Modular knots for triangle groups, Rademacher symbols, and 2-cocycles

I. Knots which behave like prime numbers (11 pages)

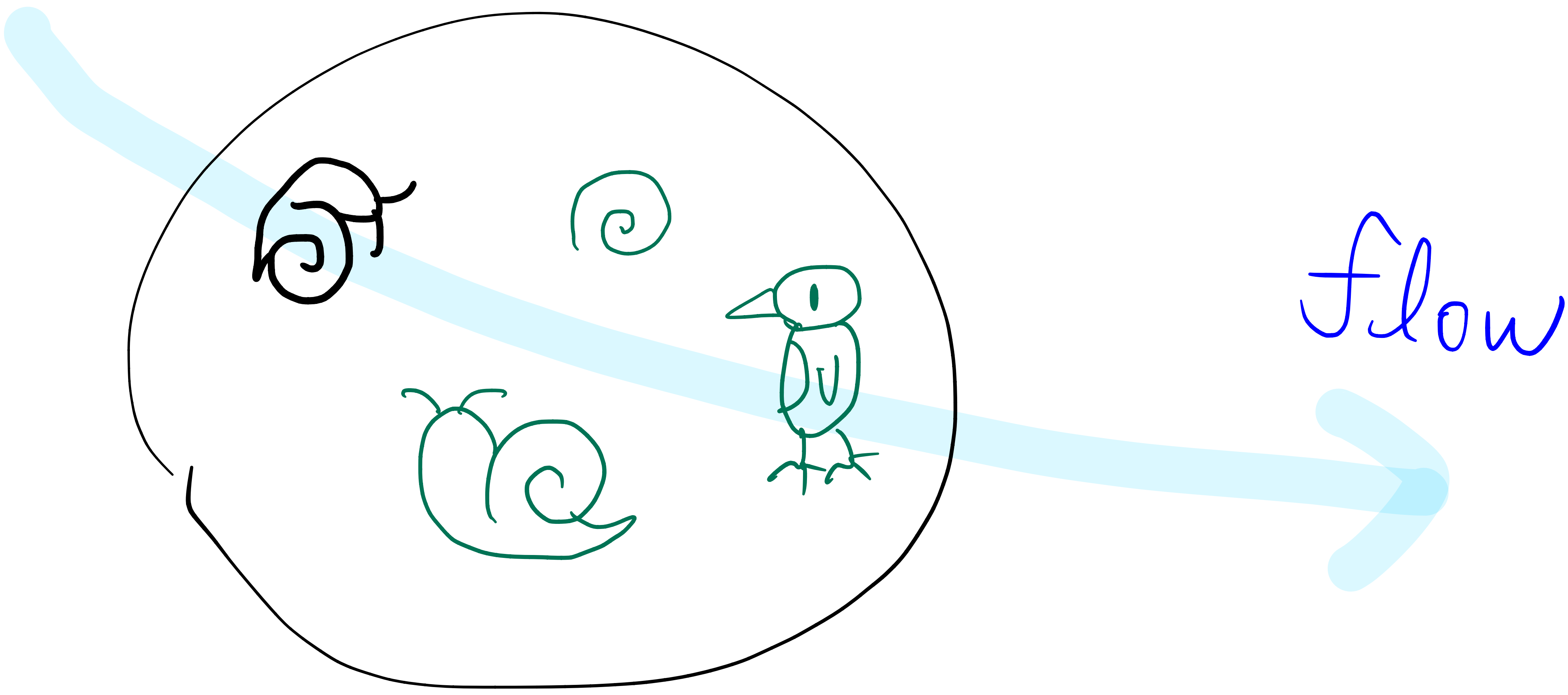
II. Rademacher symbols for triangle groups (11 pages)

