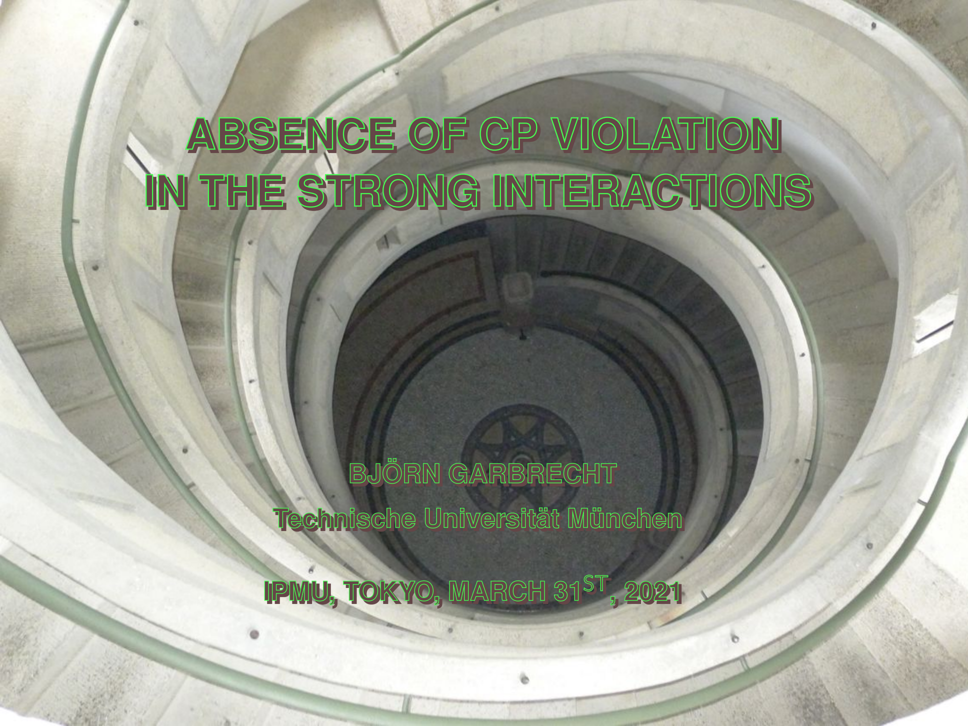

ABSENCE OF CP VIOLATION IN THE STRONG INTERACTIONS

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Outline

I. CP-odd terms in Yang–Mills theory with fermions
Specify the theory and the goal of the calculation: correlation functions for fermions in instanton backgrounds and their chiral CP phases
Boundary conditions for the path integral

II. Green's functions for fermions
Euclidean Green's function in one-instanton background

III. Continuation to Minkowski spacetime
Continuation and approximation for many instantons at fixed locations

IV. Interferences within the topological sectors
Integration over collective coordinates, e.g. instanton locations, leads to correlation functions in a fixed topological sector

V. Interferences among different topological sectors (are immaterial)
Taking the infinite-volume limit before summing over the topological sectors, there is alignment of the chiral CP phases in the fermion sector

VI. Finite (sub)volumes
Cluster decomposition and periodic boundary conditions

I. CP-odd terms in Yang–Mills theory with fermions

Consider SU(2) Yang–Mills with $N_f = 1$ massive fermions
(can be generalized to SU(N), $N_f > 1$):

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\mathrm{i}\not{D} - m\mathrm{e}^{\mathrm{i}\alpha\gamma^5})\psi + \underbrace{\frac{1}{16\pi^2}\theta\,\mathrm{tr}F_{\mu\nu}\tilde{F}^{\mu\nu}}_{\text{toplogical term}}$$

$$D_\mu = \partial_\mu - \mathrm{i}A^a_\mu\frac{\tau^a}{2} = \partial_\mu + A_\mu \qquad [D_\mu, D_\nu] = -\mathrm{i}F_{\mu\nu}$$

$$F_{\mu\nu} = F^a_{\mu\nu}\frac{\tau^a}{2} \qquad \tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

CP-odd terms:

- Topological term (θ-term where θ is an angle)
- $\bar{\psi}\mathrm{i}\gamma^5 m\sin\alpha\,\psi$

These depend on field redefinitions but there are rephasing invariant combinations.

