 z_2, \cdots, \lambda_{n-1}^{-1}\lambda_n z_n, \lambda_n^{-1} z_{n+1}, \\ & \lambda_1^{-1} w_1, \lambda_1 \lambda_2^{-1} w_2, \cdots, \lambda_{n-1}\lambda_n^{-1} w_n, \lambda_n w_{n+1}) \end{aligned}$$

Moment map is $\mu : T^*\mathbb{C}^{n+1} \to \mathbb{C}^n$

$$\mu : (\vec{z}, \vec{w}) \mapsto (z_1 w_1 - z_2 w_2, \cdots, z_n w_n - z_{n+1} w_{n+1}).$$

Example: $\mathcal{A}_n$

The holomorphic symplectic quotient is $X = \mu^{-1}(0)//_\theta G$, where $\theta : G \to \mathbb{C}^*$ is a character, or the stability condition.

Alternatively, we can view $\theta \in \mathrm{Lie}_\mathbb{R}(G)^\vee \cong \mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}$.

This $\mathbb{R}^n$ is divided into chambers by *walls*, where X is smooth if θ is in the interior of chambers, and singular if θ is on walls.

Choose a lift $\widetilde{\theta} = (\widetilde{\theta}_1, \cdots, \widetilde{\theta}_{n+1}) \in \mathbb{R}^{n+1}$. In this particular example, choice of θ is equivalent to the choice of an ordering, e.g.

$$\widetilde{\theta}_1 > \cdots > \widetilde{\theta}_{n+1}.$$

For different chambers, the X are symplectic flops to each other.

Example: $\mathcal{A}_n$

The vertex function is

$$V(q,a,z)|_{p_k} = \sum_{\substack{D_i\geq 0, i<k\\ D_i\leq 0, i>k\\ D_1+\cdots+D_{n+1}=0}} \prod_{i<k} \frac{(\hbar)_{D_i}}{(q)_{D_i}} \prod_{i>k} \frac{(1)_{D_i}}{(q\hbar^{-1})_{D_i}} \cdot \frac{(\hbar a_1\cdots a_{n+1}\hbar^{n+1-k})_{D_k}}{(qa_1\cdots a_{n+1}\hbar^{n+1-k})_{D_k}} \cdot z_1^{D_1}\cdots z_{n+1}^{D_{n+1}},$$

which lies in

$$K_{T\times\mathbb{C}_\hbar^*\times\mathbb{C}_q^*}(\mathcal{A}_n)_{loc}\left[\!\left[\frac{z_1}{z_k},\cdots,\frac{z_{k-1}}{z_k},\frac{z_k}{z_{k+1}},\cdots,\frac{z_k}{z_{n+1}}\right]\!\right]$$

(note that $z_1^{D_1}\cdots z_{n+1}^{D_{n+1}} = \prod_{i\neq k}(z_i/z_k)^{D_i}$)

$$\widetilde{V}|_{p_k} := \frac{(qa_1 \cdots a_{n+1}\hbar^{n+1-k})_\infty}{(\hbar a_1 \cdots a_{n+1}\hbar^{n+1-k})_\infty} \cdot e^{\frac{\sum_{i=1}^{n+1} \ln(\hbar^{-\delta_{i>k}} a_i^{-1}) \ln(z_{i,\sharp}/z_{k,\sharp})}{\ln q}} \cdot V|_{p_k}$$

- $\widetilde{V}|_{p_k}$ is annihilated by

$$(1-q^{-z_i\partial_{z_i}})(1-\hbar^{-1}q^{-z_j\partial_{z_j}}) - \frac{z_{i,\sharp}}{z_{j,\sharp}}(1-\hbar^{-1}q^{-z_i\partial_{z_i}})(1-q^{-z_j\partial_{z_j}})$$

  for any $i \neq j$.
- $\widetilde{V}|_{p_k} \cdot e^{-\sum_{i=1}^{n+1} \frac{\ln a_i \ln z_{i,\sharp}}{\ln q}}$ is annilhilated by

$$\prod_{i<k}(1-q^{-a_i\partial_{a_i}})\prod_{i\geq k}(1-q^{-1}\hbar q^{-a_i\partial_{a_i}})$$
$$-a_1^{-1}\cdots a_{n+1}^{-1}\prod_{i<k}(1-q\hbar^{-1}q^{-a_i\partial_{a_i}})\prod_{i\geq k}(1-q^{-a_i\partial_{a_i}})$$

Properties of vertex functions: analytic behavior

Compare the q-difference equations.

A natural guess for the conjecture would be $V = V^!$, up to the change of variables $z = a^!$, $a = z^!$.

However, this is not the case.

