On arithmetic topology and arithmetic Dijkgraaf-Witten theory

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Contents

1. Background and Motivation

Arithmetic Topology

• investigates the interaction between 3-dim. topology and number theory,

and, historically,

• is along a string of thoughts of geometrization of number theory.

1. Background and Motivation

Geometrization of Number Theory

An origin of modern number theory goes back to Gauss' Disquisitiones Arithmeticae (1801):

・ quadratic residues ・ binary quadratic forms (genus theory)

Knot theory is originated from Gauss' electro-magnetic theory (1833)

After Gauss, algebraic number theory had been developed by Kummer, Dedekind and Hilbert etc during 19th century, and, entering in the 20th century, Takagi and E. Artin established class field theory.

A principal leading this line of developments was the analogy between number fields and function fields.

1. Background and Motivation

Analogy between number fields and function fields

$$\mathbb{Z} \subset \mathbb{Q} \quad \longleftrightarrow \quad \mathbb{C}[x] \subset \mathbb{C}(x)$$

integers, rationals $\longleftrightarrow$ polynomials, rational functions

prime number $\longleftrightarrow$ irred. polynomial $\Leftrightarrow$ point on $\mathbb{C}$

Number Theory	Algebraic Geometry
number field k	function field $F(x,y)$, $f(x,y) = 0$ (F: field)
ring of integers i.e., arithmetic curve $M_k = \mathrm{Spec}(\mathcal{O}_k)$	algebraic curve (Riemann surface for $F = \mathbb{C}$) $V = \mathrm{Spec}(F[x,y]/(f(x,y)))$
prime ideal $\mathfrak{p} \in M_k$	point $P \in V$

★ This analogy is algebraic, based on that $\mathcal{O}_k$ and $F[x,y]/(f(x,y))$ are both Dedekind ring (1-dim. ring)

However,

for an algebraic curve $V = \mathrm{Spec}(F[x,y]/(f(x,y))$, say $F = \mathbb{C}$,

$$\begin{aligned} V(\mathbb{C}) &= \{(x,y) \in \mathbb{C} \times \mathbb{C} | f(x,y) = 0\} \\ &= \mathrm{Hom}(F[x,y]/(f(x,y)), \mathbb{C}) \\ &= \mathrm{Max}(F[x,y]/(f(x,y)) \end{aligned}$$

On the other hand, for an arithmetic curve $M_k = \mathrm{Spec}(\mathcal{O}_k)$,

$$\begin{aligned} M_k(\mathbb{C}) &= \mathrm{Hom}(\mathcal{O}_k, \mathbb{C}) = \{k \hookrightarrow \mathbb{C}\} : \text{ finite set} \\ &\neq \mathrm{Max}(\mathcal{O}_k) : \text{ infinite set} \end{aligned}$$

So, the analogy is not satisfactory.

1. Background and Motivation

After Gauss, entering in 20th century, topology had been developed by Poincaré, Alexander, Lefschetz, Dehn, Seifert, Reidemeister etc.

- homology and homotopy groups
- combinatorial group theory

Alexander's Thm: Any oriented connected closed 3-manifold is a finite branched cover of the 3-sphere S^3.

In the course of simplifying the complicated proof of class field theory, number theory has been influenced by topology and also by Grothendieck's topology (1950 $\sim$ 1960's; Nakayama, Tate, M. Artin, Verdier etc)

- Galois cohomology
- étale cohomology, étale homotopy

1. Background and Motivation

From topological view point,

for distinct points $P, Q \in \mathrm{Aff}^1_F = \mathrm{Spec}(F[x])$,

$$\pi_1(\mathrm{Aff}^1_F \setminus \{P\}) = \pi_1(\mathrm{Aff}^1_F \setminus \{Q\})$$
$$(\pi_1(\mathbb{C} \setminus \{P\}) = \pi_1(\mathbb{C} \setminus \{Q\}) \text{ for } F = \mathbb{C}).$$

On the other hand, for distinct prime ideals $(p), (q) \in \mathrm{Spec}(\mathbb{Z})$,

$$\pi_1(\mathrm{Spec}(\mathbb{Z} \setminus \{(p)\}) \neq \pi_1(\mathrm{Spec}(\mathbb{Z} \setminus \{(q)\}).$$

So, the analogy between number fields and function fields is not satisfactory again.

Topologically,

for $\mathfrak{p} \in \mathrm{Max}(\mathcal{O}_k)$,

$$\{\mathfrak{p}\} = \mathrm{Spec}(\mathcal{O}_k/\mathfrak{p}) = K(\widehat{\mathbb{Z}}, 1) = B_{\widehat{\mathbb{Z}}},$$
$$\widehat{\mathbb{Z}} = \text{ profinite infinite cyclic group.}$$

Similarly,

$$S^1 = K(\mathbb{Z}, 1) = B_{\mathbb{Z}},$$
$$\mathbb{Z} = \text{ infinite cyclic group.}$$

1. Background and Motivation

A number ring $\mathrm{Spec}(\mathcal{O}_k)$ is of étale cohomological dimension 3 and enjoys a sort of 3-dimensional arithmetic Poincaré duality, called Artin-Verdier duality (arithmetic analog of electro-magnetic duality).

