The Idea of Unification in Mathematics

Edward Frenkel

Department of Mathematics

University of California, Berkeley

Galileo Galilei

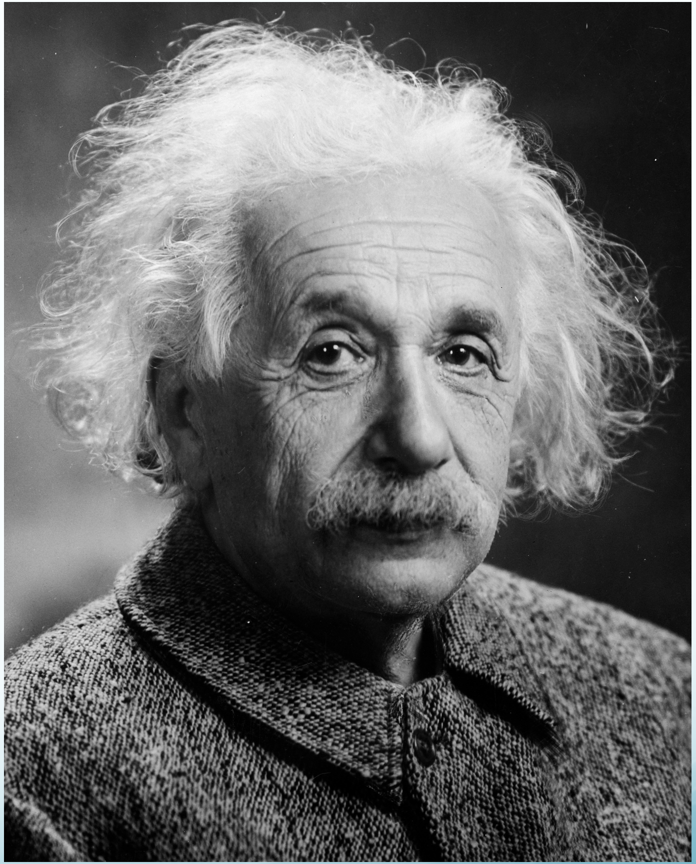
"The book of Nature is written in the language of Mathematics"

In Theoretical Physics:

- **Unified Field Theory** — early attempts by Albert Einstein to unify General Relativity (gravity) with Electromagnetism
- **Standard Model** describes 3 out of 4 known forces of nature: Electromagnetic, Weak & Strong (spectacular experimental success but some questions remain)
- **Grand Unified Theory (GUT)** — an attempt to merge these forces into a single unified force
- **Theory of Everything (TOE)** — unifying all forces

“The intellect seeking after an integrated theory cannot rest content with the assumption that there exist two distinct fields totally independent of each other by their nature.”

(Nobel Lecture, 1923)

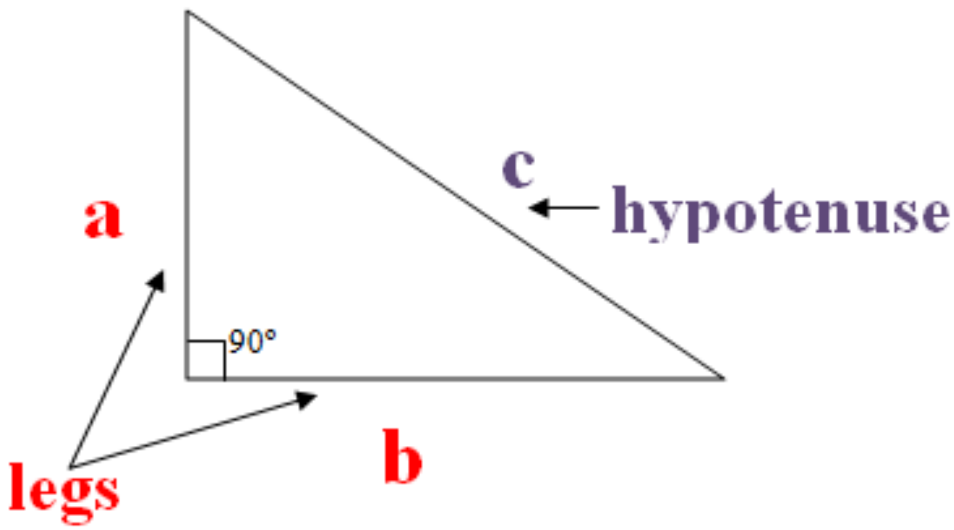


Mathematics & Physics

- Physical theories get updated in time.
- Mathematical theories *appear* to be objective, necessary, and timeless.
- Physics describes a **Universe**. But what does Math describe? There are many concepts that we don't currently find in the world around us. In what sense do they exist?
- So, what could **Unification** mean in Mathematics?

Leo Tolstoy “Anna Karenina”

Pythagoras Theorem



$$a^2 + b^2 = c^2$$

Platonic world of mathematical ideas?

- Pythagoras theorem meant the same thing to Pythagoras 2500 years ago as it does to us today, and it will mean the same 2500 years from now.
- If Pythagoras had not lived, someone else would have come up with exactly the same theorem (and many have!).

