Topological String Theory and S-duality

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Motivation and Plan

Any original result I will discuss today is based on a joint work in progress with *Surya Raghavendran*. Here are my motivations:

- S-duality is an equivalence between two different-looking physical theories. I wanted to understand a general principle.
- Based on works with C. Elliott, I conjectured a physical setup for geometric Langlands correspondence in a way different from Kapustin–Witten; I wanted to find the evidence.
- A study of twisted supersymmetric theory led to rich mathematics. Ours is a first step for twisted supergravity.

Here is an outline:

1. Type IIB Superstring theory and Topological String Theory
2. Main Definitions and Justification
3. Applications

Summary: What have we done?

- (P) mathematical understanding of S-duality of (a part of) massless sector of the physical type IIB supergravity;
- (M) recovering old conjectures and formulating new conjectures in geometric representation theory;
- (!) easy calculation of how S-duality acts on further deformations of twists of supersymmetric gauge theory;
- (!) setting up a framework that can be useful for future works; e.g., introducing a modified version of Kodaira–Spencer theory of gravity of Bershadsky–Cecotti–Ooguri–Vafa, and constructing $\mathrm{SL}_2(\mathbb{Z})$ action on it

Disclaimer: I am a string theory newbie!

Ⓟ Type IIB Superstring Theory

- Type IIB superstring theory on a 10-manifold M^{10}; need to consider the moduli spaces of Riemann surfaces;
- D-brane gauge theory for D_{2k-1}-branes wrapping on $N^{2k} \subset M^{10}$; a $2k$-dimensional field theory; e.g.,
  - D3 branes on $\mathbb{R}^4 \subset \mathbb{R}^{10}$ yield 4d $\mathcal{N} = 4$ SYM theory;
  - D5 branes on $\mathbb{R}^6 \subset \mathbb{R}^{10}$ yield 6d $\mathcal{N} = (1, 1)$ SYM theory;
- Closed string field theory on M^{10}; field theory on M^{10} describing string theory;
- Type IIB supergravity theory on a 10-manifold M^{10}; classical field theory on M^{10} realized as a low-energy limit of closed string field theory;
- Coupling between closed string field theory and D-brane gauge theory; closed string state yields a deformation of D-brane gauge theory; e.g., a twist of D-brane gauge theory;
- Existence of $\mathrm{SL}_2(\mathbb{Z})$ symmetry or S-duality

Question: How much can we capture mathematically?
Answer: Most of it, for topological string theory.

Topological Quantum Field Theory

Definition

A *d-dimensional TQFT* is a symmetric monoidal functor

$$Z\colon (\underline{\mathrm{Bord}}_d, \amalg) \to (\mathrm{Vect}_{\mathbb{C}}, \otimes)$$

Here $(\mathrm{Vect}_{\mathbb{C}}, \otimes)$ is a symmetric monoidal category of $\mathbb{C}$-vector spaces and $\underline{\mathrm{Bord}}_d$ is a category where

- an object is a closed $(d-1)$-manifold;
- a morphism is a cobordism up to diffeomorphism;
- the composition is a gluing of cobordisms;
- the monoidal structure is a disjoint union $\amalg$.

2d TQFT

A closed 1-manifold is copies of S^1. Let us write $Z(S^1) = A$.

Theorem

A 2d TQFT is the same as a commutative Frobenius algebra.

morphism in $\underline{\mathrm{Bord}}_2$	morphism in $\mathrm{Vect}_{\mathbb{C}}$
$\emptyset \to S^1$	$u\colon \mathbb{C} \to A$
$S^1 \to \emptyset$	$\mathrm{Tr}\colon A \to \mathbb{C}$
$S^1 \amalg S^1 \to S^1$	$m\colon A \otimes A \to A$
$S^1 \to S^1 \amalg S^1$	$\Delta\colon A \to A \otimes A$

One should think of this as (baby) (topological) string theory, where $Z(S^1)$ is the space of string states.

To be more precise, $Z(S^1)$ is the space of *closed* string states, where closed refers to the fact that our string S^1 has no boundary. We want to see an open string (interval) floating around as well.