$\widetilde{V}(q, z, a)$ admits very *different* analytic properties w.r.t. the Kähler and equivariant parameters.

- $\widetilde{V}(q, z, a)$ is analytic as $z \to 0$, by definition;
- $\widetilde{V}(q, z, a)$ has infinitely many poles as $a \to 0$ or ∞.

Example: $T^*\mathbb{P}^N$

$$\widetilde{V}|_{p_k} := \prod_{i\neq k} \frac{(q\hbar^{-1}a_k a_i^{-1})_\infty}{(a_k a_i^{-1})_\infty} \cdot e^{\sum_{i=1}^{n+1} \frac{\ln(a_i/a_k)\ln z_{i,\sharp}}{\ln q}} \cdot V(q,z)|_{p_k}$$

- In terms of $z_1, \cdots, z_{n+1}$, it is the product of
  - a multi-valued exp factor, and
  - an analytic function $V|_{p_k}$, considered as valued in $K_{T\times\mathbb{C}^*_\hbar\times\mathbb{C}^*_q}(T^*\mathbb{P}^n)[[z_1\cdots z_{n+1}]]$.
- In terms of $a_1, \cdots, a_{n+1}$,
  $\widetilde{V}|_{p_k}$ contains factors of the form

$$\frac{(q\hbar^{-1}a_k/a_i)_\infty}{(a_k/a_i)_\infty} \cdot \frac{(\hbar a_i/a_k)_d}{(qa_i/a_k)_d}$$

for $d \geq 0$.
It admits infinitely poles of the form q^N as $a_i/a_k \to 0$ or ∞.

Let $(X, X^!)$ be a 3d mirror pair. Let $V(q,z,a)$, $V^!(q,z^!,a^!)$ be their vertex functions, and $\widetilde{V}(q,z,a)$, $\widetilde{V}^!(q,z^!,a^!)$ be the rescaled vertex functions.

Conjecture (Aganagic–Okounov, for vertex functions)

Under the change of variables $z \mapsto a^!$, $a \mapsto z^!$, there is a (nontrivial) transition matrix $\mathcal{P} \in \mathrm{End}(K(X^T))$, such that

$$\widetilde{V}^! = \mathcal{P} \cdot \widetilde{V}.$$

Moreover, $\mathcal{P}$ is given in terms of the *elliptic stable envelope* Stab,

$$\mathcal{P}_{\mathsf{p},\mathsf{q}} \propto \frac{\mathrm{Stab}(\mathsf{p})|_{\mathsf{q}}}{\mathrm{Stab}(\mathsf{q})|_{\mathsf{q}}}.$$

3d mirror conjecture for elliptic stable envelopes

Conjecture (Aganagic–Okounov, elliptic stable envelope)

Under the change of variables $z \mapsto a^!$, $a \mapsto z^!$,

1. the elliptic stable envelopes for X and $X^!$ admit a symmetry:

$$\frac{\mathrm{Stab}(\mathrm{p})|_{\mathrm{q}}}{\mathrm{Stab}(\mathrm{q})|_{\mathrm{q}}} = \frac{\mathrm{Stab}^!(\mathrm{q})|_{\mathrm{p}}}{\mathrm{Stab}^!(\mathrm{p})|_{\mathrm{p}}};$$

2. there exists a class $\mathfrak{m} \in \mathrm{Ell}_{T\times T^!}(X \times X^!)$, called "duality interface", such that

$$\mathfrak{m}|_{X\times \mathrm{p}} = \mathrm{Stab}^!(\mathrm{p})|_{\mathrm{p}} \cdot \mathrm{Stab}(\mathrm{p}), \ \ \mathfrak{m}|_{\mathrm{q}\times X^!} = \mathrm{Stab}(\mathrm{q})|_{\mathrm{q}} \cdot \mathrm{Stab}^!(\mathrm{q}).$$

1 imples 2.

Stable envelope

Let $X \to X_0$ be a symplectic resolution, with a T-action, where $T = A \times \mathbb{C}^*_\hbar$.

Choose a cocharacter $\sigma : \mathbb{C}^* \to A$. For each connected component $Z \subset X^A$, let

$$\mathrm{Attr}_\sigma(Z) := \{x \in X \mid \lim_{t \to 0} \sigma(t) \cdot x \in Z\}$$

be the attracting set, and let $\mathrm{Attr}^f_\sigma(Z)$ be the *full attracting set*, i.e. smallest closed subset in X which is closed under taking Attr.

Partial ordering: $Z' \preceq Z \Leftrightarrow Z' \subset \mathrm{Attr}^f_\sigma(Z)$.