Toplogical term

Theta term is a total divergence

$$\frac{1}{4}\mathrm{tr}F_{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu K_\mu \qquad K_\mu = \epsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{1}{2}A_\nu\partial_\alpha A_\beta + \frac{1}{3}A_\nu A_\alpha A_\beta\right]$$

$\rightarrow$ Equivalent to a surface term, i.e. the flux of the current through the boundary of the integration volume

So does it vanish?

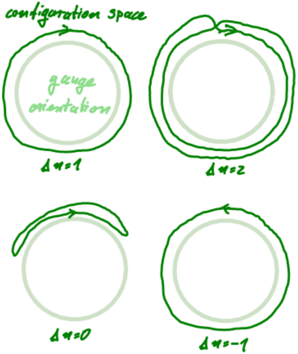
Cf. anti-instanton: $A_\mu{}^u{}_v = -\frac{\sigma_{\mu\nu}{}^u{}_v x_\nu}{x^2+\rho^2}$ (extended solution to Euclidean EOMs) [Belavin, Polyakov, Schwarz, Tyupkin (1975)]

Surface term decays as $1/|x|^3 \rightarrow$ surface integral does not need to vanish

For $x^2 \to \infty$, the field becomes a pure gauge:

$$A_\mu \to -\frac{\mathrm{i}}{g}(\partial_\mu \Omega)\Omega^{-1} \quad \text{where} \quad \Omega \in \mathrm{SU}(2)$$

$$K_\mu \to \frac{1}{6}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}[(\Omega^{-1}\partial_\nu\Omega)(\Omega^{-1}\partial_\lambda\Omega)(\Omega^{-1}\partial_\rho\Omega)]$$



Winding number

$$\Delta n = \frac{1}{16\pi^2}\int \mathrm{d}^4x F_{\mu\nu}\tilde{F}_{\mu\nu} = \frac{1}{4\pi^2}\oint_{S^3} \mathrm{d}^3\sigma K_\perp$$

Integrand is a Haar measure and maps $S^3 \to S^3$

(Anti-)instanton is a configuration with winding number $\Delta n = (-)1$

Theta term contributes to the action though being a total derivative

Boundary conditions on the path integral

The parameter θ can be viewed as an angular variable.
(Also forced by the anomalous chiral current) $\longrightarrow$
Requires $\Delta n \in \mathbb{Z}$ ("topological quantization")

This is readily built into the theory:

At **infinity** vanishing physical fields are the only boundary conditions leading to **saddle points with finite Euclidean action** ($\equiv$ solutions to the EOMs). [*cf.* Coleman (1985)]

Indeed, for pure gauge configurations $\to \Delta n \in \mathbb{Z}$ (as discussed above), i.e. these boundary conditions at infinity imply topological quantization.

The action & path integral are a first principle-definition of the theory. Nonetheless, in this context, it is also interesting to look at vacuum states defined through field functionals in canonical quantization.
$\longrightarrow$ θ-vacua

Theta vacuum [Callan, Dashen, Gross (1976); Jackiw, Rebbi (1976); Jackiw (1980)]

Now consider initial and final states, taking $x_4 \to \pm\infty$
$\to$ Construct from pure gauge configurations on these surfaces, with

$$\Delta n = \frac{1}{16\pi^2} \int d^4x F_{\mu\nu}\tilde{F}_{\mu\nu} = n_\infty - n_{-\infty} \quad \text{gauge invariant}$$

$$n_{\pm\infty} = \frac{1}{4\pi^2} \int_{x^4=\pm\infty} d^3\sigma K_\perp \quad \text{Chern–Simons number, **not** gauge invariant}$$

Gauge transformations Ω change $n_{\pm\infty}$ by same number of integer units

Boundary conditions fixed by prevacua: $\begin{array}{r} n_{-\infty} \to |n\rangle \\ n_\infty \to \langle n| \end{array}$ (field eigenstates)

Gauge invariant (up to phase) state $|\text{vac}\rangle = e^{-HT} \sum_n e^{in\theta}|n\rangle$, $T \to \infty$

Alternatively, set $|\text{vac}\rangle = e^{-HT} \sum_n |n\rangle$ and absorb θ in topological term

Consequence: In the path integral, sum over all topological sectors Δn, weigh these by $\exp(i\Delta n\theta)$

Boundary conditions on finite surfaces?