Extended 2d TQFT

Roughly, an *extended 2d TQFT* is a symmetric monoidal functor

$$Z\colon (\mathrm{Bord}_2, \amalg) \to (\mathsf{DGCat}_{\mathbb{C}}, \otimes)$$

Bord_2	$\mathsf{DGCat}_{\mathbb{C}}$
closed 2-manifold	complex number
closed 1-manifold cobordism of 1-manifolds	$\mathbb{C}$-vector space $\mathbb{C}$-linear map
closed 0-manifold cobordism of 0-manifolds	$\mathbb{C}$-linear category $\mathbb{C}$-linear functor

Theorem (Costello, Hopkins–Lurie, Lurie)

An extended 2d TQFT Z is the same as a Calabi–Yau category $Z(\mathrm{pt}) = \mathcal{C}$.

Physically speaking, $Z(\mathrm{pt}) = \mathcal{C}$ captures where an open string can end; hence the name of $\mathcal{C}$ is the category of boundary conditions; then $\mathrm{Hom}_{\mathcal{C}}(\mathcal{B}_1, \mathcal{B}_2)$ is to be interpreted as the space of open string states which end at $\mathcal{B}_1$ and $\mathcal{B}_2$.

Topological String Theory as 2d Extended TQFT

By topological string, we mean such a 2d extended TQFT determined by CY 5-category. In this case, a boundary condition is also called a D-brane. Let X be a CY 5-fold with a holomorphic volume form Ω_X. Here are the two main examples:

	A-model	B-model
$Z(\mathrm{pt}) = \mathcal{C}$	$\mathrm{Fuk}(X)$	$\mathrm{Coh}(X)$
$Z(S^1) = \mathrm{HH}(\mathcal{C})$	$\mathrm{QH}(X)$	$\mathrm{PV}(X)$

Here $\mathrm{PV}(X) = \bigoplus \mathrm{PV}^{i,j}(X)$ is the space of polyvector fields, where $\mathrm{PV}^{i,j}(X) = \Omega^{0,j}(X, \wedge^i T_X)$, with a differential $\overline{\partial}\colon \mathrm{PV}^{i,j} \to \mathrm{PV}^{i,j+1}$. For future reference, note that using the isomorphism $(-) \vee \Omega_X\colon \mathrm{PV}^{i,j}(X) \cong \Omega^{d-i,j}(X)$, one has $\partial\colon \mathrm{PV}^{i,j} \to \mathrm{PV}^{i-1,j}$.

Type IIB string theory on $M^{10} \rightsquigarrow$ Calabi–Yau 5-category $\mathcal{C}$

Example

- $\mathcal{C} = \mathrm{Coh}(X^5)$ for CY 5-fold X
- $\mathcal{C} = \mathrm{Fuk}(T^*N) \otimes \mathrm{Coh}(X^3)$ for a smooth 2-manifold N

Ⓜ Classical Field Theory and BV Formalism

A d-dimensional classical field theory is described by

- a spacetime manifold $M = M^d$;
- a space of fields $\mathcal{F}$;
- an action functional $S\colon \mathcal{F} \to \mathbb{C}$.

In what follows, we use the BV formalism, where space of fields $\mathcal{E}$ is a (-1)-shifted "symplectic" space with a differential Q and a Lie bracket $[-,-]$, giving $S(\phi) = \int_M \frac{1}{2}\langle \phi, Q\phi\rangle + \frac{1}{6}\langle \phi, [\phi, \phi]\rangle$.

Example

- Free scalar field theory has $\mathcal{E} = C^\infty(M) \oplus C^\infty(M)[-1]$ with $Q = \Delta$. This means $S(\phi) = \int_M \langle \phi, \Delta\phi\rangle$.
- Chern–Simons theory has $\mathcal{E} = \Omega^\bullet(M^3) \otimes \mathfrak{g}[1]$ with $Q = d$ and natural $[-,-]$, giving $S(A) = \int_M \frac{1}{2}\langle A, dA\rangle + \frac{1}{6}\langle A, [A, A]\rangle$.

For us, things will be mostly $\mathbb{Z}/2$-graded, although we write in such a manner that a $\mathbb{Z}$-grading is respected when possible.