$\rightsquigarrow$

$$\begin{array}{ccc} M_k = \mathrm{Spec}(\mathcal{O}_k) & \longleftrightarrow & \text{3-manifold } M \\ \overline{M}_k = M_k \cup \{\text{infinite primes}\} & \longleftrightarrow & \text{end compactified 3-manifolds } \overline{M} \end{array}$$

Alexander's Thm $\leftrightarrow$ Any number field is a finite branched extension of $\mathbb{Q}$

Thus,

prime $\{\mathfrak{p}\} = \mathrm{Spec}(\mathcal{O}_k/\mathfrak{p}) = K(\hat{\mathbb{Z}}, 1) \hookrightarrow \mathrm{Spec}(\mathcal{O}_k)$

$\longleftrightarrow$

knot $\mathcal{K}\ :\ S^1 = K(\mathbb{Z}, 1) \hookrightarrow M$

1. Background and Motivation

3-dim. Topology	Number Theory
3-manifold M	number ring $M_k = \mathrm{Spec}(\mathcal{O}_k)$
knot $\mathcal{K} : S^1 \hookrightarrow M$	maximal ideal $\{\mathfrak{p}\} = \mathrm{Spec}(\mathcal{O}_k/\mathfrak{p}) \hookrightarrow \mathrm{Spec}(\mathcal{O}_k)$
tubular n.b.d. of a knot $V_{\mathcal{K}}$	$\mathfrak{p}$-adic integer ring $V_S = \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}})$
boundary torus $\partial V_{\mathcal{K}}$	$\mathfrak{p}$-adic field $\partial V_S = \mathrm{Spec}(k_{\mathfrak{p}})$
link $\mathcal{L} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r$	finite set of prime ideals $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$
link complement $X_{\mathcal{L}} = M \setminus \mathcal{L}$	complement of a finite set of primes $X_S = M_k \setminus S$
link group $\Pi_{\mathcal{L}} = \pi_1(X_{\mathcal{L}})$	Galois gr. with given ramification $\Pi_S = \pi_1^{\acute{e}t}(X_S) = \mathrm{Gal}(k_S/k)$
$\vdots$	$\vdots$

These analogies were been discovered originally and initiated to be studied by B. Mazur, M. Kapranov, A. Reznikov and by myself independently.

$\rightsquigarrow$

Arithmetic Topology

A driving force to obtain analogous results and develop 3-dim. topology and number theory.

1. Background and Motivation

Relation with the analogy between no. fields and f'n fields

· algebraic curve over a finite field $\longleftrightarrow$ surface bundle over S^1

· Analogy between knots and primes, 3-manifolds and number ring

$=$

Time evolution of the analogy between number fields and function fields, where prime ideal looks like a trajectory (braid) of a particle on a surface.

Enhancement of arithmetic topology

Foliated dynamical systems (C. Deninger)

FDS	$\longleftrightarrow$	number ring
3-dim. closed mfd + $\begin{cases}\text{2-dim. foliation}\\ \text{transversal flow}\end{cases}$		$\overline{\mathrm{Spec}(\mathcal{O}_k)}$
closed orbit (knot)	$\longleftrightarrow$	finite prime
non-transverse compact leaf	$\longleftrightarrow$	infinite prime

Reflection

knot group $\pi_1(S^3 \setminus K) \longleftrightarrow$ prime group $\pi_1(\mathrm{Spec}(\mathbb{Z}) \setminus \{(p)\})$

$\pi_1(S^3 \setminus K)$ is finitely presented, on the other hand,

$\pi_1(\mathrm{Spec}(\mathbb{Z}) \setminus \{(p)\})$ is huger (unknown if it is finitely generated or not)

$\Longrightarrow$

A prime is a invisibly complicated knot !

3-dim. picture is a geometric approximation of a number ring and primes.

1. Background and Motivation

Hope (Dream)

$$\begin{array}{ccccc} \mathrm{Spec}(\mathcal{O}_k) & \overset{\text{analogy}}{\longleftrightarrow} & \text{3-dim. space } M & \subset & \text{bigger space } \mathfrak{X}_k \text{ ?} \\ \cup & & \cup & & \cup \\ \{\mathfrak{p}\} & \to & \text{complicated knot} & \mapsto & \text{unknot} \end{array}$$

$\mathfrak{X}_k$ is the true geometric model of $\mathrm{Spec}(\mathcal{O}_k)$.

Deninger proposed recently a candidate $\mathfrak{D}_k$ of such a space $\mathfrak{X}_k$ of infinite dimension, equipped with dynamical sysytem ($\mathbb{R}$-action).

$$\begin{cases} \mathfrak{D}_k \sim \mathrm{Spec}(W_{\mathrm{rat}}(\mathcal{O}_k))(\mathbb{C}), \\ W_{\mathrm{rat}}(R) \subset W(\mathcal{O}_k) = \mathbb{Z} \otimes_{\mathbb{F}_1} \mathcal{O}_k. \end{cases}$$

☆ Note $\pi_1(\mathbb{F}_1) = \mathbb{R}$. Arithmetic Topology $\overset{\text{Deninger}}{\longleftrightarrow}$ $\mathbb{F}_1$-geometry

Basic Dictionary of AT

3-dim. Topology	Number Theory
oriented, connected 3-manifold M	number ring $M_k = \mathrm{Spec}(\mathcal{O}_k)$
end of M	infinite primes
closed 3-manifold $\overline{M}$	compactified $\overline{M}_k$
knot $\mathcal{K} : S^1 \hookrightarrow X$	prime ideal $\{\mathfrak{p}\} = \mathrm{Spec}\ (\mathcal{O}_K/\mathfrak{p}) \hookrightarrow \overline{X}_k$
link $\mathcal{L} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r$	finite set of prime ideals $S = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_r\}$
tubular n.b.d. of a knot $V_{\mathcal{K}}$	$\mathfrak{p}$-adic integer ring $V_{\mathfrak{p}} = \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}})$
boundary torus $\partial V_{\mathcal{K}}$	$\mathfrak{p}$-adic field $\partial V_{\mathfrak{p}} = \mathrm{Spec}(k_{\mathfrak{p}})$
peripheral group $\Pi_{\mathcal{K}} = \pi_1(\partial V_{\mathcal{K}})$	local absolute Galois group $\Pi_{\mathfrak{p}} = \pi_1(\mathrm{Spec}(k_{\mathfrak{p}})) = \mathrm{Gal}(\overline{k_{\mathfrak{p}}}/k_{\mathfrak{p}})$