J. Ueki @TDU

w/ T. Matsusaka @Nagoya

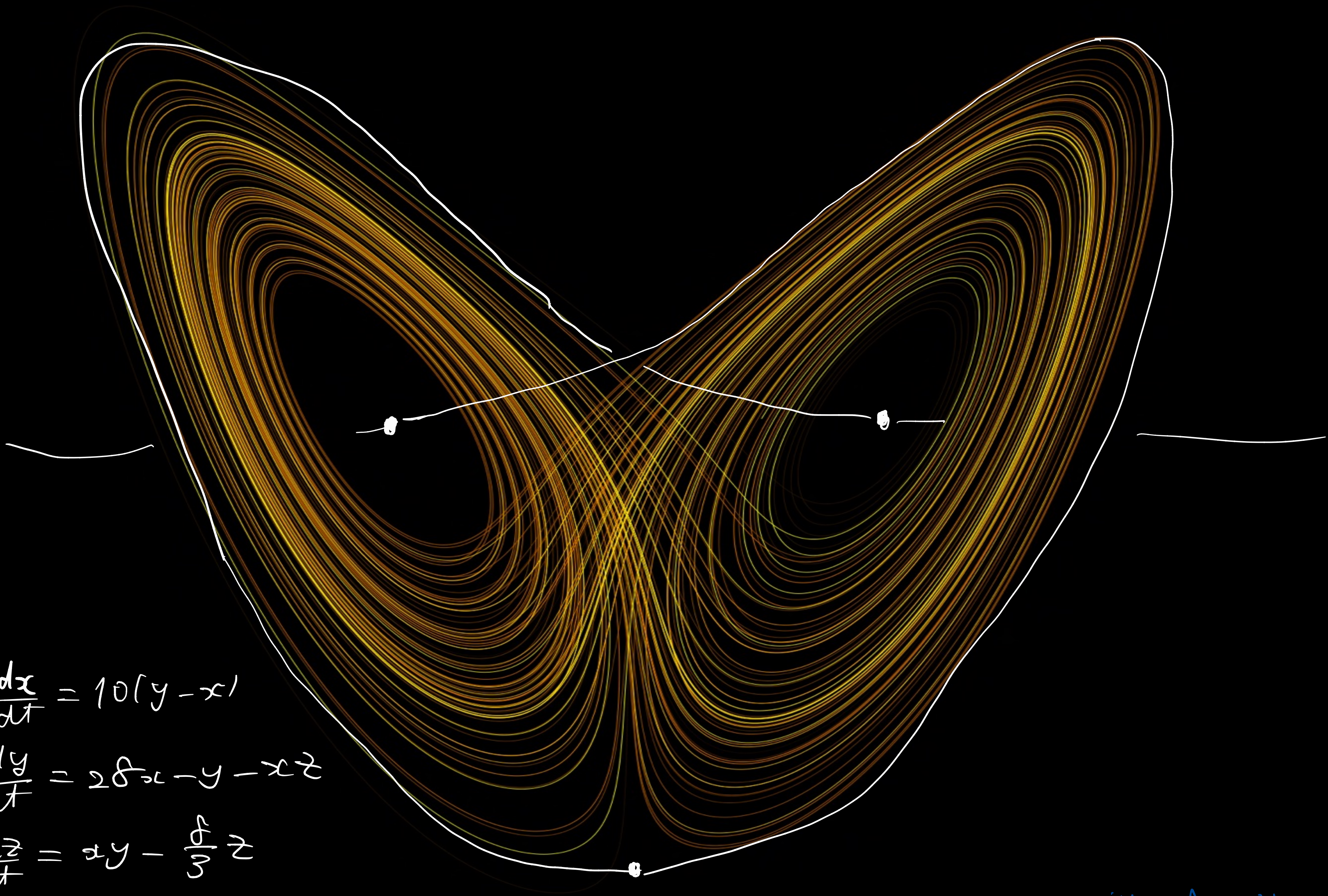
20210330

@ IPMU MS-Seminar



open system

Lorenz attractor [Lorenz 1963 Deterministic Nonperiodic Flow]



$$\begin{cases} \dfrac{dx}{dt} = 10(y-x) \\ \dfrac{dy}{dt} = 28x - y - xz \\ \dfrac{dz}{dt} = xy - \dfrac{8}{3}z \end{cases}$$

-picture from Wikipedia

Q. An analogue of {prime numbers} ?

- nice conditions for

$$\mathcal{K} = \bigcup_{i \in \mathbb{N}} K_i \quad \text{in } S^3$$

$[K_i] \longleftrightarrow [\mathrm{Frob}_p]$

in the fundamental grp π_1

- examples

[U. 2021 BLMS]

$\boxed{\text{the Chebotarev law}}$ $\Rightarrow$ stably generic $\Rightarrow$ very admissible

[Mazur 2012] Def. Q. const

[McMullen 2013] an example

[U.] many examples

[Mihara 2019]

[Niibo 2014]

[Niibo Ueki 2019 TAMS]

∴ ray class grp

∴ idelic class field theory

sophisticated

elementary

I-§1 the Cheb. law

Gauß
$\mathrm{lk} \bmod 2 \longleftrightarrow \left(\frac{p}{q}\right)$ Legendre's quadratic symbol

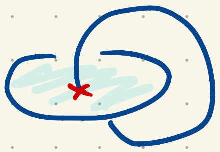
Definition 1 (The Chebotarev law). Let $(K_i) = (K_i)_{i \in \mathbb{N}_{>0}}$ be a sequence of disjoint knots in a 3-manifold M. For each $n \in \mathbb{N}_{>0}$ and $j > n$, put $L_n = \cup_{i \leqslant n} K_i$ and let $[K_j]$ denote the conjugacy class of K_j in $\pi_1(M - L_n)$. We say that (K_i) *obeys the Chebotarev law* if the density equality

$$\lim_{\nu \to \infty} \frac{\#\{n < j \leqslant \nu \mid \rho([\underbrace{K_j}_{\text{Frob } p}]) = C\}}{\nu} = \frac{\#C}{\#G}$$

holds for any $n \in \mathbb{N}_{>0}$, any surjective homomorphism $\rho : \pi_1(M - L_n) \twoheadrightarrow G$ to any finite group, and any conjugacy class $C \subset G$.

A countably infinite link $\mathcal{K}$ is said to be *Chebotarev* if it obeys the Chebotarev law with respect to some order.

Eg mod 2 linking number

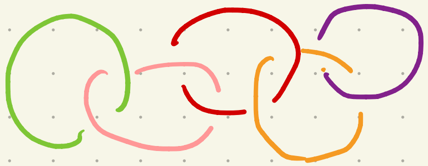


$$\rho : \pi_1(M - K_1) \to \mathbb{Z}/2\mathbb{Z}$$
$$[K_j] \longmapsto \mathrm{lk}(K_j, K_1) \bmod 2$$

$$\mathrm{lk}(K_j, K_1) = \begin{cases} 0 \\ 1 \end{cases} \bmod 2$$

the densities of K_j's are both $\frac{1}{2}$.

Quiz mod 2 Olympic link



find the density!

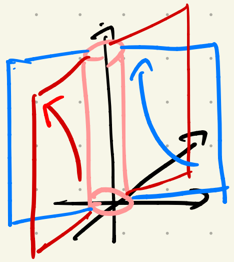
Please Answer: on the Chat

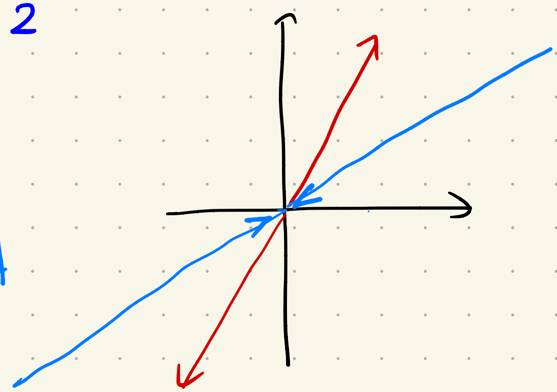
§ 2 McMullen's constr.

<u>Quiz</u> $A = \begin{pmatrix}1&1\\1&2\end{pmatrix}$

Find the eigen values
vectors

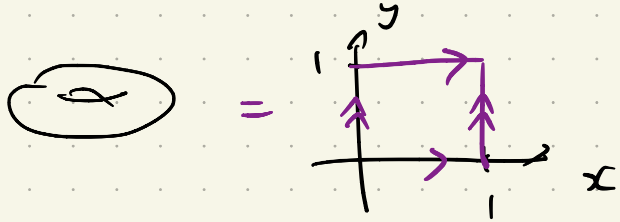
$\mathrm{tr} = 3 > 2$





$A = \begin{pmatrix}1&1\\1&2\end{pmatrix} \curvearrowright T = \mathbb{R}^2/\mathbb{Z}^2$

$\in SL_2\mathbb{Z}$



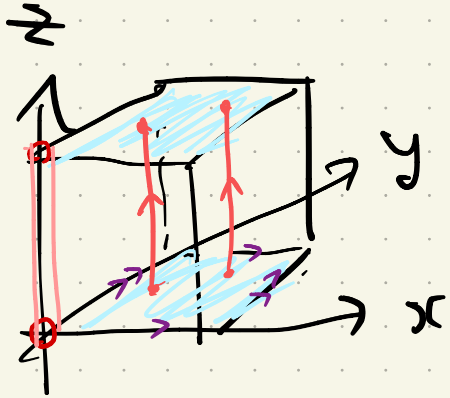
$(T\setminus\{0\})\times[0,1] \,/\, (x,0)\sim(Ax,1)$

$\hat{=}\ S^3 \setminus$  the figure-eight knot

$K = 4_1$

homeo
non-trivial

⭑ every closed orbits has a tub nbd of this shape.
(Anosov flow)



- glue  and  via $\varphi : x \longmapsto Ax$
  the monodromy map

- $\varphi^t : (x,y,z) \longmapsto (x,y,z+t)$, $t\in\mathbb{R}$
  defines a flow ($\mathbb{R}$-action). : the suspension flow of φ.