Do we **discover** mathematics

or do we **invent** it?

Kurt Gödel

Mathematical ideas "form an objective reality of their own, which we cannot create or change, but only perceive and describe."

If we ever meet aliens, will they have the same math as us?

Solaris–like intelligence

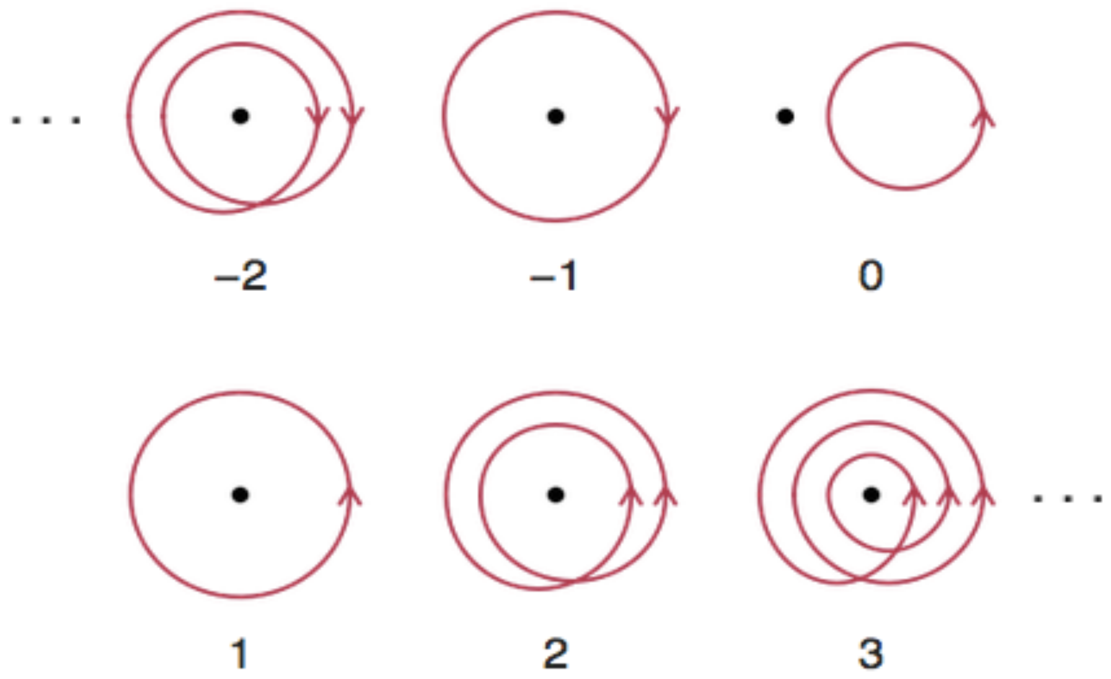
Would Solaris be able to

discover whole numbers?

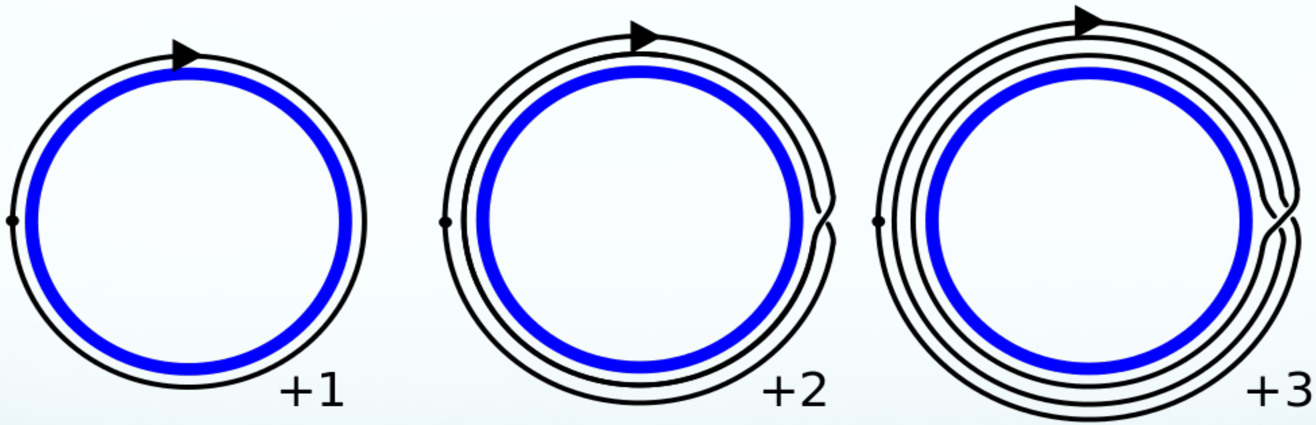
Counting

But counting “similar looking” objects is **not** the only way to discover whole numbers!