D-brane Gauge Theory

(P) Open strings ending on branes $\mathcal{B}$ yield D-brane gauge theory.

(M) [Brav–Dyckerhoff] The moduli $\mathcal{M}_{\mathcal{C}}$ of objects is $(2-d)$ shifted symplectic and $\mathbb{T}_{\mathcal{B}}[-1]\mathcal{M}_{\mathcal{C}} \cong \mathbb{R}\,\mathrm{End}_{\mathcal{C}}(\mathcal{B})$ for $\mathcal{B} \in \mathcal{C}$.

D-brane gauge theory on $N^{2k} \subset M^{10} \rightsquigarrow \mathcal{E} = \mathbb{R}\,\mathrm{End}_{\mathcal{C}}(\mathcal{B})[1]$

$\mathcal{C}$ a DG category $\rightsquigarrow$ associative and hence Lie on $\mathbb{R}\,\mathrm{End}_{\mathcal{C}}(\mathcal{B})$
$\mathcal{C}$ a CY category $\rightsquigarrow$ a shifted symplectic structure on $\mathbb{R}\,\mathrm{End}_{\mathcal{C}}(\mathcal{B})[1]$

Example

- If $\mathcal{C} = \mathrm{Coh}(X)$, then N D-branes on $Y \subset X$, or $\mathcal{B} = \mathcal{O}_Y^N \in \mathrm{Coh}(X)$, gives $\mathcal{E} = \Omega^{0,\bullet}(Y, \wedge^\bullet N_{X/Y}) \otimes \mathfrak{gl}_N[1]$. N D3 on $\mathbb{C}^2 \subset \mathbb{C}^5$ give $\mathcal{E}^{\mathrm{Hol}}_{\mathrm{D3}}(\mathbb{C}^2) := \Omega^{0,\bullet}(\mathbb{C}^2)[\varepsilon_1, \varepsilon_2, \varepsilon_3] \otimes \mathfrak{gl}_N[1]$, or the holomorphic twist of 4d $\mathcal{N} = 4$ $\mathfrak{gl}_N$ gauge theory.
- If $\mathcal{C} = \mathrm{Fuk}(\mathbb{R}^4) \otimes \mathrm{Coh}(X^3)$, a D-brane should be of the form $\mathbb{R}^2 \times Y \subset \mathbb{R}^4_A \times X_B$. Then N D3 branes on $\mathbb{R}^2 \times \mathbb{C} \subset \mathbb{R}^4 \times \mathbb{C}^3$ yield $\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}) := \Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}_N[1]$, or the holomorphic-topological twist.

Closed String Field Theory

Recall $Z(S^1)$ is the space of closed string states, but note that

Ⓟ The worldsheet theory, being coupled with gravity theory, should be invariant under $\mathrm{Diff}(S^1)$. This motivates $Z(S^1)^{S^1}$.

Ⓜ Here $Z(S^1) = \mathsf{HH}(\mathcal{C})$ admits an S^1-action which corresponds to so-called Connes' B operator, so $Z(S^1)^{S^1} = \mathrm{Cyc}(\mathcal{C})$.

Ⓜ [Brav–Rozenblyum] $\mathbb{T}_{\mathcal{C}}[-1]\mathcal{M}_{\mathrm{CY}} \cong \mathrm{Cyc}^\bullet(\mathcal{C})[1]$ where $\mathcal{M}_{\mathrm{CY}}$ is the moduli space of Calabi–Yau categories.

Closed string field theory on $M^{10} \rightsquigarrow \mathcal{E} = \mathrm{Cyc}^\bullet(\mathcal{C})[2]$
where $\mathcal{E}$ is understood in the framework of [Butson–Y.].

Example (Bershadsky–Cecotti–Ooguri–Vafa, Costello–Li)

If $\mathcal{C} = \mathsf{Coh}(X^5)$, then $Z(S^1) \cong \mathsf{PV}(X)$ and $B = \partial$; hence the corresponding closed string field theory is given by $(\ker \partial \subset \mathsf{PV}(X)[2], \overline{\partial})$ or $\mathcal{E} = (\mathsf{PV}(X)[\![t]\!][2], \overline{\partial} + t\partial)$.