For simplicity, we state it in the case where σ is chosen generically, such that $X^\sigma = X^A$ consists of isolated fixed points.

Definition (Maulik–Okounkov)

There is a unique map of $H_T^*(\mathrm{pt})$-modules $\mathrm{Stab}_\sigma^H : H_T^*(X^A) \to H_T^*(X)$, such that for any $\mathrm{p} \in X^A$,

1. $\mathrm{Stab}_\sigma^H(\mathrm{p})$ is supported on $\mathrm{Attr}_\sigma^f(\mathrm{p})$;
2. $\mathrm{Stab}_\sigma^H(\mathrm{p})|_{\mathrm{p}} = \pm e(T_{\mathrm{p}}^- X)$;
   ($T_{\mathrm{p}}^- X$ is the "half" in $T_{\mathrm{p}} X$ which pairs negatively with σ, and the sign $\pm$ is determined by a choice of polarization)
3. for any $\mathrm{q} \prec \mathrm{p}$, $\mathrm{Stab}_\sigma^H(\mathrm{p})|_{\mathrm{q}}$ is divisible by h. Equivalently, $\deg_A \mathrm{Stab}_\sigma^H(\mathrm{p})|_{\mathrm{q}} < \deg_A \mathrm{Stab}_\sigma^H(\mathrm{q})|_{\mathrm{q}} = \dim X/2$.

Example: $T^*\mathbb{P}^n$

Choose $\sigma(t) = (t^{\sigma_1}, \cdots, t^{\sigma_{n+1}})$, with $\sigma_1 < \cdots < \sigma_{n+1}$, ordering $p_1 > \cdots > p_{n+1}$.

$H^*_T(T^*\mathbb{P}^n) = \mathbb{C}[h, b_1, \cdots, b_{n+1}, H]/\prod_{i=1}^{n+1}(H + b_i)$, where $\hbar = e^h$, $a_i = e^{b_i}$.

$T_{p_k}X = \sum_{i\neq k}(\frac{a_i}{a_k} + \hbar^{-1}\frac{a_k}{a_i})$, $N^-_{p_k} = \sum_{i<k}\frac{a_i}{a_k} + \sum_{i>k}\hbar^{-1}\frac{a_k}{a_i}$.

$$\boxed{\mathrm{Stab}^H_\sigma(p_k) = \prod_{i<k}(H + b_i)\prod_{i>k}(h - H - b_i).}$$

$\mathrm{Stab}^H_\sigma(p_k)|_{p_j} = 0$, for $j < k$;
$\mathrm{Stab}^H_\sigma(p_k) = \prod_{i<k}(b_i - b_k)\prod_{i>k}(h - b_i + b_k)$;
$\mathrm{Stab}^H_\sigma(p_k)|_{p_j} = \prod_{i<k}(b_i - b_j)\prod_{i>k,i\neq j}(h + b_j - b_i) \cdot h$, for $j > k$.

K-theoretic stable envelope

Need an extra choice – the *slope* $\mathcal{L} \in \mathrm{Pic}(X)_{\mathbb{R}}$.

Definition (Maulik–Okounkov)

There is a unique map of $K_T^*(\mathrm{pt})$-modules $\mathrm{Stab}_\sigma^K : K_T(X^A) \to K_T(X)$, such that for any $\mathrm{p} \in X^A$,

1. $\mathrm{Stab}_\sigma^K(\mathrm{p})$ is supported on $\mathrm{Attr}_\sigma^f(\mathrm{p})$;
2. $\mathrm{Stab}_\sigma^K(\mathrm{p})|_{\mathrm{p}} \propto \bigwedge^\bullet(T_{\mathrm{p}}^- X)^\vee$;
   ($T_{\mathrm{p}}^- X$ is the "half" in $T_{\mathrm{p}}X$ which pairs negatively with σ, and the equality is up to a det factor determined by a choice of polarization)
3. for any $\mathrm{q} \prec \mathrm{p}$,

$$\deg_A \mathrm{Stab}_\sigma(\mathrm{p})|_{\mathrm{q}} \otimes \mathcal{L}|_{\mathrm{p}} \subset \deg_A \mathrm{Stab}_\sigma(\mathrm{q})|_{\mathrm{q}} \otimes \mathcal{L}|_{\mathrm{q}},$$

   where $\deg_A$ means to take the Newton polygon of a Laurent polynomial in a_i's.

Example: $T^*\mathbb{P}^n$

Choose $\sigma(t) = (t^{\sigma_1}, \cdots, t^{\sigma_{n+1}})$, with $\sigma_1 < \cdots < \sigma_{n+1}$, ordering $p_1 > \cdots > p_{n+1}$.