Fixing boundary conditions up to gauge transformations on a finite three-surface also leads to distinct homotopy classes

However: Fixed boundary conditions (up to gauge) at a finite surface are not physical:

$|\mathrm{vac}\rangle$ can be represented by a wave functional $\Psi(\vec{A}^a)$ [Jackiw (1980)] $\longrightarrow$

Restriction to **finite** surfaces requires **integration over boundary conditions** obtained either from

(i) weighting field eigenstates with $\Psi(\vec{A}^a)$ or

(ii) matching to path integral in the outside volume

Problems with (i): wave functional unknown & in general no topological quantization (no integer Δn)

Option (ii): To be discussed here in the context of cluster decomposition

Way out: Construct $|\mathrm{vac}\rangle = \mathrm{e}^{-HT} \sum_n \mathrm{e}^{\mathrm{i}n\theta}|n\rangle$, $T \to \infty$ in terms of field eigenstates $|n\rangle$ or use path integral with **boundary conditions fixed at infinity** in first place

When requiring physical observables (and manifestly $\Delta n \in \mathbb{Z}$)

$\longrightarrow$ Must impose vanishing physical boundary conditions at infinity

Fermion mass & topological term $\leftrightarrow$ correlations & effective operators

CP-odd Lagrangian terms:

$$\mathcal{L} \supset -\sum_{j=1}^{N_f} \bar{\psi}_j m_j \mathrm{e}^{\mathrm{i}\alpha_j\gamma^5}\psi_j + \frac{1}{16\pi^2}\theta\,\mathrm{tr}F_{\mu\nu}\tilde{F}^{\mu\nu}$$

Chiral symmetry of the fermions is anomalous $\longrightarrow$
Rephasing invariant: $\bar{\theta} = \theta + \bar{\alpha}$, where $\bar{\alpha} = \sum_{j=1}^{N_f}\alpha_j$ [Fujikawa (1979,80)]

Instanton effects described by effective 't Hooft vertex (main subject of this talk): ['t Hooft (1976,86)]

$$\mathcal{L} + \frac{1}{16\pi^2}\theta\,\mathrm{tr}F_{\mu\nu}\tilde{F}^{\mu\nu} \to \mathcal{L} - \Gamma_{N_f}\mathrm{e}^{\mathrm{i}\xi}\prod_{j=1}^{N_f}(\bar{\psi}_j P_\mathrm{L}\psi_j) - \Gamma_{N_f}\mathrm{e}^{-\mathrm{i}\xi}\prod_{j=1}^{N_f}(\bar{\psi}_j P_\mathrm{R}\psi_j)$$

(Γ_{N_f} some coefficient)

$\xi = \theta$ (in general misaligned with masses) $\rightarrow CP$ violation
$\xi = -\bar{\alpha}$ (present claim, aligned with mass terms) $\rightarrow$ no CP violation

Note: Both comply with $\bar{\theta}$ being the only rephasing invariant CP phase

The effective vertex is chosen such that it generates the following **correlation functions** at tree level:

$$\langle\prod_{j=1}^{N_f}\psi_j(x_j)\bar{\psi}_j(x_j')\rangle_{\rm inst} = \left(\mathrm{e}^{-\mathrm{i}\xi}\prod_{j=1}^{N_f}P_{\mathrm{L}j}+\mathrm{e}^{\mathrm{i}\xi}\prod_{j=1}^{N_f}P_{\mathrm{R}j}\right)\bar{H}(x_1,\ldots,x_1',\ldots)$$

Cf. leading contribution to two-point function

$$\langle\psi_i(x)\psi_j(x')\rangle = \mathrm{i}S_{0\mathrm{inst}\,ij}(x,x')$$

$$\mathrm{i}S_{0\mathrm{inst}\,ij}(x,x') = (-\gamma^\mu\partial_\mu+\mathrm{i}m_i\mathrm{e}^{-\mathrm{i}\alpha_i\gamma^5})\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p(x-x')}\frac{\delta_{ij}}{p^2-m_i^2+\mathrm{i}\epsilon}$$

Again: $\xi=\theta/\xi=-\bar{\alpha}$ implies CP-violation/no CP-violation

Take $N_f=1$ from here onwards

Recall: Obtain correlation functions from Green's functions in instanton background, then interfere all instanton configurations. First, within one topological sector, then over the different sectors.