Supergravity

(P) Supergravity is a theory of low-energy limit of closed string field theory where we see neither non-perturbative effects nor non-propagating fields.

Supergravity on M^{10} $\rightsquigarrow$ non-propagating part of $\mathrm{Cyc}^\bullet(\mathcal{C})[2]$

The non-propagating part of BCOV theory can be identified:

Definition

Let (X, Ω_X) be a Calabi–Yau d-fold. A minimal BCOV theory is $\mathcal{E}_{\mathrm{m}}(X) = \mathcal{E}_{\mathrm{mBCOV}}(X) = \bigoplus_{i+k\leq d-1} t^k \,\mathsf{PV}^{i,\bullet}(X)$.

Example

If $\mathcal{C} = \mathsf{Coh}(X^3)$ (or $\mathcal{C} = \mathrm{Fuk}(\mathbb{R}^4) \otimes \mathsf{Coh}(X^3)$), then it is $\mathcal{E}_{\mathrm{m}}(X^3)$ (or $\Omega^\bullet(\mathbb{R}^4) \otimes \mathcal{E}_{\mathrm{m}}(X^3)$), where $\mathcal{E}_{\mathrm{m}}(X^3)$ is

$\underline{-2}$	$\underline{-1}$	$\underline{0}$	$\underline{1}$	$\underline{2}$
$\mathsf{PV}^{0,\bullet}$				
	$\mathsf{PV}^{1,\bullet}$	$\to t\,\mathsf{PV}^{0,\bullet}$		
		$\mathsf{PV}^{2,\bullet}$	$\to t\,\mathsf{PV}^{1,\bullet}$	$\to t^2\,\mathsf{PV}^{0,\bullet}$

Coupling of Open and Closed Sectors

Coupling of closed string field theory and D-brane gauge theory
$\rightsquigarrow$ closed-open map $\mathrm{Cyc}^\bullet(\mathcal{C})[1] \dashrightarrow \mathrm{Cyc}^\bullet(\mathbb{R}\,\mathsf{End}_{\mathcal{C}}(\mathcal{F}))[1]$

Theorem (Kontsevich, Willwacher–Calaque)

The formality $\mathsf{PV}(X) \overset{\simeq}{\dashrightarrow} \mathsf{HH}(\mathsf{Coh}(X))$ *gives*
$\mathsf{PV}(X) \dashrightarrow \mathsf{HH}(\mathbb{R}\,\mathsf{End}_{\mathcal{C}}(\mathcal{F}))$ *for* $\mathcal{F} \in \mathsf{Coh}(X)$, *or its cyclic version.*

Example

- If $\mathcal{C} = \mathsf{Coh}(X^5)$, for N D3 branes on $\mathbb{C}^2_{z_1,z_2} \subset \mathbb{C}^5_{z_1,z_2,w_1,w_2,w_3}$, or $\mathcal{E}^{\mathrm{Hol}}_{\mathrm{D3}} = \Omega^{0,\bullet}(\mathbb{C}^2_{z_1,z_2})[\varepsilon_1,\varepsilon_2,\varepsilon_3] \otimes \mathfrak{gl}_N[1]$, the CO map is

$$\mathsf{PV}(\mathbb{C}^2_{z_1,z_2} \times \mathbb{C}^3_{w_1,w_2,w_3}) \to \mathsf{HH}(\Omega^{0,\bullet}(\mathbb{C}^2_{z_1,z_2})[\varepsilon_1,\varepsilon_2,\varepsilon_3]),$$