$K_T(T^*\mathbb{P}^n) = \mathbb{C}[\hbar^{\pm 1}, a_1^{\pm 1}, \cdots, a_{n+1}^{\pm 1}, s^{\pm 1}] / \prod_{i=1}^{n+1}(1 - a_i s)$.

$T_{p_k}X = \sum_{i \neq k}(\frac{a_i}{a_k} + \hbar^{-1}\frac{a_k}{a_i})$, $N_{p_k}^- = \sum_{i<k}\frac{a_i}{a_k} + \sum_{i>k}\hbar^{-1}\frac{a_k}{a_i}$.

Choose slope $\mathcal{L} = s^r$, where $r \in \mathbb{R}$.

$$\boxed{\mathrm{Stab}_\sigma^K(p_k) \propto \prod_{i<k}(1 - a_i s)\prod_{i>k}(1 - \hbar a_i s) \cdot s^{\lfloor r \rfloor + 1/2}.}$$

Properties of stable envelopes

- Depend locally constantly in σ and $\mathcal{L}$.
- Wall-crossing properties: define R-matrices

$$R_{\sigma,\sigma'} := \mathrm{Stab}_\sigma \circ \mathrm{Stab}_{\sigma'}^{-1} \in \mathrm{End}(H_T^*(X^A)) \text{ or } \mathrm{End}(K_T(X^A)).$$

  and similar for $\mathcal{L}$. They satisfy Yang–Baxter equations, and can be used to construct quantum groups.
- In the "stable basis", the q-difference equation in equivariant parameters resembles qKZ equation, and the q-diference operator in Kähler parameters resembers quantum dynamical Weyl operators.
- Its inverse is easy to describe:

$$\mathrm{Stab}_{\sigma,\mathcal{L},T^{1/2}}^{-1} = \mathrm{Stab}_{-\sigma,\mathcal{L}^{-1},T_{opp}^{1/2}}.$$

Elliptic stable envelope

Elliptic stable envelope is the generalization of the above in the equivariant elliptic cohomology. The slope $\mathcal{L}$ is replaced by a Kähler parameter z.

Let $E = \mathbb{C}^*/q^{\mathbb{Z}}$ be the elliptic curve.

Definition (Aganagic–Okounkov '16, Okounkov '20)

For any $\mathrm{p} \in X^A$, $\mathrm{Stab}_\sigma(\mathrm{p})$ is a section of a certain line bundle $\mathcal{T}(\mathrm{p})$ (explicitly described by q-quasi-periods of its sections) on $\mathrm{Ell}_T(X) \times E^{\mathrm{rk}\,\mathrm{Pic}(X)}$, such that

1. $\mathrm{Stab}_\sigma(\mathrm{p})$ is supported on $\mathrm{Attr}^f_\sigma(\mathrm{p})$;
2. $\mathrm{Stab}_\sigma(\mathrm{p})|_{\mathrm{p}} \propto \Theta(T^-_{\mathrm{p}}X)$;
   ($T^-_{\mathrm{p}}X$ is the "half" in $T_{\mathrm{p}}X$ which pairs negatively with σ)
3. holomorphicity condition
   (describes possible poles of $\mathrm{Stab}_\sigma(\mathrm{p})$)

Example: $T^*\mathbb{P}^n$

Choose $\sigma(t) = (t^{\sigma_1}, \cdots, t^{\sigma_{n+1}})$, with $\sigma_1 < \cdots < \sigma_{n+1}$, ordering $p_1 > \cdots > p_{n+1}$.

$\operatorname{Spec} K_T(T^*\mathbb{P}^n) = \operatorname{Spec} \mathbb{C}[\hbar^{\pm 1}, a_1^{\pm 1}, \cdots, a_{n+1}^{\pm 1}, s^{\pm 1}] / \prod_{i=1}^{n+1}(1 - a_i s)$. This is the union of $(n+1)$ copies of $(\mathbb{C}^*)^{\mathrm{rk}\, T}$, each given by the equation $(1 - a_i s = 0)$. The intersection is transversal (implied by the fact that $T^*\mathbb{P}^n$ is a GKM manifold).

$\mathrm{Ell}_T(T^*\mathbb{P}^n)$ is the union of $(n+1)$ copies of abelian varieties $E^{\mathrm{rk}\, T}$, where the k-th component is

$$\mathrm{Ell}_T(p_k) = \left(\operatorname{Spec} \mathbb{C}[\hbar^{\pm 1}, a_1^{\pm 1}, \cdots, a_{n+1}^{\pm 1}, s]/(1 - a_k s)\right) / q^{\mathbb{Z}^{\mathrm{rk}\, T}} = E^{\mathrm{rk}\, T}.$$

Example: $T^*\mathbb{P}^n$

Let z be the coordinates on $E^{\mathrm{rk}\,\mathrm{Pic}(X)} = E$.