[McMullen 2013] : There are countably infinite number of closed orbits.
They obey the Cheb law if ordered by length, in a generic metric!

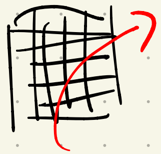
Thm [McMullen 2013]

topologically mixing

The closed orbits of a pseudo Anosov flow on a closed 3-mfd obey the Cheb. law if ordered by length.

prf $\exists$ Markov section

a rectangle , "flow box"



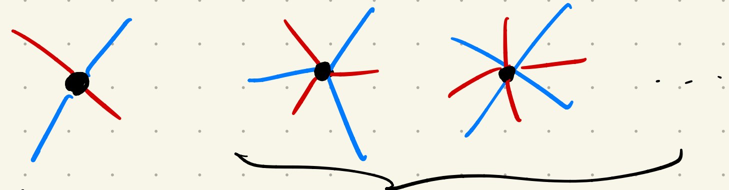
Parry Pollicott's

zeta func. of symbolic dynamics

& some topology.

pseudo Anosov

2-pronged $\qquad$ $k \geqq 3$ - pronged



regular closed orbits $\qquad$ singular closed orbits (finitely many)

Thm ⟨Nilsen-Thurston classif.⟩

$\varphi \circlearrowright \Sigma$ a surface

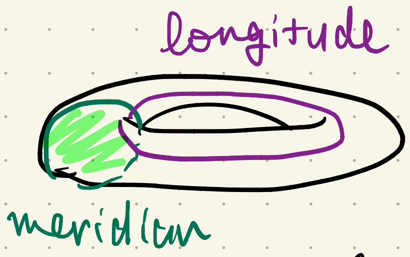
- periodic
- reducible
- pseudo Anosov

Thm (Thurston) if genus $\geqq 2$, then

φ: pseudo Anosov

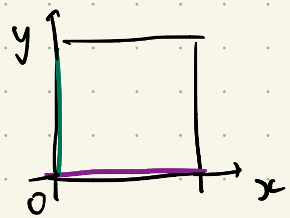
$\underset{\text{iff}}{\Longleftrightarrow}$ the mapping torus Σ_φ is hyperbolic.

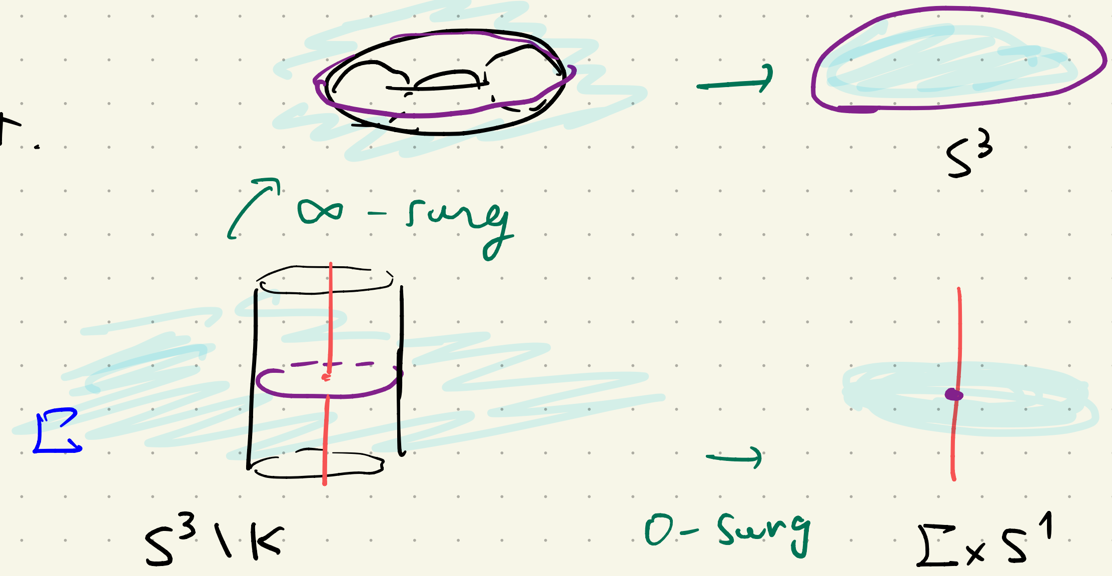
Lem the Cheb. law persists under a Dehn surgery along a closed orbit.

solid torus $V =$ 

$(M - V) \cup_r V$

slope $= r \in \mathbb{Q} \cup \{\infty\}$





$= (\Sigma \setminus \{0\} \times [0,1]) / \sim$

McMullen's argument for the fig-eight

① the (pseudo)-Anosov flow extends to a (pseudo)-Anosov flow on the result of the 0-filling.

② Hence by Thm, the orbits on the torus bundle obeys the Cheb. law.

③ By the ∞-surgery along the 0-orbit, we obtain a Cheb. link in S^3 containing the fig. eight.

// $\cup$(closed orbits)

Rem [Ghrist] this infinite link contains every type of knots and links.
(by template + Morse-Smale dynamics.)

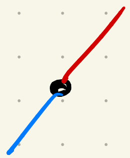
Claim (McMullen) Thm (V.)

The same construction applies to $\forall K$ hyp. fib. knot in S^3

Rem $S^3 \setminus K \cong \Sigma_\varphi$, φ : pseudo-Anosov map does not necessarily extends to a pseudo-Anosov map on the capped surface $\hat{\Sigma}_\varphi$.