  where the RHS is $\mathsf{HH}(\mathcal{O}(\mathbb{C}^{2|3})) \cong \mathbb{C}[z_i,\partial_{z_i},\varepsilon_j,\partial_{\varepsilon_j}]$, is given by $z_i,\partial_{z_i},w_j,\partial_{w_j} \mapsto z_i,\partial_{z_i},\partial_{\varepsilon_j},\varepsilon_j$.
- If $\mathcal{C} = \mathsf{Fuk}(\mathbb{R}^4) \otimes \mathsf{Coh}(X)$, for $\mathbb{R}^2 \times \mathbb{C}_z \subset \mathbb{R}^4 \times \mathbb{C}^3_{z,w_1,w_2}$, or $\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}} = \Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C}_z)[\varepsilon_1,\varepsilon_2] \otimes \mathfrak{gl}_N[1]$, the CO map is $\mathsf{PV}(\mathbb{C}_z \times \mathbb{C}^2_{w_1,w_2}) \to \mathsf{HH}(\Omega^{0,\bullet}(\mathbb{C}_z)[\varepsilon_1,\varepsilon_2])$ given by $z,\partial_z,w_j,\partial_{w_j} \mapsto z,\partial_z,\partial_{\varepsilon_j},\varepsilon_j$.

Modification of BCOV Theory

Definition

Minimal BCOV theory with potential $\widetilde{\mathcal{E}}_{\mathrm{m}}(X)$ is a cochain complex

$$
\begin{array}{ccccc}
\underline{-2} & \underline{-1} & \underline{0} & \underline{1} & \underline{2} \\
\mathrm{PV}^{0,\bullet} & & & & \\
& \mathrm{PV}^{1,\bullet} \rightarrow & t\,\mathrm{PV}^{0,\bullet} & & \\
& & \mathrm{PV}^{3,\bullet} & & \\
& & \mathrm{PV}^{2,\bullet} \longrightarrow & t\,\mathrm{PV}^{1,\bullet} \rightarrow & t^2\,\mathrm{PV}^{0,\bullet}
\end{array}
$$

with additional structures.

(M) There is a "map" $\Phi\colon \widetilde{\mathcal{E}}_{\mathrm{m}} \to \mathcal{E}_{\mathrm{m}}$ that has $\partial\colon \mathrm{PV}^{3,\bullet} \to \mathrm{PV}^{2,\bullet}$, respecting structures of interest.

(P) The modification amounts to introducing Ramond–Ramond forms as a potential for Ramond–Ramond field strengths.

S-duality

Definition/Theorem (Raghavendran–Y.)

Let (X, Ω_X) be a Calabi–Yau 3-fold. Recall

$$\mathrm{SL}_2(\mathbb{Z}) = \left\langle S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \;\middle|\; S^4 = 1,\ (ST)^3 = S^2 \right\rangle .$$

Then there is an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\widetilde{\mathcal{E}}_{\mathrm{m}}(X) = \mathrm{PV}^{0,\bullet}(X)[2] \oplus \left(\mathrm{PV}^{1,\bullet}(X)[1] \to t\,\mathrm{PV}^{0,\bullet}(X)\right) \oplus \mathrm{PV}^{3,\bullet}(X)$:

$$S \mapsto \begin{pmatrix} & & -(-) \vee \Omega_X \\ & \mathrm{Id} & \\ (-) \wedge \Omega_X^{-1} & & \end{pmatrix}, \quad T \mapsto \begin{pmatrix} \mathrm{Id} & & (-) \vee \Omega_X \\ & \mathrm{Id} & \\ & & \mathrm{Id} \end{pmatrix}$$

For example, $\alpha \in \mathrm{PV}^{0,\bullet}(X) \rightsquigarrow S(\alpha) = \alpha \wedge \Omega_X^{-1} \in \mathrm{PV}^{3,\bullet}(X)$ and $\gamma \in \mathrm{PV}^{3,\bullet}(X) \rightsquigarrow T(\gamma) = \alpha + \gamma \in \mathrm{PV}^{0,\bullet}(X) \oplus \mathrm{PV}^{3,\bullet}(X)$ where α is such that $\gamma = \alpha \wedge \Omega_X^{-1}$.