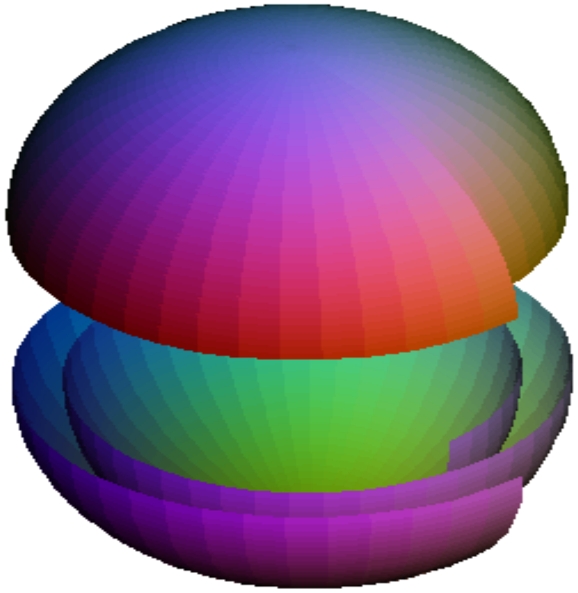
We can discover them through **winding**



Circle wrapping onto itself



- Likewise, a sphere can wrap onto itself multiple times — that’s how Solaris like consciousness can discover whole numbers
- “Homotopy groups” (topology)