Eg (Gabai) $K = 8_{20}$

the 0-surg is not hyperbolic



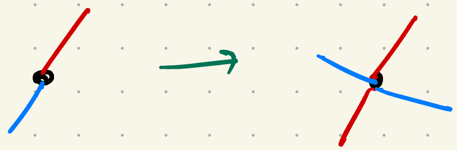
1-pronged orbit may exist.

< generalized pseudo-Anosov flow >

How to justify ①

Dehn-Fried-Goodman

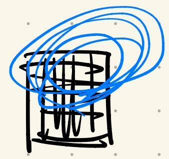
$\exists$ rational surgery yields pseudo Anosov flow



How to justify ②

McMullen's Thm extends.

$\begin{cases} \text{- topologically mixing } \left(\Leftrightarrow \exists i,j \ \frac{l(k_i)}{l(k_j)} \notin \mathbb{Q}\right) \\ \text{- } \exists \text{ Markov section} \end{cases}$

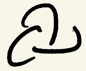


//

Cor modular knots $\underset{\cdot}{=}$ Lorentz knots

& the missing trefoil

also obeys the Cheb. law

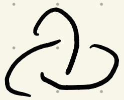
§3. Modular knots and  obey the Cheb law.

$PSL_2\mathbb{Z}\backslash\mathbb{H}$ with 1 cusp
: modular orbifold 2 cone pts

$T^1(PSL_2\mathbb{Z}\backslash\mathbb{H})$ unit tangent bundle

$:= \{ X : \text{a tan. vec} \mid \|X\| = 1 \}$

$\cong \{ |z|^2 + |w|^2 = 1 \} \setminus \{ z^3 = w^2 \}$ in $\mathbb{C}^2$

$\cong S^3 \setminus$  trefoil

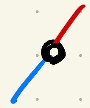
$\cong SL_2\mathbb{Z}\backslash SL_2\mathbb{R}$

"the geodesic flow" is defined by

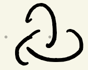
$\varphi^t : M \mapsto M \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad t \in \mathbb{R}.$

classically known to be Anosov.

closed orbits are called modular knots.

- cusp orbit = the missing trefoil.
- $\exists$ a nice compactification: so that  around the cusp.

By the similar argument,

{closed orbits} $\cup$ {  }

obey the Cheb. law.

[Ghys 2007 ICM]

$\{ \gamma \in SL_2\mathbb{Z} \mid |\mathrm{tr}\,\gamma| > 2 \text{ primitive} \}$

$\longrightarrow\!\!\!\!\rightarrow$ {modular knots C_γ}

Rademacher symbol $\Psi(\gamma)$

$= lk(\text{trefoil}, C_\gamma)$

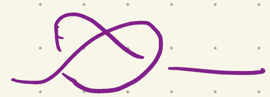
Cor. density thm for Ψ mod m:

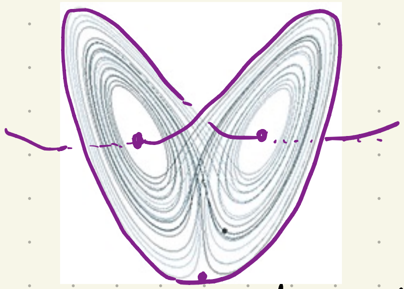
Corollary. *Suppose that γ runs through primitive hyperbolic elements of $\mathrm{SL}_2\mathbb{Z}$. For any $m \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}/m\mathbb{Z}$, we have the density equality*

$$\lim_{\nu\to\infty} \frac{\#\{\gamma \mid |\mathrm{tr}\,\gamma| < \nu,\ \Psi(\gamma) = k \text{ in } \mathbb{Z}/m\mathbb{Z}\}}{\#\{\gamma \mid |\mathrm{tr}\,\gamma| < \nu\}} = \frac{1}{m}.$$

- [Ghys 2007 ICM] + [Bonatti Pinsky 2020] etc.

modular knots $\xleftrightarrow{1:1}$ Lorenz knots

(with the missing trefoil) $\xleftrightarrow{\text{Conj}}$ ()



Lorenz attracter

classic parameters:

$$\begin{cases} \dfrac{dx}{dt} = 10(y-x) \\ \dfrac{dy}{dt} = 28x - y - xz \\ \dfrac{dz}{dt} = xy - \dfrac{8}{3}z \end{cases}$$

today's latter half

@ If we replace $SL_2\mathbb{Z}$ by a general $\Gamma_{p,q}$ the triangle gp

trefoil $\rightsquigarrow$ (p,q)-torus knot

$\rightsquigarrow$ gen. of various thm's [Dehornoy Pinsky] [Matsusaka U.]

Q. [idelic CFT for 3-mfds]

Can we translate the Global reciprocity law into

" Artin L = Hecke L " ?

Artin L = Parry Pollicott.

Hecke L = ? (I want a nice one)

The Rademacher symbol

The discriminant function $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ with $q = e^{2\pi\sqrt{-1}z}$, $z \in \mathbb{H}^2$ is a well-known modular function of weight 12. *The Dedekind symbol* Φ and *the Rademacher symbol* Ψ are the functions $\mathrm{SL}_2\,\mathbb{Z} \to \mathbb{Z}$ satisfying the equalities

$$\log\Delta(\gamma z) - \log\Delta(z) = \begin{cases} 6\log(-(cz+d)^2) + 2\pi i\Phi(\gamma) & \text{if } c \neq 0, \\ 2\pi i\Phi(\gamma) & \text{if } c = 0, \end{cases}$$

$$\Psi(\gamma) = \Phi(\gamma) - 3\mathrm{sgn}(c(a+d))$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\,\mathbb{Z}$ acting on $z \in \mathbb{C}$ via the Möbius transformation $\gamma z = \dfrac{az+b}{cz+d}$. Here we take a branch of the logarithm so that $-\pi \leqslant \mathrm{Im}\log z < \pi$ holds. This Ψ factors through the conjugacy classes of $\mathrm{PSL}_2\mathbb{Z}$ and satisfies $\Psi(\gamma^{-1}) = -\Psi(\gamma)$ for any γ. (We may find in many literatures various confusions about the convention of the Rademacher symbol. Our convention is based on Matsusaka's quite thorough investigation; See [Mat20].)

The Rademacher symbol Ψ is known to be a highly ubiquitous function. Indeed, Atiyah proved the equivalence of seven definitions rising from very distinct contexts [Ati87], whereas Ghys gave further characterizations (cf.[BG92]) especially by using modular knots [Ghy07, Sections 3.3–3.5] (see also [DIT17, Appendix]), proving that *for each primitive hyperbolic $\gamma \in \mathrm{SL}_2\,\mathbb{Z}$, the linking number between the modular knot C_γ and the missing trefoil K coincides with the Rademacher symbol*, namely,

$$\mathrm{lk}(C_\gamma, K) = \Psi(\gamma)$$

Thm. Modular knots C_γ obey the Cheb. law.