2. Dictionary and Results

link complement $X_{\mathcal{L}} = \overline{M} \setminus \mathcal{L}$	complement of a finite set of primes $\overline{X}_S = \overline{M}_k \setminus S$
link group $\Pi_{\mathcal{L}} = \pi_1(X_{\mathcal{L}})$	Galois gr. with given ramification $\Pi_S = \pi_1^{\acute{e}t}(\overline{X}_S) = \mathrm{Gal}(k_S/k)$
$\partial : C_2(M) \to Z_1(M)$ $\Sigma \mapsto \partial\Sigma$	$k^\times \to I_k$ $a \mapsto (a) = a\mathcal{O}_k$
1st homology group $H_1(M)$	ideal class group $Cl(k)$
2nd homology group $H_2(M)$	unit group $\mathcal{O}_k^\times$

More elaborate analogies were obtained by T. Mihara and J. Ueki in recent years.

2. Dictionary and Results

Classical invariants$\cdots$ defined by using knot, link, 3-manifold groups and Galois groups and their representations.

- Topology $\Rightarrow$ Number Theory: **Higher order linking numbers and multiple quadratic residue symbols**

link group $\Pi_{\mathcal{L}} = \pi_1(X_{\mathcal{L}})$ for $\mathcal{L} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r$	Galois gr. with given ramification $\Pi_S = \pi_1^{ét}(\overline{X}_S) = \mathrm{Gal}(k_S/k)$ for $S = \{p_1, \ldots, p_r\}$
Milnor' Theorem $\Pi_{\mathcal{L}}^{\text{pro-}l} = \langle x_1, \ldots, x_r \mid$ $[x_1, y_1] = \cdots = [x_r.y_r] = 1\rangle$ x_i = meridian of $\mathcal{K}_i$ y_i = longitude of $\mathcal{K}_i$	Koch's Theorem $\Pi_S^{\text{pro-}l} = \langle x_1, \ldots, x_r \mid$ $x_1^{p_1-1}[x_1, y_1] = \cdots = x_r^{p_r-1}[x_r.y_r] = 1\rangle$ x_i = monodromy over p_i y_i = Frobenius auto. over p_i
Higher order linking numbers (Milnor invariants) $\mathrm{lk}(\mathcal{K}_1, \cdots, \mathcal{K}_n)$	?

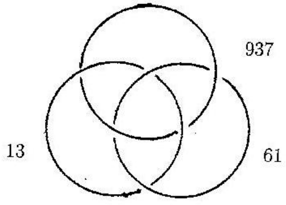
2. Dictionary and Results

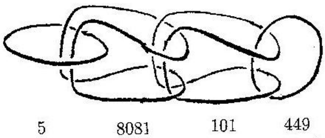
Arithmetic higher order mod 2 linking numbers of primes (M.)

We can introduce arithmetic higher order mod 2 linking numbers $\mathrm{lk}_2(1\cdots n)$ and the multiple quadratic residue symbols $[p_1,\ldots,p_n] = (-1)^{\mathrm{lk}_2(1\cdots n)}$ so that

- $[p_1,p_2] = \left(\frac{p_1}{p_2}\right)$ (quadratic residue symbol),
- $[p_1,\cdots,p_n]$ describes the decomposition of p_n in a certain nilpotent extension K_n of $\mathbb{Q}$ branched over $p_1,\ldots,p_{n-1}$.

Ex.(Borromean primes, Milnor primes (Amano))





2. Dictionary and Results

Applications

Higher order generalization of Gauss' genus theory (M.)

Multiple quadratic residue symbols $[p_{i_1}, \ldots, p_{i_n}]$'s describes the 2-class group of $\mathbb{Q}(\sqrt{p_1 \cdots p_r})$.

Cf. Recent related works by A. Smith, P. Koymans and C. Pagono on Higher Genus Theory for the 2-class groups of multi-quadratic fields $\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_r})$.

Pro-p link groups of number fields (Mizusawa)

Deep study of certain pro-p Galois groups of number fields with Koch type presentation, called pro-p link groups of number fields, and arithmetic higher order linking numbers, including primes over p, with application to Iwasawa theory.

Y. Mizusawa, On pro-p link groups of number fields, Transactions of AMS, **372** (2019), no. 10, 7225-7254.

2. Dictionary and Results

Digression:

Quadratic reciprocity law $[p_1, p_2] = [p_2, p_1]$ $p_1, p_2 \equiv 1 \bmod 4$	The law of action-reaction $(e, b) = (b, e)$ e = electric charge, b = magnetic charge
Class field theory Artin-Verdier duality	Maxwell theory electro-magnetic duality
multiple quadratic reciprocity law $[p_{\sigma(1)}, \ldots, p_{\sigma(n)}] = [p_1, \ldots, p_n]$	?

Ques. Is there (electro-magnetic) duality with nilpotent gauge groups ?

- Number Theory $\Rightarrow$ Topology: **Alexander invariants and Iwasawa zeta functions**

$\Pi_{\mathcal{K}}^{\mathrm{ab}} = \mathrm{Gal}(X_{\mathcal{K}}^{\infty}/X_{\mathcal{K}})$ $= \langle \mathfrak{m} \rangle = \mathbb{Z}$ infinite cyclic cover $X_{\mathcal{K}}^{\infty} \to X_{\mathcal{K}}$	$\Pi_{p}^{\mathrm{ab}} = \mathrm{Gal}(X_{p}^{\infty}/X_{p})$ $\approx \langle \tau \rangle = \mathbb{Z}_p$ cyclotomic $\mathbb{Z}_p$-extension $X_p^{\infty} \to X_p$ $(\mathbb{Q}(\sqrt[p^{\infty}]{1})/\mathbb{Q})$
Alexander module over $\mathbb{Z}[\alpha]$ $H_1(X_{\mathcal{K}}^{\infty})$	Iwasawa module $\mathbb{Z}_p[[\tau]]$ $H_1(X_p^{\infty})$
Alexander polynomial $=$ Lefschetz zeta function $A_K(t) = \det(t - \mathfrak{m} \mid H_1(X_{\mathcal{K}}^{\infty}))$	Iwasawa polynomial $=$ p-adic zeta function $I_p(t) = \det(t - \tau \mid H_1(X_p^{\infty}))$
twisted Alexander polynomial for a repr. $\Pi_{\mathcal{K}} \to GL_n(\mathbb{C})$	twisted Iwasawa polynomial for a repr. $\Pi_p \to GL_n(\mathbb{Z}_p)$