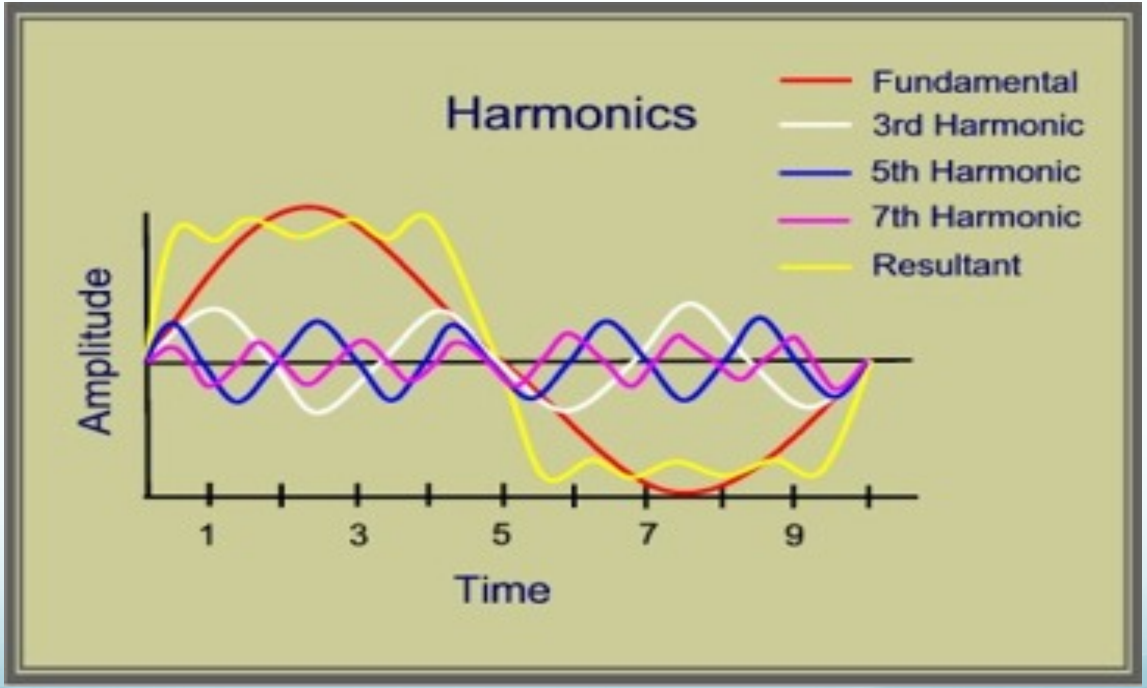
Math as a giant jigsaw puzzle

There are different continents

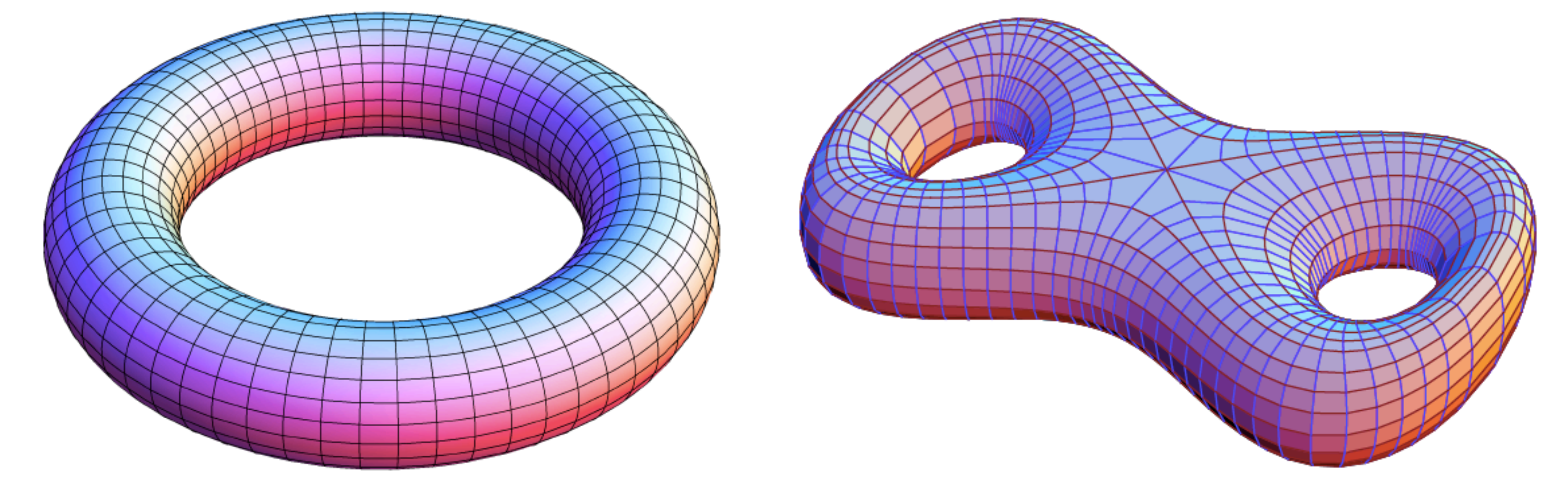
of Mathematics

Number Theory

Harmonic Analysis



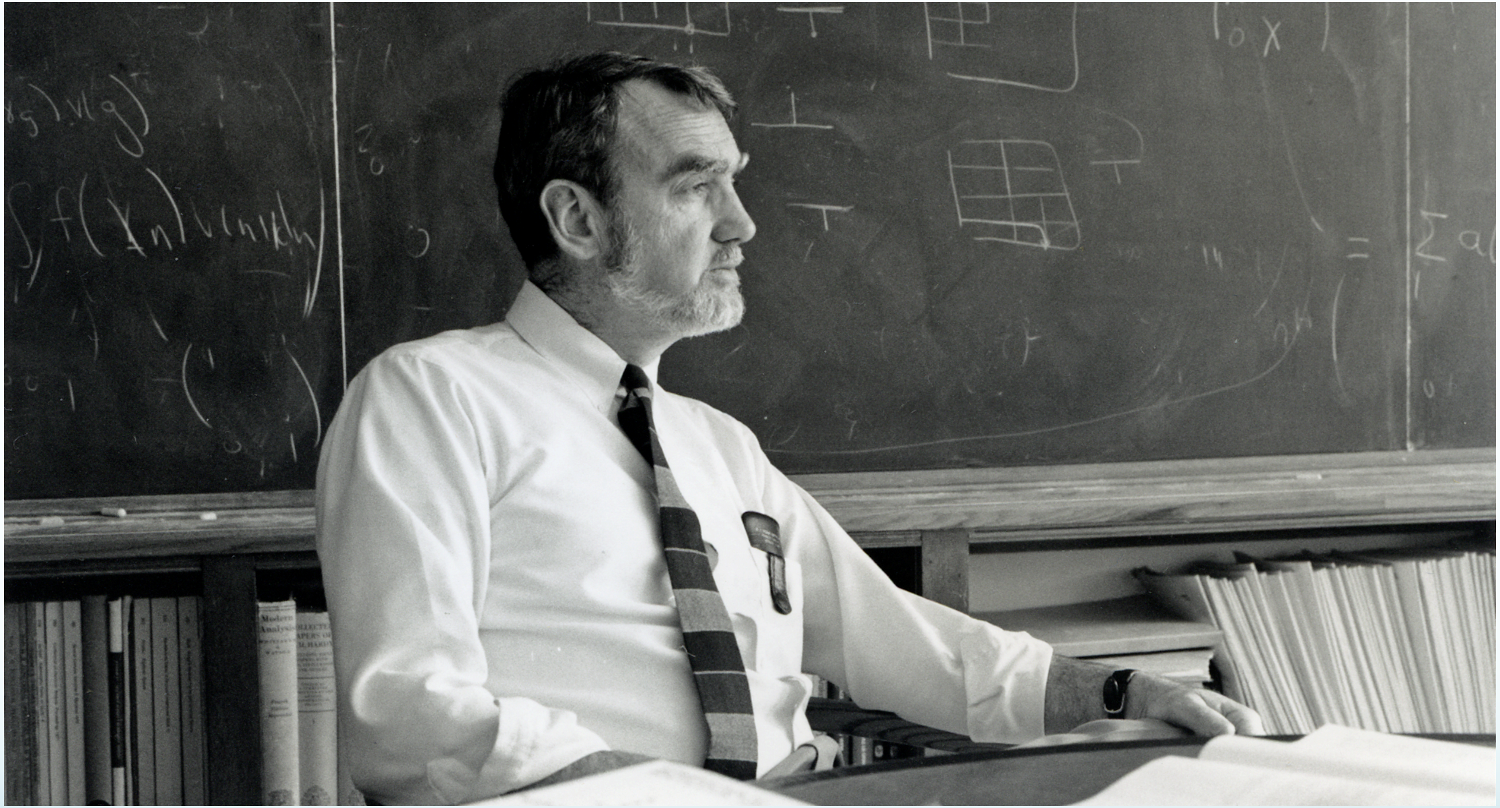
Geometry



Langlands Program — a big project aimed at finding common patterns in

Number Theory **Harmonic Analysis**

Geometry

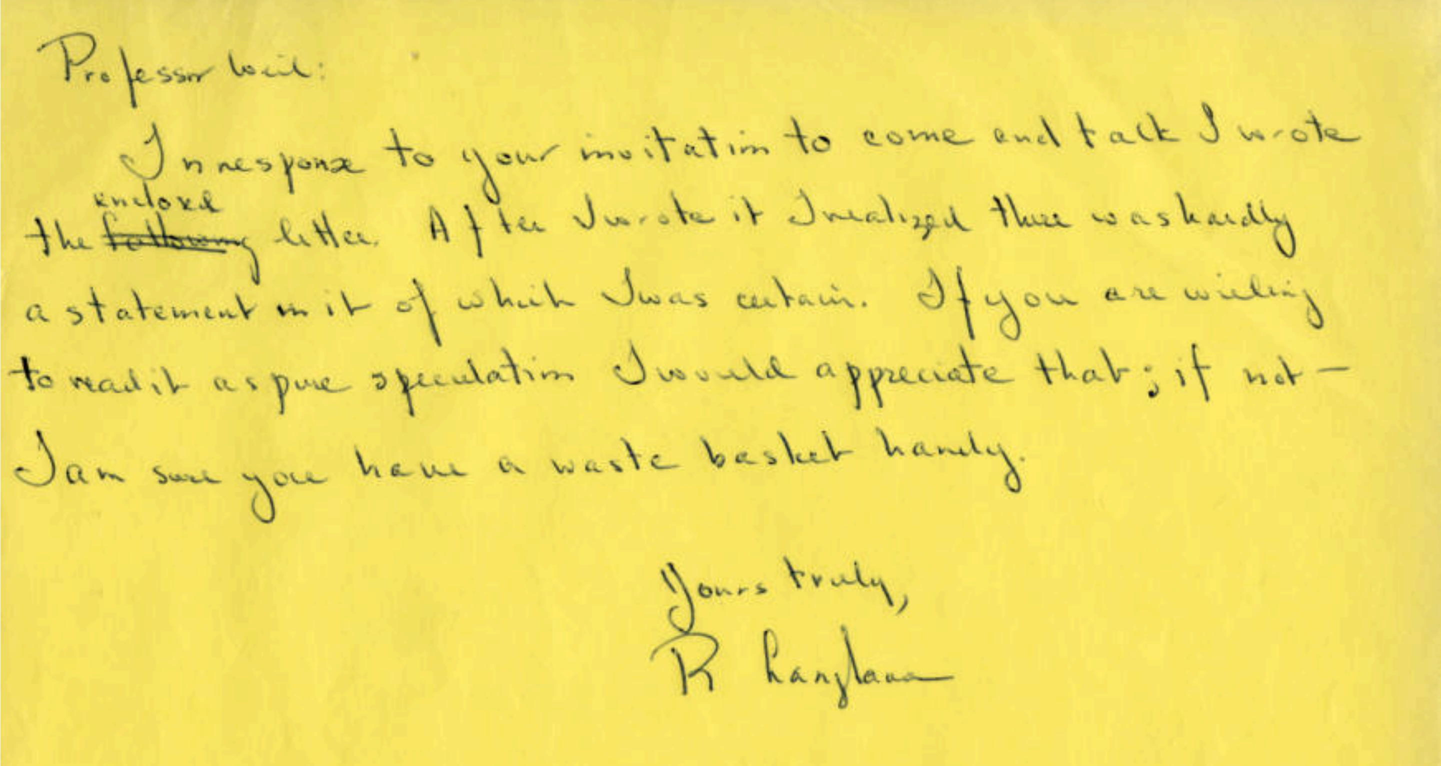


Robert Langlands at his office at the Institute for Advanced Study, 1999
(photo: Jeff Mozzochi)

Langlands Program — building bridges between different continents of Mathematics

Langlands Program — building bridges between different continents of Mathematics

Unification — finding hidden connections between areas of Math that seem far apart



Professor Weil:

In response to your invitation to come and talk I wrote the ~~following~~ enclosed letter. After I wrote it I realized there was hardly a statement in it of which I was certain. If you are willing to read it as pure speculation I would appreciate that; if not — I am sure you have a waste basket handy.