COROLLARY *Suppose that γ runs through primitive hyperbolic elements of $\mathrm{SL}_2\,\mathbb{Z}$. For any $m \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}/m\mathbb{Z}$, we have the density equality*

$$\lim_{\nu\to\infty} \frac{\#\{\gamma \mid |\operatorname{tr}\gamma| < \nu,\ \Psi(\gamma) = k \text{ in } \mathbb{Z}/m\mathbb{Z}\}}{\#\{\gamma \mid |\operatorname{tr}\gamma| < \nu\}} = \frac{1}{m}.$$

~ Intermission / commercial ~

Friday Tea Time

Zoom Seminar

Number theory and adjacent topics

once per two weeks 2021 Spring-Summer

New speakers are welcome!

I Ueki @ TDU

T Matsusaka @ Nagoya

II.§1 the triangle group $\Gamma_{p,q} = \Gamma(p,q,\infty)$

$2 \leq p < q$ coprime integers. put $r = pq - p - q$.

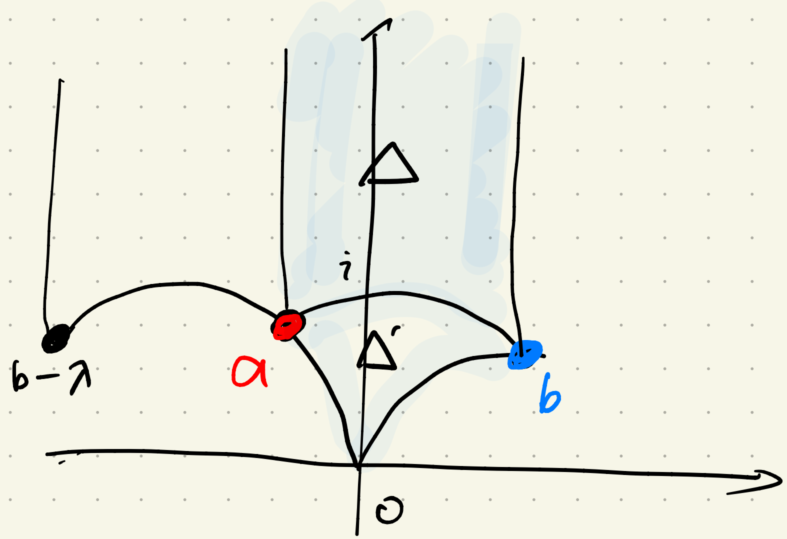
$\Gamma_{p,q} < SL_2\mathbb{R}$ a Fuchsian grp of the 2nd kind

generated by

$$\begin{cases} T_{p,q} = \begin{pmatrix} 1 & 2\left(\cos\frac{\pi}{p} + \cos\frac{\pi}{q}\right) \\ 0 & 1 \end{pmatrix}, \\ S_p = \begin{pmatrix} 0 & -1 \\ 1 & 2\cos\frac{\pi}{p} \end{pmatrix}, \\ U_q = \begin{pmatrix} 2\cos\frac{\pi}{q} & -1 \\ 1 & 0 \end{pmatrix}. \end{cases}$$

We have
- $T_{p,q} = -U_q S_p$
- $S_p^{\,p} = U_q^{\,q} = -I_2$

The fundamental domain $\Delta(p,q,\infty) \cup \Delta'(p,q,\infty)$



$a = e^{\pi i(1-\frac{1}{p})}$, $b = e^{\pi i/q}$, $i\infty$

are fixed pts of S_p, U_q, $\pm T_{p,q}$.

$$\mathrm{Vol}(\Gamma_{p,q}\backslash\mathbb{H}) = 2\pi\frac{pq-p-q}{pq} = 2\pi\frac{r}{pq}.$$

$$\Gamma_{p,q} \cong \mathbb{Z}/2p\mathbb{Z} \ast_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/2q\mathbb{Z}$$ the amalgamated product

$$H^2(\Gamma_{p,q}) \cong \mathbb{Z}/2pq\mathbb{Z}.$$

§2. Modular forms for $\Gamma_{p,q}$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R}$, $z \in \mathbb{H}$

- the automorphic factor $j(\gamma, z) := cz + d$

  2-cocycle cond:

  $$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z)\, j(\gamma_2, z)$$

- the slash operator of weight $k \in \mathbb{Z}$ on $f : \mathbb{H} \to \mathbb{C}$ is def'd by

  $$(f|_k\gamma)(z) = j(\gamma, z)^{-k} f(\gamma z).$$

- $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic modular form of weight k for $\Gamma_{p,q}$ if

  i) $f|_k\gamma = f$

  ii) $f(z) = \sum_{n=0}^{\infty} a_n q_\lambda^n$ a Fourier expans. with $\begin{cases} \lambda = 2(\cos\frac{\pi}{p} + \cos\frac{\pi}{q}) \\ q_\lambda = e^{2\pi i z/\lambda} \end{cases}$

- $f : \mathbb{H} \to \mathbb{C}$ is a harmonic Maass form if

  i) $f|_k\gamma = f$ $\mathcal{H}_k(\Gamma_{p,q}) := \left\{ \begin{matrix} \text{h.M.f's} \\ \text{for } \Gamma_{p,q} \end{matrix} \right\}$

  ii) $\Delta_k f = 0$

  iii) $\exists \alpha > 0$, s.t., $f(x+iy) = O(y^\alpha)$ as $y \to \infty$ unif. in $x \in \mathbb{R}$.

$$\left[\begin{array}{l} \Delta_k = -\xi_{2-k} \circ \xi_k \\ \quad = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \\ \xi_k(f) = 2iy^k \overline{\frac{\partial}{\partial \bar{z}} f} \end{array}\right.$$

- $\xi_k(f|_k\gamma) = (\xi_k f)|_{2-k}\gamma$.

- Fourier expans is explicitly given by Whittaker functions.