2. Dictionary and Results

Iwasawa theoretic and p-adic study of covers of links (Hillman-Matei-M., Kadokami-Mizusawa, Sugiyama, Ueki, Tange)

- p-adic study of homology growth of covers of links (H-M-M, Kadokami-Mizusawa).
- Topological analog of Iwasawa main conjecture (Sugiyama).
- p-adic dynamical study of Iwasawa invariants for covers of links, Kida's formula (Ueki).
- Twisted homology growth of covers of knots (Tange).

2. Dictionary and Results

1-dim. tautological repr. $\rho_{\mathcal{K}} : \Pi_{\mathcal{K}} \to \Pi_{\mathcal{K}}^{\mathrm{ab}} \subset \mathbb{C}[\langle \mathfrak{m} \rangle]^{\times}$	1-dim. universal deformation $\rho_p : \Pi_p \to \Pi_p^{\mathrm{ab}} \subset \mathbb{Z}_p[[\langle \tau \rangle]]^{\times}$
$\mathfrak{A}_{\mathcal{K}} = H_1(\Pi_{\mathcal{K}}, \rho_{\mathcal{K}})$ is a coherent sheaf on $\mathfrak{X}_{\mathcal{K}} = \mathrm{Spec}(\mathbb{C}[\langle \alpha \rangle])$ s.t. $A_{\mathcal{K}}(t) \in \Gamma(\mathfrak{X}_{\mathcal{K}}, \mathfrak{A}_{\mathcal{K}})$	$\mathfrak{I}_p = H_1(\Pi_p, \rho_p)$ is a coherent sheaf on $\mathfrak{X}_p = \mathrm{Spec}(\mathbb{Z}_p[[\langle \tau \rangle]])$ s.t. $I_p(t) \in \Gamma(\mathfrak{X}_p, \mathfrak{I}_p)$
?	deformation theory for higher dim. repr.'s of Π_p and the Selmer module

Deformations of knot group representations (Kitayama, M., Takakura, Tange, Terashima, Tran, Ueki)

- Deformation theory for SL_2-representations of a knot group.
- Study of Alexander invariants associated to the universal deformations of SL_2-representations of knot groups.

2, Dictionary and Results

holonomy repr's of $\Pi_{\mathcal{K}}$ (K: hyperbolic knot)	ordinary p-adic modular repr's of Π_p
deformations of hyperbolic str's on $X_{\mathcal{K}}$	deformations of p-adic ordinary modular forms

Chern-Simons variations of MHS (M., Terashima)

Complex Chern-Simons invariants $CS(\rho)$ gives a variation of MHS on the deformation space $\rho \in \mathfrak{D}_K$ of hyperbolic structures.

3. Two Basic Questions

Two basic questions in order to develop the analogies further:

Q 1. Number Theory ⇒ 3-dim. Topology :

What set of knots in S^3 is a topological analog of the set of all primes 2, 3, 5, 7, . . . ?
(★ Note that the set $\{2, 3, 5, 7, \ldots\}$ has marvelous nice structures such as the product formula, Hilbert reciprocity law, etc)

Can we develop an idèlic theory for 3-manifolds ?

Q 2. 3-dim Topology ⇒ Number Theory :

What are arithmetic analogs of quantum invariants ?

What are arithmetic analogs of Witten invariants (partition functions) in (2+1)-dim. TQFT ?

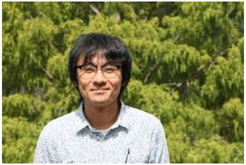
4. Idèlic Class Field Theory for 3-Manifolds

As for **Q 1**, Niibo-Ueki and Mihara gave answers and established idèlic class field theory for 3-manifolds.

- H. Niibo and J. Ueki,
Idèlic class field theory for 3-manifolds and very admissible link,
Transactions AMS, **371**, 2019, 8467–8488.

- T. Mihara,
Cohomological approach to class field theory in arithmetic topology,
Canadian J. Math., **71**, 2019, 891–935.

stably generic link



4. Idèlic Class Field Theory for 3-Manifolds

Very admissible link, Stably generic link (Niibo-Ueki, Mihara)

A link $\mathcal{L}$ of countably many components in a 3-manifold M is called a very admissible link if for any finite branched cocer $h : N \to M$ branched over a finite link in $\mathcal{L}$, the components of $h^{-1}(\mathcal{L})$ generates $H_1(N)$.
A link $\mathcal{L}$ is called a stably generic link if for any finite branched cocer $h : N \to M$ branched over a finite link in $\mathcal{L}$ and for a finite link L of $h^{-1}(\mathcal{L})$, the components of $h^{-1}(\mathcal{L}) \setminus L$ generates $H_1(N \setminus L)$.

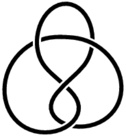
So stably generic $\Rightarrow$ very admissible.

Thm. (Niibo-Ueki, Mihara)

Stably generic link (hence very admissible link) exists.