Yours truly,

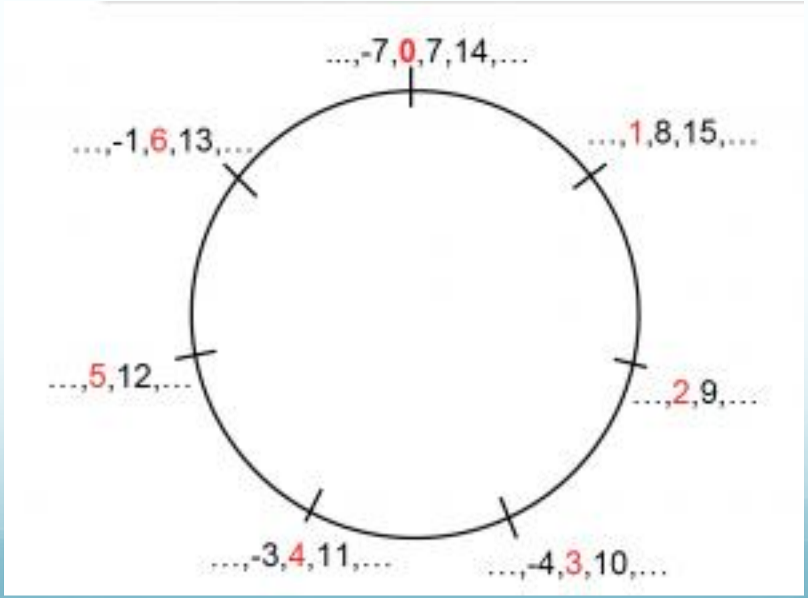
R. Langlands

Cover page of Langlands' letter to André Weil, 1967

(from the archive of the Institute for Advanced Study)

Arithmetic modulo primes

We can do arithmetic *modulo* any number; for example, a **prime number**, such as 2, 3, 5, 7, 11, 13, ...



Elliptic Curves mod **p**

- Cubic equation, such as

$$y^2 + y = x^3 - x^2$$

- Look for solutions modulo ***every*** prime number **p**

- Count the number of solutions for ***every*** prime **p**

Example

- Let **p=5**. What are the solutions?

$\boldsymbol{X} = 0$, $\boldsymbol{Y} = 0$ (both sides = 0)

$\boldsymbol{X} = 1$, $\boldsymbol{Y} = 0$ (both sides = 0)

$\boldsymbol{X} = 0$, $\boldsymbol{Y} = 4$ (LHS = 20, RHS = 0)

$\boldsymbol{X} = 1$, $\boldsymbol{Y} = 4$ (LHS = 20, RHS = 0)

$$y^2 + y = x^3 - x^2$$

- So we have 4 solutions **modulo 5**

Counting Problem

prime p	number of solutions	a(p)=p−#
2	4	−2
3	4	−1
5	4	1
7	9	−2
11	10	1
13	9	4

$$y^2 + y = x^3 - x^2$$

Miracle

- These numbers **a(p)** can be described all at once in the language of **Harmonic Analysis!**
- Namely, they are coefficients in the infinite series

$$q(1-q)^2(1-q^{11})^2(1-q^2)^2(1-q^{22})^2(1-q^3)^2(1-q^{33})^2(1-q^4)^2\ldots$$

$$= q-2q^2-q^3+2q^4+q^5+2q^6-2q^7-2q^9-2q^{10}+q^{11}-2q^{12}+4q^{13}+\ldots$$

Finding order in seeming chaos

- Denote by **b(p)** the coefficient in front of the **p**-th power of **q** in this series:

$$q(1-q)^2(1-q^{11})^2(1-q^2)^2(1-q^{22})^2(1-q^3)^2(1-q^{33})^2(1-q^4)^2\ldots$$

- Then **a(p) = b(p)** for all primes **p**.

Finding order in seeming chaos

Colossal compression of information:

Just one line of code gives us a simple rule for solving the counting problem, and for all primes at once!

Finding order in seeming chaos

Colossal compression of information:

Just one line of code gives us a simple rule for solving the counting problem, and for all primes at once! That's what I meant by "finding hidden connections."

There is more...

So, these numbers **a(p)** of solutions of the cubic equation mod **p** appear as coefficients of the infinite series:

$$q(1-q)^2(1-q^{11})^2(1-q^2)^2(1-q^{22})^2(1-q^3)^2(1-q^{33})^2(1-q^4)^2\ldots$$

$$= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13} + \ldots$$

- This infinite series actually converges if $|q| < 1$.

  i.e. on the complex unit disc

- We get a function on the unit disc, which is called a

  modular form

- It has special transformation properties under

  the group ***PSL(2,Z)*** of symmetries of the unit disc

Symmetries of the Unit Disc

Harmonic Analysis on the circle

- Basic harmonics: sin(nx), cos(nx), n integer
- What they have in common: they are invariant under shifts

$$x \longrightarrow x + 2\pi$$

- Group of symmetries is **Z** (integers w.r.t addition):

$$x \longrightarrow x + 2\pi M, \quad M \text{ arbitrary integer}$$

and the fundamental domain: interval $[0, 2\pi]$

Shimura-Taniyama-Weil Conjecture

- This correspondence between a cubic equation and a modular form has a vast generalization:
- A correspondence between **elliptic curves** over **Q** and **modular forms** (of weight 2).
- The first version of the conjecture was formulated by **Yutaka Taniyama** in 1955 at the historic Tokyo-Nikkō Int'l Symposium on Algebraic Number Theory.
- Proved in 1995 by A. Wiles & R. Taylor (stable case).