Harmonic Maass forms of wt $2k$ for $\Gamma_{p,q}$

(containing the Eisenstein sereses)

$C_{p,q} - \frac{\log y}{\mathrm{vol}} + \frac{y}{\lambda} + \ominus + \ominus$

very complex ed

" Im. formulas

Def

$$E_{2k}^{(p,q)}(z,s) = \frac{1}{\lambda^s} \sum_{\gamma \in (\Gamma_{p,q})_\infty \backslash \Gamma_{p,q}} \frac{\mathrm{Im}(\gamma z)^{s-k}}{j(\gamma, z)^{2k}}$$

- abs + unif conv on cpt subsets on $z \in \mathbb{H}$, $\mathrm{Re}(s) > 1$.
- $\Delta_{2k} E_{2k}^{(p,q)}(z,s) = (s-k)(1-k-s)\, E_{2k}^{(p,q)}(z,s)$.

Prop $E_0^{(p,q)}(z,s)$ has a meromorph. continuation around $s=1$ with a single pole there with the residue

$$\mathrm{Res}_{s=1} E_0^{(p,q)}(z,s) = \frac{1}{\mathrm{vol}(\Gamma_{p,q} \backslash \mathbb{H})}$$

Def

$$\mathcal{L}_{p,q}(z) := \lim_{s \to 1} \left(E_0^{(p,q)}(z,s) - \frac{1}{\mathrm{vol}} \frac{1}{s-1} \right)$$

$$E_2^{(p,q),*}(z) := \xi_0 \mathcal{L}_{p,q}$$

$$= \frac{-1}{\mathrm{vol}} \frac{1}{y} + \frac{1}{\lambda} + \sum_{n=1}^{\infty} d(n) q_\lambda^n$$

Prop $E_2^{(p,q),*}$: a harmonic Maass form of wt 2 for $\Gamma_{p,q}$

$\mathcal{H}_2(\Gamma_{p,q}) = (1 \text{ dim } \mathbb{C}\text{-}\mathrm{sp} \text{ gen by })$

Def. $E_2^{(p,q)}(z) :=$ the hol. part of $E_2^{(p,q),*}(z)$

Lem for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{p,q}$

$$(cz+d)^{-2} E_2^{(p,q)}(\gamma z) - E_2^{(p,q)}(z) = \frac{pq}{r} \frac{c}{\pi i (cz+d)}.$$

$r = pq - p - q$

The Rademacher symbol $\Psi_{p,q}$

$$F_{p,q}(z) := \frac{2\pi i r z}{\lambda} - 4\pi r \sum_{n=1}^{\infty} c_{p,q}(n) q_\lambda^n$$

so that $\frac{d}{dz} F_{p,q}(z) = 2\pi i r E_2^{(p,q)}(z)$ holds.

Put $R_{p,q}(\gamma, z) := F_{p,q}(\gamma z) - F_{p,q}(z)$

$$\parallel$$

$$\log \Delta_{p,q}(\gamma z) - \log \Delta_{p,q}(z).$$

By $\boxed{\text{Lem}}$,

$$\frac{d}{dz}\Big(R_{p,q}(\gamma, z) - 2pq \underbrace{\log j(\gamma, z)}_{\mathrm{Im} \in [-\pi, \pi)}\Big) = 0$$

$\therefore \exists\, \Psi_{p,q} : \Gamma_{p,q} \longrightarrow \mathbb{C}$ s.t.

$$R_{p,q}(\gamma, z) = 2pq \log j(\gamma, z) + 2\pi i \Psi_{p,q}(\gamma).$$

Prop. $\Psi_{p,q}(T_{p,q}) = r$, $\Psi_{p,q}(S_p) = -q$,

- $\mathrm{Im}\, \Psi_{p,q} \subset \mathbb{Z}$. $\quad \Psi_{p,q}(U_q) = -p$.

Rem $\Psi_{2,3}$ is the symbol for $SL_2\mathbb{Z}$.

A cusp form of weight $2pq$

Def $\Delta_{p,q}(z) = \exp F_{p,q}(z)$ on $\mathbb{H}$

Thm $\Delta_{p,q}(z)$ is a cusp form of wt $2pq$

(i.e., $\Delta_{p,q}|_{2pq}\gamma = \Delta_{p,q}$).

with a Fourier expans.

$$\Delta_{p,q}(z) = q_\lambda^r + O(q_\lambda^{r+1})$$

Def The Rademacher symbol is a function $\Psi_{p,q} : \Gamma_{p,q} \to \mathbb{Z}$ satisfying

$$\log \Delta_{p,q}(\gamma z) - \log \Delta_{p,q}(z) = 2pq \log j(\gamma, z) + 2\pi i \Psi_{p,q}(\gamma).$$

- $\frac{d}{dz} \log \Delta_{p,q}(z) = 2\pi i r E_2^{(p,q)}(z)$.

- $\mathcal{L}_{p,q}(z) := \lim_{s \to 1} \Big(E_0^{(p,q)}(z, s) - \frac{1}{\mathrm{vol}(\Gamma_{p,q} \backslash \mathbb{H})} \frac{1}{s-1} \Big)$

$$= -\frac{1}{\mathrm{vol}} \log\left(y |\Delta_{p,q}(z)|^{1/pq}\right) + C_{p,q}.$$

§3 Cycle integrals

Let $\gamma \in \Gamma_{p,q}$

- hyperbolic ($|\mathrm{tr}\,\gamma| > 2$)

- γ has two real fixed pts

  $w_\gamma > w'_\gamma$ on $\mathbb{R}$.

$$M_\gamma^{-1} \gamma M_\gamma = \begin{pmatrix} j(\gamma, w_\gamma) & 0 \\ 0 & j(\gamma, w'_\gamma) \end{pmatrix} =: \begin{pmatrix} \xi_\gamma & 0 \\ 0 & \xi_\gamma^{-1} \end{pmatrix}$$

$$M_\gamma = \frac{1}{\sqrt{w_\gamma - w'_\gamma}} \begin{pmatrix} w_\gamma & w'_\gamma \\ 1 & 1 \end{pmatrix} \in SL_2\mathbb{R}.$$

Assume $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- primitive ($\gamma = \sigma^n \Rightarrow n = 1$)
- assume $\mathrm{tr}\,\gamma > 2$, $c > 0$. then $\xi_\gamma > 1$.

Let S_γ denote the geodesic on $\mathbb{H}$ connecting w_γ and $w_{\gamma'}$ and $\overline{S_\gamma}$ the image in $\Gamma_{p,q}\backslash\mathbb{H}$.

then

Thm

$$\boxed{\int_{\overline{S_\gamma}} E_2^{(p,q),*}(z)\,dz = \frac{1}{r}\Psi_{p,q}(\gamma).}$$

prf $E_2^{(p,q),*}(z) = E_2^{(p,q)}(z) - \frac{1}{\mathrm{vol}} \frac{1}{\mathrm{Im}\,z}$.

take $z_0 \in S_\gamma$ and consider the int'l $\int_{z_0}^{\gamma z_0}$ along S_γ.