Example of stably generic link (Ueki):

$\mathcal{K}$ = figure eight knot $\subset S^3$:



$$\begin{cases} X_K = S^3 \setminus K = (\Sigma \times [0,1])/(x,1) \sim (\tau(x),1), \ \Sigma = \text{ punctured torus,} \\ \tau : \Sigma \to \Sigma : \text{ monodromy} \end{cases}$$

ϕ_t: suspended flow of τ on $X_{\mathcal{K}}$.

Thm. (Ueki)

The set $\{$closed of orbits of $\phi^t\} \cup \{\mathcal{K}\}$ is a stably generic link.

Local theory.

Local class field theory for $\partial V_{\mathcal{K}}$ (Niibo-Ueki)

:
There is a canonical isomorphism (Hurewicz isom.)

$$\rho_{\mathcal{K}} : H_1(\partial V_{\mathcal{K}}) \xrightarrow{\sim} \mathrm{Gal}(\partial V_{\mathcal{K}}^{\mathrm{ab}}/\partial V_{\mathcal{K}})$$

s.t.

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}[\mathfrak{m}_{\mathcal{K}}] & \to & H_1(\partial V_{\mathcal{K}}) & \stackrel{v_{\mathcal{K}}^{\mathrm{h}}}{\to} & H_1(V_{\mathcal{K}}) = \mathbb{Z}[\mathfrak{l}_{\mathcal{K}}] & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \mathrm{Gal}(\partial V_{\mathcal{K}}^{\mathrm{ab}}/\partial V_{\mathcal{K}}^{\mathrm{ur}}) & \to & \mathrm{Gal}(\partial V_{\mathcal{K}}^{\mathrm{ab}}/\partial V_{\mathcal{K}}) & \to & \mathrm{Gal}(\partial V_{\mathcal{K}}^{\mathrm{ur}}/\partial V_{\mathcal{K}}) & \to & 0 \end{array}$$

Cf. Local class field theory for $k_{\mathfrak{p}}$:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_{\mathfrak{p}}^{\times} & \to & k_{\mathfrak{p}}^{\times} & \stackrel{v_{\mathfrak{p}}}{\to} & \mathbb{Z} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cap & & \\ 0 & \to & \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}^{\mathrm{ur}}) & \to & \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ab}}/k_{\mathfrak{p}}) & \to & \mathrm{Gal}(k_{\mathfrak{p}}^{\mathrm{ur}}/k_{\mathfrak{p}}) = \hat{\mathbb{Z}} & \to & 0 \end{array}$$

4. Idèlic Class Field Theory for 3-Manifolds

Cohomological local class field theory for $\partial V_{\mathcal{K}}$ (Mihara)

:
There is a canonical isomorphism

$$\rho_{\mathcal{K}} : H^1(\partial V_{\mathcal{K}}) \stackrel{\sim}{\longrightarrow} \mathrm{Gal}(\partial V_{\mathcal{K}}^{\mathrm{ab}}/\partial V_{\mathcal{K}})$$

s.t.

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}[\mathfrak{m}_{\mathcal{K}}] & \to & H^1(\partial V_{\mathcal{K}}) & \stackrel{v_{\mathcal{K}}}{\to} & H_1(V_{\mathcal{K}}) = \mathbb{Z}[\mathfrak{l}_{\mathcal{K}}] & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \mathrm{Gal}(\partial V_{\mathcal{K}}^{\mathrm{ab}}/\partial V_{\mathcal{K}}^{\mathrm{ur}}) & \to & \mathrm{Gal}(\partial V_{\mathcal{K}}^{\mathrm{ab}}/\partial V_{\mathcal{K}}) & \to & \mathrm{Gal}(\partial V_{\mathcal{K}}^{\mathrm{ur}}/\partial V_{\mathcal{K}}) & \to & 0 \end{array}$$

Rem.
• This local class field theory is analogous to Tate's arithmetic local duality: $H^1(k_{\mathfrak{p}}, \mu_n) \simeq H^1(k_{\mathfrak{p}}, \mathbb{Z}/n\mathbb{Z})^*$.
• This local class field theory is essentially same as the isomorphism $H_1(\partial V_{\mathcal{K}}) \simeq H_1(\mathrm{Jac}(\partial V_{\mathcal{K}}))$, which is, taking $\mathrm{Hom}(-, \mathbb{C}^\times)$, same as the abelian S-duality

flat $\mathbb{C}^\times$-connections on $\partial V_{\mathcal{K}}$ $\Leftrightarrow$ flat $\mathbb{C}^\times$-connections on $\mathrm{Jac}(\partial V_{\mathcal{K}})$

Global theory.

$\mathcal{L}$: stably generic link in an ori. conn. closed 3-manifold M.
Idèle group for $(M, \mathcal{L})$:
$J_{\mathcal{L}} := \{(a_{\mathcal{K}}) \in \prod_{\mathcal{K} \in \mathcal{L}} H^1(\partial V_{\mathcal{K}}) \,|\, v_{\mathcal{K}}(a_{\mathcal{K}}) = 0$ for almost all $\mathcal{K}\}$
Idèle class group for $(M, \mathcal{L})$:
$C_{\mathcal{L}} := \mathrm{Coker}(\Delta : H_2(X_{\mathcal{L}}, \partial V_{\mathcal{L}}) \simeq H^1(X_{\mathcal{L}}) \to J_{\mathcal{L}})$

Idèlic class field theory for a 3-manifold M (Niibo-Ueki, Mihara)

$\prod_{\mathcal{K} \in \mathcal{L}} \rho_{\mathcal{K}} : J_{\mathcal{L}} \to \prod_{\mathcal{K} \in \mathcal{L}} \mathrm{Gal}(\partial V_{\mathcal{K}}^{\mathrm{ab}}/\partial V_{\mathcal{K}}) \to \mathrm{Gal}(X_{\mathcal{L}}^{\mathrm{ab}}/X_{\mathcal{L}})$ induces the isomorphism

$$C_{\mathcal{L}} \xrightarrow{\sim} \mathrm{Gal}(X_{\mathcal{L}}^{\mathrm{ab}}/X_{\mathcal{L}})$$

Rem. and Ques..