Consistency Checks

P S-duality is an action of $S \in \mathsf{SL}_2(\mathbb{Z})$ on type IIB string theory induced from the diagram

$$\mathrm{M}[\overset{\mathsf{SL}_2(\mathbb{Z}) \circlearrowright}{S^1_{\mathrm{M}} \times S^1_r} \times M^9] \xrightarrow[\simeq]{\mathrm{red}_M} \mathrm{IIA}[S^1_r \times M^9] \xrightarrow[\simeq]{\mathbf{T}} \mathrm{IIB}[\overset{\mathsf{SL}_2(\mathbb{Z}) \circlearrowright}{S^1_{1/r}} \times M^9]$$

- M stands for M-theory;
- IIA stands for type IIA string theory;
- red_M is an equivalence from the "fact" that a circle reduction of M-theory is equivalent to type IIA theory;
- T-duality $\mathbf{T}$ is an equivalence between type II string theories;
- $\mathsf{SL}_2(\mathbb{Z})$-action on M-theory is on $S^1_M \times S^1_r$;
- $\mathsf{SL}_2(\mathbb{Z})$-action on IIB string theory is transferred from the $\mathsf{SL}_2(\mathbb{Z})$-action on M-theory through equivalences.

We find its twisted versions (based on [Costello–Li]).

P We have some further consistency checks with "twisted supergravity".

Summary

- S-duality is a duality of type IIB string theory.
- By simplifying type IIB string theory to topological string theory, we construct S-duality operation on closed string states or supergravity theory. In particular, we obtain $\mathrm{SL}_2(\mathbb{Z})$ action on a version of BCOV theory.
- Our interest is duality between D-brane gauge theories, or more precisely, deformations of D-brane gauge theory.
- Through closed-open map as well as the map from modified BCOV theory to minimal BCOV theory, modified BCOV theory and deformations of D-brane gauge theory are related.

From now on, we let $\mathcal{C} = \mathrm{Fuk}(\mathbb{R}^4) \otimes \mathrm{Coh}(\mathbb{C}^3)$ and consider N D3 branes on $\mathbb{R}^2 \times \mathbb{C}_z \subset \mathbb{R}^4 \times \mathbb{C}^3_{z,w_1,w_2}$ to get $\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}_z) = \Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}_N[1]$. Then for

$$\overset{S}{\circlearrowright}\widetilde{\mathcal{E}}_{\mathrm{m}} \xrightarrow{\Phi} \mathcal{E}_{\mathrm{m}} \xrightarrow{\mathrm{CO}} \mathrm{HH}(\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}_z))$$

we compare deformations of HT twist by S-dual elements.

S-duality gives Geometric Langlands: $F = w_1$

Based on [Elliott–Y.]

$$\begin{array}{ccccc}
\widetilde{\mathcal{E}}_{\mathrm{m}} & \xrightarrow{\Phi} & \mathcal{E}_{\mathrm{m}} & \xrightarrow{\mathsf{CO}} & \mathsf{HH}(\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}_z)) \\
w_1 & \longmapsto & w_1 & \longmapsto & \partial_{\varepsilon_1} \\
\Downarrow S & & & & \\
w_1 \partial_z \partial_{w_1} \partial_{w_2} & \longmapsto & \partial_{w_2} \wedge \partial_z & \longmapsto & \varepsilon_2 \partial_z
\end{array}$$

Recall $\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}_z) = \Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}_N[1]$. Globalizing with replacing $\mathbb{R}^2 \times \mathbb{C}$ by $\Sigma \times C$, one has $\mathrm{EOM}^{\mathrm{HT}}_{\mathrm{D3}}(\Sigma \times C) = \underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}}, T^*[1]\,\mathrm{Higgs}_G(C))$, aka B-model with target Hitchin moduli. Here ε_1 is responsible for $T^*[1]$ and ε_2 makes C into C_{Dol}. Hence we have the following deformations

$$(\mathrm{B}, \mathrm{Bun}_G(C)_{\mathrm{dR}}) \xleftarrow{\partial_{\varepsilon_1}} (\mathrm{B}, \mathrm{Higgs}_G(C)) \xrightarrow{\varepsilon_2 \partial_z} (\mathrm{B}, \mathrm{Flat}_G(C))$$

giving an equivalence between $\mathrm{D}(\mathrm{Bun}_G(C)) := \mathrm{QCoh}(\mathrm{Bun}_G(C)_{\mathrm{dR}})$ and $\mathrm{QCoh}(\mathrm{Flat}_G(C))$ for $G = \mathrm{GL}_N$. This gives geometric Langlands without considering A-model at all.