Tokyo–Nikkō symposium, 1955

- The Shimura–Taniyama–Weil Conjecture implies

  Fermat's Last Theorem, which took 350 years to prove:

  (Ken Ribet, 1986)

  The equation

  $$\mathbf{X}^{n} + \mathbf{Y}^{n} = \mathbf{Z}^{n} \qquad n=3,4,5,\ldots$$

  has no positive integer solutions X, Y, Z

- And it's only a tiny special case of the general Langlands Conjectures!

Langlands Program

Difficult questions of **Number Theory** (such as counting numbers of solutions of algebraic equations) may be reformulated in terms of more easily tractable questions of **Harmonic Analysis**.

Shimura-Taniyama-Weil Conjecture

cubic equations {numbers of solutions a_p}	$\longleftrightarrow$	modular forms {coefficients b_p}

Langlands Correspondence

representations of the Galois group	$\longleftrightarrow$	automorphic functions

A twist: Langlands dual group

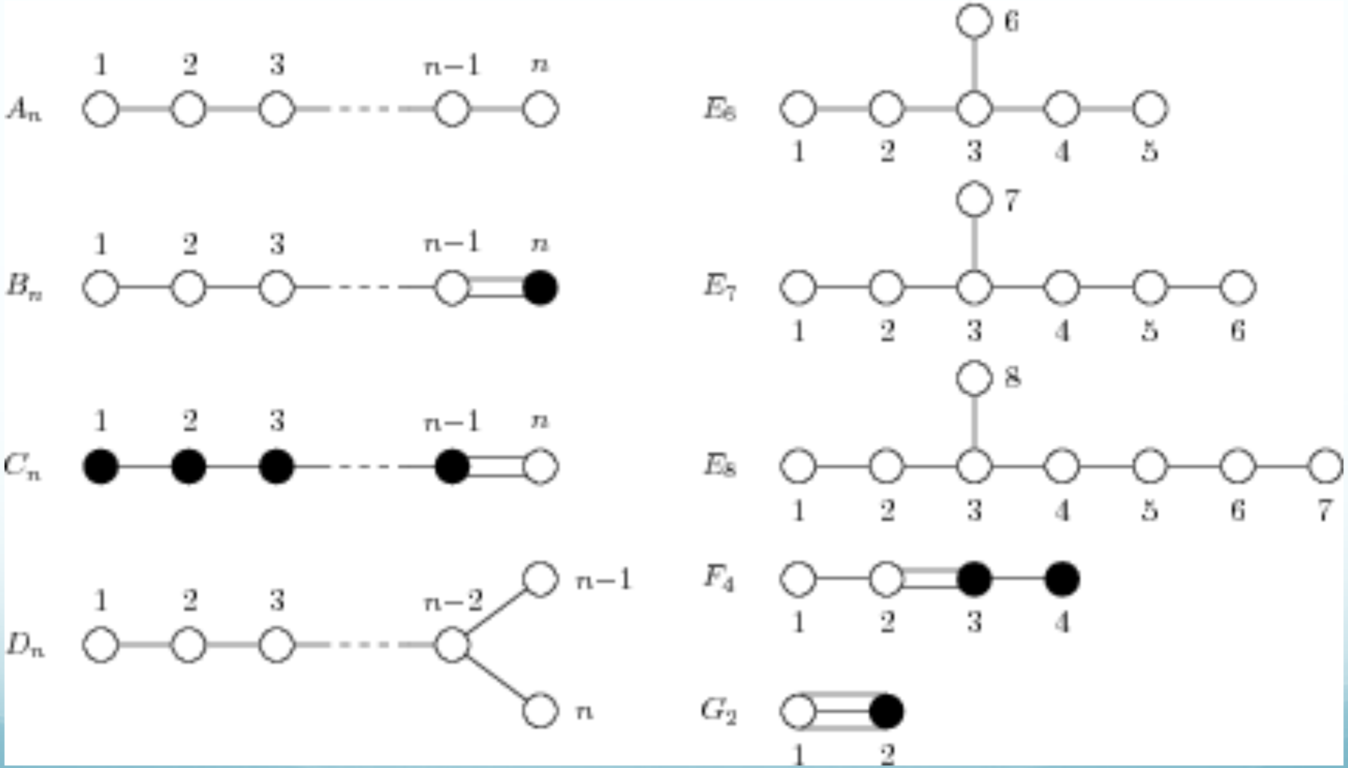
On the side of **Number Theory** we have a Lie group **G**

But on the side of **Harmonic Analysis** another Lie group appears:

The **Langlands dual** group **^{L}G**

This is still a Big Mystery!