We showed Hilbert type reciprocity law for Deninger's 3-dimensional foliated dynamical system (J. Kim, M., Noda, Terashima).
Can we develop enhanced idèlic class field theory for FDS ?
Then, can we have an analytic interpretation for idèlic class field theory by introducing Artin and Hecke type L-functions for FDS ?

(Is there any relation with 3d/3d duality in physics ?)

5. Arithmetic Dijkgraaf-Witten TQFT

As for **Q 2**: **Rough answer**:

invariants in 3-dim. topology $\longleftrightarrow$ zeta functions in number theory (and their special values)

Ex. Alexander polynomial = Lefschetz zeta function.
Jones polynomial = partition function in Chern-Simons gauge theory.

partition functions in physics $\longleftrightarrow$ zeta functions in number theory

partition function $= \int_{\text{all fields}} e^{S(A)} \mathcal{D}A$, $S(A)$ = action f'nal.

Riemann's $\zeta(s) = \int_{\text{all idèles}} e^{\Lambda(x)} dx$, $\Lambda(x) = \log ||x||^s \varphi(x)$ = action f'nal.

Ex. $U(1)$ CS partition f'n $\leftrightarrow$ Gaussian sum $\sim L(\chi, 1)$.

5. Arithmetic Dijkgraaf-Witten TQFT

Q 2:

3-dim. Topology		Number Theory
Quantum invariants	$\longleftrightarrow$	?
Witten invariants	$\longleftrightarrow$	?

Recall:
A framework to produce quantum invariants for knots, 3-manifolds is (2+1)-dim. Topological Quantum Field Theory (TQFT)

Ex. (Witten)
Jones polynomial $\Leftarrow$ Chern-Simons TQFT with gauge group $\mathrm{SU}(2)$

★ For **Q 2**, we want an arithmetic analog of Chern-Simons TQFT.

5. Arithmetic Dijkgraaf-Witten TQFT

Arithmetic Chern-Simons Theory (Minhyong Kim, 2006)

3-dim. TQFT	Number Theory
Chern-Simons theory with compact gauge group	arithmetic Chern-Simons Theory with finite or p-adic gauge gr.
Chern-Simons functional on the space of connections	arithmetic Chern-Simons functional on the space of Galois representations

Dijkgraaf-Witten TQFT

= Chern-Simons TQFT with finite gauge group

★ "path integral" is replaced by a finite sum

5. Arithmetic Dijkgraaf-Witten TQFT

Question:

DW TQFT	Arithmetic DW. TQFT
CS f'nal	arithmetic CS f'nal (M. Kim)
TQFT structure $\Sigma \rightsquigarrow \mathcal{H}_\Sigma$ $M \rightsquigarrow \mathcal{Z}_M \in \mathcal{H}_{\partial M}$	?
quantum Hilbert space $\mathcal{H}_\Sigma$	?
DW invariant $\mathcal{Z}_M$	?

5. Arithmetic Dijkgraaf-Witten TQFT

Arithmetic Dijkgraaf-Witten TQFT (J. Kim-Hirano-M.)

DW TQFT	Arithmetic DW TQFT
CS f'nal	arithmetic CS f'nal (M. Kim)
TQFT structure $\Sigma \rightsquigarrow \mathcal{H}_\Sigma$ $M \rightsquigarrow \mathcal{Z}_M \in \mathcal{H}_{\partial M}$	arithmetic TQFT structure $\Sigma_S \rightsquigarrow \mathcal{H}_S$ $X_S \rightsquigarrow \mathcal{Z}_S \in \mathcal{H}_S$
quantum Hilbert space $\mathcal{H}_\Sigma$	arithmetic quantum Hilbert space $\mathcal{H}_S$
DW invariant $\mathcal{Z}_M$	arithmetic DW invariant $\mathcal{Z}_S$

$$\Sigma_S = \partial V_S = \mathrm{Spec}(k_{\mathfrak{p}_1}) \cup \cdots \cup \mathrm{Spec}(k_{\mathfrak{p}_r}),$$
$$X_S = \mathrm{Spec}(\mathcal{O}_k) \setminus S,\ \partial X_S = \partial V_S$$

5. Arithmetic Dijkgraaf-Witten TQFT

Dijkgraaf-Witten TQFT (Review)

G: a finite group
$c \in Z^3(G, \mathbb{R}/\mathbb{Z})$: 3-cocycle

For an oriented compact manifold X with a fixed finite triangulation,
$\mathcal{F}_X$: the space of gauge fields
$\mathcal{G}_X = \mathrm{Map}(X, G)$: the gauge group
$\mathcal{F}_X / \mathcal{G}_X = \mathrm{Hom}(\pi_1(X), G)/G$ (X: connected)

The following construction is due to K. Gomi.