S-duality between Superconformal Deformations: $F = zw_2$

	0	1	2	3	4	5	6	7	8	9
	u		v		z		w_1		w_2	
K D5		$\times$	$\times$		$\times$	$\times$	$\times$	$\times$		
N D3	$\times$	$\times$			$\times$	$\times$				

$$\widetilde{\mathcal{E}}_{\mathrm{m}} \xrightarrow{\quad \Phi \quad} \mathcal{E}_{\mathrm{m}} \xrightarrow{\quad \mathsf{CO} \quad} \mathsf{HH}(\mathcal{E}^{\mathrm{HT}}_{\mathrm{D5}}(\mathbb{R}^2 \times \mathbb{C}^2_{z,w_1}))$$

$$zw_2 \longmapsto zw_2 \longmapsto z\partial_{\varepsilon_2}$$

$$\Downarrow S$$

$$zw_2\partial_z\partial_{w_1}\partial_{w_2} \mapsto w_2\partial_{w_1}\partial_{w_2} + z\partial_z\partial_{w_1} \longmapsto \varepsilon_1\varepsilon_2\partial_{\varepsilon_2} + \varepsilon_1 z\partial_z$$

The deformation $z\partial_{\varepsilon_2}$ turns HT twist of 6d $\mathcal{N} = (1,1)$ theory to 4d CS theory on $\mathbb{R}^2 \times \mathbb{C}_{w_1}$ [Costello–Yagi]: it follows from

$$\Omega^{0,\bullet}(\mathbb{C}_{w_1}) \otimes \left(\Omega^{0,\bullet}(\mathbb{C}_z)\varepsilon_2 \xrightarrow{z\partial_{\varepsilon_2}} \Omega^{0,\bullet}(\mathbb{C}_z)\right) \cong \Omega^{0,\bullet}(\mathbb{C}_{w_1})$$

The appearance of (truncated) Yangian on the 1d defect can be understood as its S-dual 3d $\mathcal{N} = 4$ theory configuration, where the Yangian is the quantized Coulomb branch algebra.

New Examples of S-dual Theories: $F = w_1 w_2$

$$\widetilde{\mathcal{E}}_{\mathrm{m}} \xrightarrow{\Phi} \mathcal{E}_{\mathrm{m}} \xrightarrow{\mathsf{CO}} \mathrm{HH}(\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}(\mathbb{R}^2 \times \mathbb{C}_z))$$

$$w_1 w_2 \longmapsto w_1 w_2 \longmapsto \partial_{\varepsilon_1}\partial_{\varepsilon_2}$$

$$\Downarrow S$$

$$w_1 w_2 \partial_z \partial_{w_1} \partial_{w_2} \mapsto w_1 \partial_z \partial_{w_1} - w_2 \partial_z \partial_{w_2} \mapsto \pi = \partial_{\varepsilon_1} \partial_z \varepsilon_1 - \partial_{\epsilon_2} \partial_z \varepsilon_2$$

- As $(\mathbb{C}[\varepsilon_1, \varepsilon_2], \partial_{\varepsilon_1}\partial_{\varepsilon_2})$ is Clifford algebra $\mathrm{Cl}(\mathbb{C}^2) \cong \mathrm{End}(\mathbb{C}^{1|1})$, the element $\partial_{\varepsilon_1}\partial_{\varepsilon_2}$ deforms $\Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}_N[1]$ into $\Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C}) \otimes \mathfrak{gl}_{N|N}[1]$ which is 4d Chern–Simons theory with gauge group $\mathrm{GL}_{N|N}$.
- The category of line defects of 4d Chern–Simons theory is known, in terms of modules over Yangian, quantum affine algebras, and elliptic quantum groups for $C = \mathbb{C}$, $\mathbb{C}^\times$, and E.
- The element π gives a particular deformation $\mathrm{Coh}(\mathrm{Higgs}_G(C), \pi)$ of $\mathrm{Coh}(\mathrm{Higgs}_G(C))$ in terms of difference modules as a category of boundary conditions.
- There should be an action of monoidal category of line defects on category of boundary conditions.

Thanks for your attention!