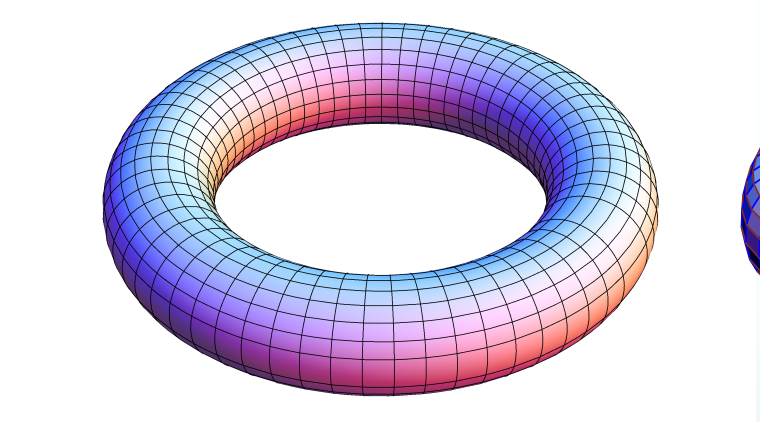
Dynkin Diagrams

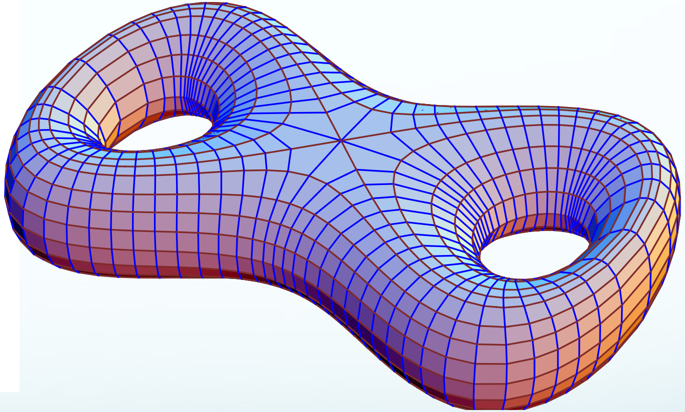


Rosetta Stone of Math

- **Andre Weil**: letter to his sister, **Simone Weil**, written from prison in 1940.
- The role of **analogy** in mathematics. Specifically, between these 3 areas:
- Number Theory
- Curves over finite fields
- Riemann surfaces

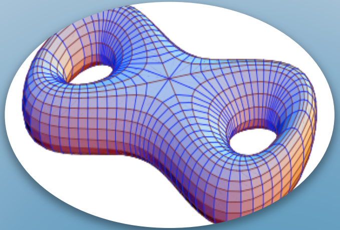
Riemann Surfaces





Rosetta Stone

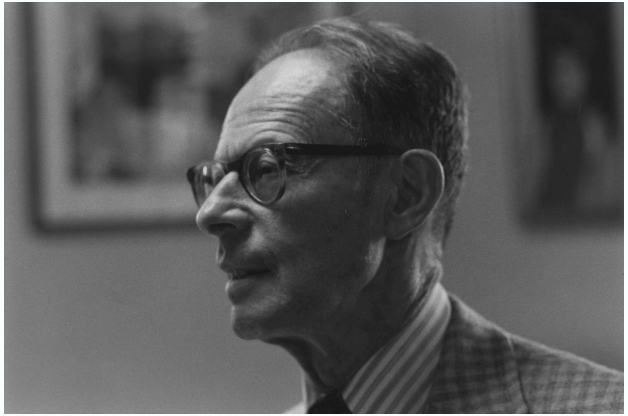
$$y^2 + y = x^3 - x^2$$



Number Theory

Curves over finite fields

Riemann surfaces



Andre Weil: "*My work consists in deciphering a trilingual text; of each of the three columns I have only disparate fragments; I have some ideas about each of the three languages: but I also know that there are great differences in meaning from one column to another… In the several years I have worked at it, I have found little pieces of the dictionary.*"

- It turns out that the Langlands Program patterns can be observed throughout the 3 columns of the Rosetta stone.
- Translating them between different columns of the Rosetta stone helps us to better understand their meaning.
- Moreover, the *same* patterns also appear in quantum physics (specifically, in the study of the **electromagnetic duality**) – so Quantum Physics appears as the 4th column!

- It turns out that the Langlands Program patterns can be observed throughout the 3 columns of the Rosetta stone.
- Translating them between different columns of the Rosetta stone helps us to better understand their meaning.
- Moreover, the *same* patterns also appear in quantum physics (specifically, in the study of the **electromagnetic duality**) – so Quantum Physics appears as the 4th column!
- Recently, with **Pavel Etingof & David Kazhdan**, we found a new **"flavor"** of the Langlands Program:
- **Analytic Langlands Correspondence for Riemann surfaces**

I will talk about this new work at the IPMU Conference

Number Theory, Strings, and Quantum Physics

May 31—June 5, 2021

Please tune in! ☺