Key ingredient – transgression hom.

$$\tau_X : C^3(G, \mathbb{R}/\mathbb{Z}) \longrightarrow C^{3-d}(\mathcal{G}_X, \mathrm{Map}(\mathcal{F}_X, \mathbb{R}/\mathbb{Z})) \quad (d = \dim(X))$$

$$\begin{cases} \lambda_\Sigma := \tau_X(c) \text{ for a surface } \Sigma \\ CS_M := \tau_X(c) \text{ for a 3-manifold } M \end{cases}$$

5. Arithmetic Dijkgraaf-Witten TQFT

Classical theory:

$$\begin{array}{rcl} \text{oriented closed surface } \Sigma & \rightsquigarrow & \lambda_\Sigma \in Z^1(\mathcal{G}_\Sigma, \mathrm{Map}(\mathcal{F}_\Sigma, \mathbb{R}/\mathbb{Z})), \\ \text{oriented compact 3-manifold } M & \rightsquigarrow & CS_M \in C^0(\mathcal{G}_M, \mathrm{Map}(\mathcal{F}_M, \mathbb{R}/\mathbb{Z})) \end{array}$$

s.t. $$dCS_M = \mathrm{res}^* \lambda_{\partial M}$$

$\mathrm{res} : \mathcal{F}_M \ (\mathrm{resp.}\ \mathcal{G}_M) \to \mathcal{F}_{\partial M} \ (\mathrm{resp.}\ \mathcal{G}_{\partial M})$: restriction map

$\lambda_\Sigma : \mathcal{G}_\Sigma \to \mathrm{Map}(\mathcal{F}_\Sigma, \mathbb{R}/\mathbb{Z})$: Chern-Simons 1-cocycle

$CS_M : \mathcal{F}_M \to \mathbb{R}/\mathbb{Z}$: Chern-Simons functional

CS 1-cocycle $\lambda_\Sigma \rightsquigarrow$ $\mathcal{G}_\Sigma$-equiv. complex line bundle L_Σ on $\mathcal{F}_\Sigma$

prequantum line bundle $\overline{L}_\Sigma$ on $\mathcal{F}_X/\mathcal{G}_X$.

5. Arithmetic Dijkgraaf-Witten TQFT

Quantum theory:

oriented closed surface Σ $\rightsquigarrow$ quantum Hilbert space $\mathcal{H}_\Sigma$,
oriented compact 3-manifold M $\rightsquigarrow$ partition function $\mathcal{Z}_M \in \mathcal{H}_{\partial M}$.

quantum Hilbert space:

$$\mathcal{H}_\Sigma := \Gamma(\mathcal{F}_\Sigma/\mathcal{G}_\Sigma, \overline{L}_\Sigma)$$

Dijkgraaf-Witten invariant:

$$\mathcal{Z}_M(\rho) = \frac{1}{\#G} \sum_{\substack{\tilde{\rho} \in \mathcal{F}_M \\ \mathrm{res}(\tilde{\rho}) = \rho}} e^{2\pi\sqrt{-1}CS_M(\tilde{\rho})}$$

Ex.

M is closed and c is trivial $\Rightarrow$ $\mathcal{Z}_M = \dfrac{1}{\#G}\#\mathrm{Hom}(\pi_1(M), G)$

5. Arithmetic Dijkgraaf-Witten theory

k: a number field
$\mathcal{O}_k$: the ring of alg. integers in k

For a prime ideal $\mathfrak{p}$ $(\neq 0)$ of $\mathcal{O}_k$,
$k_{\mathfrak{p}}$: the $\mathfrak{p}$-adic field
$\mathcal{O}_{\mathfrak{p}}$: the ring of $\mathfrak{p}$-adic integers

$X_k := \mathrm{Spec}(\mathcal{O}_k)$
$\overline{X}_k := X_k \cup \{\text{infinite prime}\}$
For a finite set of prime ideals $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$,
$\overline{X}_S := \overline{X}_k \setminus S$

$\Pi_{\mathfrak{p}} := \pi_1(\mathrm{Spec}(k_{\mathfrak{p}})) = \mathrm{Gal}(\overline{k}_{\mathfrak{p}}/k_{\mathfrak{p}})$
$\Pi_S := \pi_1(\overline{X}_S) = \mathrm{Gal}(k_S/k)$
(k_S: max. Galois ext. of k unramified outside S)

5. Arithmetic Dijkgraaf-Witten TQFT

N : a fixed integer > 1
G : finite group, $c \in Z^3(G, \mathbb{Z}/N\mathbb{Z})$
Assume k contains a primitive N-th root of unity $\zeta_N = e^{\frac{2\pi\sqrt{-1}}{N}}$.

$$\mathcal{F}_S := \prod_{i=1}^{r} \mathrm{Hom}_{\mathrm{cont}}(\Pi_{\mathfrak{p}_i}, G) \curvearrowleft \mathcal{G}_S := G \quad \text{diagonal conjugte action}$$
$$\mathcal{F}_{\overline{X}_S} := \mathrm{Hom}_{\mathrm{cont}}(\Pi_S, G) \curvearrowleft \mathcal{G}_{\overline{X}_S} := G \quad \text{conjugate action}$$

Key ingredients (due to M. Kim):
- ·2nd Galois cohomology (Brauer group) of the local field $k_{\mathfrak{p}}$
- · Conjugate action on group cocycle

5. Arithmetic Dijkgraaf-Witten TQFT

Arithmetic Dijkgraaf-Witten TQFT (J. Kim-Hirano-M.)

We can construct the following correspondences:

- arithmetic CS 1-cocycle:
  $\Sigma_S = \mathrm{Spec}(k_{\mathfrak{p}_1}) \cup \cdots \cup \ \mathrm{Spec}(k_{\mathfrak{p}_r}) \rightsquigarrow \lambda_S \in Z^1(\mathcal{G}_S, \mathrm{Map}(\mathcal{F}_S, \mathbb{Z}/N\mathbb{Z}))$
- arithmetic CS functional:
  $\overline{X}_S \rightsquigarrow CS_{\overline{X}_S} \in C^0(\mathcal{G}_{\overline{X}_S}, \mathrm{Map}(\mathcal{F}_{\overline{X}_S}, \mathbb{Z}/N\mathbb{Z}))$
  s.t. $dCS_M = \mathrm{res}^* \lambda_{\partial M}$ ($\mathrm{res} : \mathcal{F}_{\overline{X}_S} \to \mathcal{F}_S$: restriction)
- arithmetic prequantum line bundle L_S and arithmetic quantum space:
  $\Sigma_S = \mathrm{Spec}\ k_{\mathfrak{p}_1} \sqcup \cdots \sqcup \ \mathrm{Spec}\ K_{\mathfrak{p}_r} \rightsquigarrow \mathcal{H}_S = \Gamma(\mathcal{F}_S/\mathcal{G}_S, \overline{L}_S)$
- arithmetic DW invariant:
  $\overline{X}_S \rightsquigarrow \mathcal{Z}_{\overline{X}_S} \in \mathcal{H}_S$