- $\int_{z_0}^{\gamma z_0} E_2\,dz = \frac{1}{2\pi i} \frac{1}{r} R_{p,q}(\gamma, z_0)$.

- $-\frac{1}{\mathrm{Vol}} \int_{z_0}^{\gamma z_0} \frac{dz}{\mathrm{Im}\,z} = \cdots$

  $= -\frac{pq}{2\pi r} \cdot (-2i \log j(\gamma, z_0))$.

$$\int_{z_0}^{\gamma z_0} E_2^{*}\,dz = \frac{1}{2\pi i} \frac{1}{r}(R - 2pq \log j(\gamma, z_0)) = \frac{\Psi}{r}$$

//

§4 $\widetilde{SL_2\mathbb{R}}$ and a 2-cocycle

the universal covering group $\widetilde{SL_2\mathbb{R}}$ of $SL_2\mathbb{R}$ may be defined by the following exact seq:

$$0 \to \underset{\substack{\text{ıı} \\ \pi_1 SL_2\mathbb{R}}}{\mathbb{Z}} \to \widetilde{SL_2\mathbb{R}} \to SL_2\mathbb{R} \to 1.$$

Thm an explicit formula with use of "sgn".

Def/Prop a 2-cocycle $W : (SL_2\mathbb{R}) \longrightarrow \{0, \pm 1\}$ is def'd by

$$W(\gamma_1, \gamma_2) = \frac{1}{2\pi i}\left(\log j(\gamma_1, \gamma_2 z) + \log j(\gamma_2, z) - \log j(\gamma_1\gamma_2, z)\right)$$

w/ $\operatorname{Im}\log \in [-\pi, \pi)$

Prop [Asai 1970]
$\widetilde{SL_2\mathbb{R}}$ is a central ext. of $SL_2\mathbb{R}$ by $\mathbb{Z}$ corresponding to W;

$\widetilde{SL_2\mathbb{R}} = SL_2\mathbb{R} \times \mathbb{Z}$ as a set and

$(\gamma_1, n_1)\cdot(\gamma_2, n_2) = (\gamma_1\gamma_2, n_1 + n_2 + W(\gamma_1, \gamma_2))$.

$0 \neq [W] \in H^2_{bdd}(SL_2\mathbb{R}; \mathbb{R}) \cong \mathbb{R}$.

the bounded cohomology

highly interesting!

$[W|_{\Gamma_{p,q}}] \in H^2(\Gamma_{p,q}; \mathbb{Z}) \cong \mathbb{Z}/2pq$

$H^1(\Gamma_{p,q}; \mathbb{Z}) = 0$

$\therefore \exists ! f : \Gamma_{p,q} \to \mathbb{Z}$ s.t.

$$2pq\, W(\gamma_1, \gamma_2) = \delta^1 f(\gamma_1, \gamma_2) = f(\gamma_1\gamma_2) - f(\gamma_1) - f(\gamma_2)$$

$$\boxed{\textbf{Thm} \quad f = \Psi_{p,q}}$$

§5 Additive char $\chi_{p,q} : \tilde{\Gamma}_{p,q} \to \mathbb{Z}$

$P : \widetilde{SL_2\mathbb{R}} \twoheadrightarrow SL_2\mathbb{R}$; $(\gamma, n) \mapsto \gamma$

$\tilde{\Gamma}_{p,q} := P^{-1}(\Gamma_{p,q}) < \widetilde{SL_2\mathbb{R}}$

for each $\gamma \in SL_2\mathbb{R}$,

write $\tilde{\gamma} := (\gamma, 0) \in \widetilde{SL_2\mathbb{R}} \text{ `` = '' } SL_2\mathbb{R} \times \mathbb{Z}$

$\tilde{\Gamma}_{p,q}$ is gen'd by $\tilde{S}_p$ and $\tilde{U}_q$.

(gp hom)

Define an additive char $\chi_{p,q} : \tilde{\Gamma}_{p,q} \to \mathbb{Z}$

by $\chi_{p,q}(\tilde{S}_p) = -q$, $\chi_{p,q}(\tilde{U}_q) = -p$.

and a function $V_{p,q} : \Gamma_{p,q} \to \mathbb{Z}$

($\neq$ gp hom) by $\gamma \longmapsto \chi_{p,q}(\tilde{\gamma})$.

we have $\chi_{p,q}(\gamma, n) = \chi_{p,q}(\tilde{\gamma} \cdot (I, 1)^n)$

$= V_{p,q}(\gamma) - 2npq$.

Lem $V_{p,q}(\gamma_1\gamma_2) = V_{p,q}(\gamma_1) + V_{p,q}(\gamma_2)$

$+ 2pq\, W(\gamma_1, \gamma_2)$.

Thm

- $\psi_{p,q} = V_{p,q}$
- $\psi_{p,q}(\gamma) = \chi_{p,q}(\gamma, n) + 2npq \quad \forall n \in \mathbb{Z}$

§6 "Original" Rademacher symbol

Define

$$\Psi_{p,q}(\gamma) := \psi_{p,q}(\gamma) + \frac{pq}{2} \operatorname{sgn}(\gamma)(1 - \operatorname{sgn}(\operatorname{tr}\gamma))$$

Then, we have

- $\Psi_{p,q}(-\gamma) = \Psi_{p,q}(\gamma)$
- $\Psi_{p,q}(\gamma^{-1}) = -\Psi_{p,q}(\gamma)$
- If $|\operatorname{tr}\gamma| > 2$, then $\Psi_{p,q}(\gamma^n) = n\Psi_{p,q}(\gamma)$.
- $\Psi_{p,q}(g^{-1}\gamma g) = \Psi_{p,q}(\gamma)$

Rem $\Psi_{2,3}$ = the original one in [Rademacher 1956].