Sections of the line bundle $\mathcal{T}(p_k)$, over $E^{\mathrm{T}} \times E^{\mathrm{rk}\,\mathrm{Pic}(X)}$, can be written in terms of theta functions in $z, a_i, \hbar$.

Let $\vartheta(x) := (x^{1/2} - x^{-1/2})(qx)_\infty(\frac{q}{x})_\infty$.

$$\mathrm{Stab}_\sigma(p_k) = \prod_{i<k} \vartheta(a_i s) \cdot \frac{\vartheta(z\hbar^{k-1-n} a_k s)}{\vartheta(z\hbar^{k-1-n})} \cdot \prod_{i>k} \vartheta(\hbar a_i s).$$

3d mirror conjecture for vertex functions

Let $(X, X^!)$ be a 3d mirror pair. Let $V(q, z, a)$, $V^!(q, z^!, a^!)$ be their vertex functions, and $\widetilde{V}(q, z, a)$, $\widetilde{V}^!(q, z^!, a^!)$ be the rescaled vertex functions.

Conjecture (Aganagic–Okounov, for vertex functions)

Under the change of variables $z \mapsto a^!$, $a \mapsto z^!$, there is a nontrivial transition matrix $\mathcal{P} \in \mathrm{End}(K(X^T))$, such that

$$\widetilde{V}^! = \mathcal{P} \cdot \widetilde{V}.$$

Moreover, $\mathcal{P}$ is given in terms of the *elliptic stable envelope* Stab,

$$\mathcal{P}_{\mathsf{p},\mathsf{q}} \propto \frac{\mathrm{Stab}(\mathsf{p})|_{\mathsf{q}}}{\mathrm{Stab}(\mathsf{q})|_{\mathsf{q}}}.$$

3d mirror conjecture for elliptic stable envelopes

Conjecture (Aganagic–Okounov, for elliptic stable envelopes)

Under the change of variables $z \mapsto a^!$, $a \mapsto z^!$,

1. the elliptic stable envelopes for X and $X^!$ admit a symmetry:

$$\frac{\mathrm{Stab}(\mathsf{p})|_{\mathsf{q}}}{\mathrm{Stab}(\mathsf{q})|_{\mathsf{q}}} = \frac{\mathrm{Stab}^!(\mathsf{q})|_{\mathsf{p}}}{\mathrm{Stab}^!(\mathsf{p})|_{\mathsf{p}}};$$

2. there exists a class $\mathfrak{m} \in \mathrm{Ell}_{T \times T^!}(X \times X^!)$, called "duality interface", such that

$$\mathfrak{m}|_{X \times \mathsf{p}} = \mathrm{Stab}^!(\mathsf{p})|_{\mathsf{p}} \cdot \mathrm{Stab}(\mathsf{p}), \ \ \mathfrak{m}|_{\mathsf{q} \times X^!} = \mathrm{Stab}(\mathsf{q})|_{\mathsf{q}} \cdot \mathrm{Stab}^!(\mathsf{q}).$$

Results

Theorem

- *Conjectures for vertex functions and elliptic stable envelopes are proved for*
  - *hypertoric varieties [Smirnov–Z, '19];*
  - $T^*Fl(1, \cdots, n)$ *[Dinkins, '20].*
- *Conjecture for elliptic stable envelopes is proved for*
  - T^*Gr *[Rimányi–Smirnov–Varchenko–Z, '18];*
  - $T^*Fl(1, \cdots, n)$ *[Rimányi–Smirnov–Varchenko–Z, '18].*
  - T^*G/B *[Rimányi–Weber, '20]*

Idea of proof (vertex functions)

Step 1 Identify the q-difference equations.

For known cases, this relies strongly on the combinatorial undertanding of the targets, and the q-difference equations.

For hypertorc varieties, q-difference equations are expressed in terms of circuits/cocircuits structures of the hypertoric data.

For $T^*Fl(1, \cdots, n)$, this follows from the results of P. Koroteev and A. Zeitlin, where q-difference operators are expressed in terms of Macdonald operators, via a duality between qKZ and tRS (trigonometric Ruijsenaars-Schneider) system.

Idea of proof (vertex functions)

Step 2 Apply the result of Aganagic–Okounkov '16, that $\mathrm{Stab} \cdot V$ satisfies the same asymptotic behavior as the mirror vertex function $\widetilde{V}^!$.

The conjecture follows by the uniqueness of solutions of q-difference equations.

Thank you!