$$\mathcal{H}_S := \{\psi : \mathcal{F}_S \to F \mid \psi(\rho_S.g_S) = \zeta_N^{\lambda_S(g,\rho_S)}\psi(\rho_S) \text{ for } \rho_S \in \mathcal{F}_S, g \in G\} = \Gamma(\mathcal{F}_S/\mathcal{G}_S, \overline{L}_S).$$

$$\mathcal{Z}_{\overline{X}_S}(\rho_S) := \frac{1}{\#G} \sum_{\substack{\tilde{\rho} \in \mathcal{F}_{\overline{X}_S} \\ \mathrm{res}_S(\tilde{\rho}) = \rho_S}} \zeta_N^{CS_{\overline{X}_S}(\tilde{\rho})}$$

5. Arithmetic Dijkgraaf-Witten TQFT

Arithmetic Dijkgraaf-Witten TQFT (J. Kim-Hirano-M.)

DW TQFT	Arithmetic DW TQFT
ori. closed surface $\Sigma \rightsquigarrow \lambda_\Sigma$	$\Sigma_S = \mathrm{Spec}(k_{\mathfrak{p}_1}) \sqcup \cdots \sqcup \mathrm{Spec}(k_{\mathfrak{p}_r}) \rightsquigarrow \lambda_S$
ori. 3-manifold $M \rightsquigarrow CS_M$	$\overline{X}_S = \overline{X}_k \setminus S \rightsquigarrow CS_{\overline{X}_S}$
$dCS_M = \mathrm{res}^* \lambda_{\partial M}$	$dCS_{\overline{X}_S} = \mathrm{res}^* \lambda_S$
prequantum line bundle L_Σ	arith. prequantum line bundle L_S
$\Sigma \rightsquigarrow \mathcal{H}_\Sigma$ $\mathcal{H}_\Sigma = \Gamma(\mathcal{F}_\Sigma/\mathcal{G}_\Sigma, \overline{L}_\Sigma)$	$\Sigma_S \rightsquigarrow \mathcal{H}_S$ $\mathcal{H}_S = \Gamma(\mathcal{F}_S/\mathcal{G}_S, \overline{L}_S)$
$M \rightsquigarrow \mathcal{Z}_M \in \mathcal{H}_{\partial M}$ $\mathcal{Z}_M(\varrho) = \frac{1}{\#G} \sum_{\substack{\rho \in \mathcal{F}_M \\ \mathrm{res}(\rho)=\varrho}} e^{2\pi\sqrt{-1}CS_M(\rho)}$	$\overline{X}_S \rightsquigarrow \mathcal{Z}_{\overline{X}/S} \in \mathcal{H}_S$ $\mathcal{Z}_{\overline{X}_S}(\rho_S) := \frac{1}{\#G} \sum_{\substack{\rho \in \mathcal{F}_{\overline{X}_S} \\ \mathrm{res}_S(\rho)=\rho_S}} \zeta_N^{CS_{\overline{X}_S}(\rho)}$

5. Arithmetic Dijkgraaf-Witten TQFT

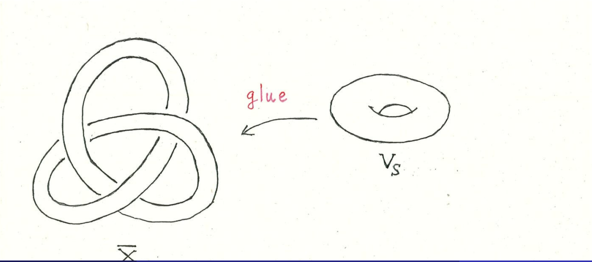
Gluing formula ("Decomposition formula" by Kim etc)

We can define the Chern-Simons functionals for $\overline{X}_k = \mathrm{Spec}(\mathcal{O}_k) \cup \{\text{infinite primes}\}$ and $V_{\mathfrak{p}} = \mathrm{Spec}(\mathcal{O}_{\mathfrak{p}})$ (Hirano, Lee-Park).

For $\rho : \pi_1(\overline{X}_k) \to G$, $\rho_S : \Pi_S \to \pi_1(\overline{X}_k) \to G$, $\tilde{\rho}_{\mathfrak{p}} : \pi_1(V_{\mathfrak{p}}) \to \pi_1(\overline{X}_k) \to G$
$\Rightarrow CS_{\overline{X}_k}(\rho), CS_{V_S}(\tilde{\rho}_S) := \sum_{i=1}^{r} CS_{V_{\mathfrak{p}}}(\tilde{\rho}_{\mathfrak{p}_i})$ are defined.

Gluing formula

$$CS_{\overline{X}_k}(\rho) = CS_{V_S}(\tilde{\rho}_S) - CS_{\overline{X}_S}(\rho_S)$$



Questions for future study

• The arithmetic quantum space $\mathcal{H}_S$ is a finite dim. vector F-space. Note that $\mathcal{H}_S$ is an arithmetic analog of the space of comformal blocks.

Can one have a dimension formula ? Is there a "canonical" basis of $\mathcal{H}_S$? cf. Verlinde's formula.

• Take $G = GL_n(\mathbb{F}_p)$ so that a Galois repr. ρ would correspond to a automorphic form (for example, when ρ is odd and irreducible).

Is there any relation between the arithmetic Dijkgraaf-Witten invariant some invariant attached to the automorphic form ?

Observe that arithmetic DW invariants are kind (variant) of (non-abelian) Gaussian sums.

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