§7 Modular knots for $\Gamma_{p,q}$

the unit tangent bundle

$$T_1(\Gamma_{p,q}\backslash \mathbb{H}) \cong \underset{\text{Lens space}}{L(r,p-1)} - \underset{\text{a knot}}{\overline{K_{p,q}}}$$

$$\cong \tilde{\Gamma}_{p,q}\backslash \widetilde{SL_2\mathbb{R}}$$

"the geodesic flow" is def'd by

$$\varphi^t : M \longmapsto M\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix} \quad t\in\mathbb{R}.$$

$$\{\gamma \in \Gamma_{p,q} ; \text{hyperbolic, primitive}\}$$
$$\longrightarrow \{\text{closed orbits } C_\gamma\}$$

Thm

$$\mathrm{lk}(C_\gamma, \overline{K_{p,q}}) = \frac{1}{r}\Psi_{p,q}(\gamma)$$

$$\in \frac{1}{r}\mathbb{Z}$$

$$\tilde{\Gamma}_{p,q} \overset{ab}{\twoheadrightarrow} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/r\mathbb{Z}$$

$$G_r := \ker$$

$$G_r\backslash \widetilde{SL_2\mathbb{R}} \cong S^3 - \underset{(p,q)\text{-torus knot}}{K_{p,q}}$$

: the $\mathbb{Z}/r\mathbb{Z}$-cover of $T_1(\Gamma_{p,q}\backslash\mathbb{H})$.

$$\{(\gamma, n) \in G_r \text{ "}{=}\text{" } \Gamma_{p,q}\times \mathbb{Z}/r\mathbb{Z}\}$$
$$\longrightarrow \{\text{closed orbits } C_{(\gamma,n)}\}$$

Thm

$$\mathrm{lk}(C_{(\gamma,n)}, K_{p,q}) = \frac{1}{\gcd(n,r)}\Psi_{p,q}(\gamma)$$

$$\in \mathbb{Z}$$

prf

we invoke

- automorphic forms on $\widetilde{SL_2\mathbb{R}}$ of rational degree [Milnor 1975]

- group theory [Tsanov 2013]

$$
\begin{tikzcd}
SL_2\mathbb{R} & \bullet \arrow[l] \arrow[d, dashed] & \widetilde{SL_2\mathbb{R}} \arrow[l, "Z(\Gamma')\cong\mathbb{Z}"', "Z(Gr)\cong\mathbb{Z}"] \arrow[d, "Gr'\cong\tilde{\Gamma}'"] \arrow[dl, "Gr"] \arrow[dll, "\tilde{\Gamma}"'] \\
L(r,s)-\overline{K_{p,q}} \arrow[from=u, "\Gamma_{p,q}"'] & S^3-K_{p,q} \arrow[l] & L_\infty = X_\infty \arrow[l, "H_r\cong\mathbb{Z}"'] \arrow[ll, bend left, "H\cong\mathbb{Z}"]
\end{tikzcd}
$$

$$W_\infty(\gamma, n) := \Delta^{1/r}_{p,q}(\gamma i)\left(\frac{e^{-4\pi i n}}{j(\gamma,i)^2}\right)^{\frac{pq}{r}}$$

$$
\begin{tikzcd}
Gr\backslash\widetilde{SL_2\mathbb{R}} \arrow[r, "W_\infty"] \arrow[d] & \mathbb{C}^\times \arrow[d, "z\mapsto z^r"] \\
\Gamma_{p,q}\backslash SL_2\mathbb{R} \arrow[r, "\tilde{\Delta}_{p,q}"] & \mathbb{C}^\times
\end{tikzcd}
\quad \circlearrowright
$$

$$\tilde{\Delta}_{p,q}(\gamma) = j(\gamma,i)^{-2pq}\Delta_{p,q}(\gamma, i)$$

taking $H_1(\ )$, we obtain

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[d] & 1 \arrow[d, mapsto] \\
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} & r
\end{tikzcd}
.
$$

for a knot $k\subset S^3\setminus K_{p,q}$,

winding number

$$\mathrm{lk}(k, K_{p,q}) = \mathrm{Ind}(W_\infty(k), 0)$$

$$= \frac{1}{2\pi i}\int_{W_\infty(k)}\frac{dz}{z}$$

$$= \cdots = \frac{1}{\gcd(n,r)}\Psi_{p,q}(\gamma).$$

Thm

cycle integral

Further problems

[Ghys 2007 ICM]

<u>Thm</u> "$lk = \Psi$" for $(p,q)=(2,3)$

✓ 1st proof

△ 2nd proof

Atiyah's result + characterization by the Euler cocycle

△ 3rd proof Lorenz attractor, template theory, explicit formula

$$\gamma \sim \pm S_2 U_3^{\varepsilon_1} S_2 U_3^{\varepsilon_2} \cdots S_2 U_3^{\varepsilon_n}$$

$\varepsilon_i \in \{\pm 1\}$ $\quad lk(C_\gamma, K_{2,3}) = \sum_i \varepsilon_i$

[Dehornoy 2015] [Dehornoy Pinsky 2018]

@ Atiyah's seven other def's

- Sarnak–Mozzochi (uses the Selberg trace formula)

Proposition [Sar08, Sar10, Moz13]. *Suppose that γ runs through primitive hyperbolic elements in $\mathrm{SL}_2\,\mathbb{Z}$. Then for any $-\infty \leqslant a \leqslant b \leqslant \infty$, we have*

$$\lim_{\nu\to\infty} \frac{\#\{\gamma \mid \ell(\gamma) < \nu,\ a \leqslant \dfrac{\Psi(\gamma)}{\ell(\gamma)} \leqslant b\}}{\#\{\gamma \mid \ell(\gamma) < \nu\}} = \frac{\mathrm{Tan}^{-1}(\frac{\pi b}{3}) - \mathrm{Tan}^{-1}(\frac{\pi a}{3})}{\pi}.$$

Galois flow?

$$X = \mathbb{C}^2 \setminus \{z^3 = w^2\}$$

$$1 \to \pi_1^{\bar{\mathbb{Q}}}(X) \to \pi_1^{\mathbb{Q}}(X) \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to 1$$

$\pi_1^{\bar{\mathbb{Q}}}(X) \cong \hat{\pi}_1(S^3 \setminus K)$ the profinite completion of the $\pi_1^{top}(X)$.

modular flow may be seen as a "Galois flow".

Can we parametrize $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by modular knots?

cf

$\mathbb{P}^1 \setminus \{0, 1, \infty\} \xrightarrow{\text{iterated integral}}$ mult. zeta values

$\updownarrow$

$\otimes \mathbb{Q}_\ell$: $H^*_{\text{ét}}(X, \mathbb{Q}_\ell) \curvearrowleft \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

Modular knots

$\int_\gamma$ [illegible] dz

$S^3 \setminus K \to$ Rademacher symbol $\Psi(\gamma)$

$\|$

$lk(C_\gamma, K)$

C_γ: closed orbits of Gal ?

quantum analogue ?

cf Pappas 2020 arXiv   p-adic hyp vol

Chern Simons inv