II. Green's functions for fermions

Euclidean Green's function $S^{\mathrm{E}}(x^{\mathrm{E}}, x^{\mathrm{E}\prime})$ satisfies

$$(\not{D}^{\mathrm{E}} + m_{\mathrm{R}} + \mathrm{i}\gamma^5 m_{\mathrm{I}}) S^{\mathrm{E}}(x^{\mathrm{E}}, x^{\mathrm{E}\prime}) = \delta^4(x^{\mathrm{E}} - x^{\mathrm{E}\prime})$$

Spectral sum (first massless case):

$$\not{D}^{\mathrm{E}} \hat{\psi}^{\mathrm{E}}_{\lambda} = \left(\not{\partial}^{\mathrm{E}} + \gamma^{\mathrm{E}}_m A^{\mathrm{E}}_m\right) \hat{\psi}^{\mathrm{E}}_{\lambda} = \lambda^{\mathrm{E}} \hat{\psi}^{\mathrm{E}}_{\lambda}$$

$$\longrightarrow \quad S^{\mathrm{E}}(x^{\mathrm{E}}, x^{\mathrm{E}\prime}) = \sum\!\!\!\!\!\!\!\int_{\lambda^{\mathrm{E}}} \frac{\hat{\psi}^{\mathrm{E}}_{\lambda}(x^{\mathrm{E}}) \hat{\psi}^{\mathrm{E}\dagger}_{\lambda}(x^{\mathrm{E}\prime})}{\lambda^{\mathrm{E}}}$$

Spectral sum for $m = 0$ is ill-defined because of the fermionic zero mode $\lambda^{\mathrm{E}} = 0$ in the instanton background

Winding number: $\Delta n = \dfrac{1}{16\pi^2}\displaystyle\int \mathrm{d}^4x F_{\mu\nu}\tilde{F}_{\mu\nu}$

Euclidean index theorem: Δn equals difference between number of right-handed and left-handed zero modes

$\rightarrow$ One left (right)-handed zero-mode for $\Delta n = -1$ ($\Delta n = 1$)

Left-handed zero mode ['t Hooft (1976)]

$$\hat{\psi}^{\mathrm{E}}_{0\mathrm{L}}(x^{\mathrm{E}}) = \begin{pmatrix} \chi^{\mathrm{E}}_0(x^{\mathrm{E}}) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}, \quad \text{where} \quad \chi^{\mathrm{E}}_0(x^{\mathrm{E}}) = \frac{\varrho u}{\pi\left[\varrho^2 + (x^{\mathrm{E}})^2\right]^{\frac{3}{2}}}, \quad u^{\alpha b} = \varepsilon^{\alpha b}$$

Include mass @ first order in perturbation theory ($\Delta n = -1$ background) [Shifman, Vainshtein, Zakharov (1979)]

$$S^{\mathrm{E}}(x^{\mathrm{E}}, x^{\mathrm{E}\prime}) = \frac{\hat{\psi}^{\mathrm{E}}_0(x^{\mathrm{E}})\hat{\psi}^{\mathrm{E}\dagger}_0(x^{\mathrm{E}\prime})}{m\mathrm{e}^{-\mathrm{i}\alpha}} + \sum_{\lambda^{\mathrm{E}}\neq 0} \frac{\hat{\psi}^{\mathrm{E}}_\lambda(x^{\mathrm{E}})\hat{\psi}^{\mathrm{E}\dagger}_\lambda(x^{\mathrm{E}\prime})}{\lambda^{\mathrm{E}}}$$

For $\alpha \neq 0$ and arbitrary m, can use linear combinations of $\hat{\psi}_\lambda^{\mathrm{E}}$ and $\gamma^5 \hat{\psi}_\lambda^{\mathrm{E}}$ as solutions to the eigenvalue problem

Eigenvalues are then given by

$$\xi_\pm^{\mathrm{E}}(\lambda^{\mathrm{E}}) = m_{\mathrm{R}} \pm \sqrt{(\lambda^{\mathrm{E}})^2 - m_{\mathrm{I}}^2}$$

$\rightarrow$ No perturbative approximation needed if full massless spectrum is known

III. Continuation to Minkowski spacetime

Analytic continuation:

$$x_4 \to \mathrm{e}^{-\mathrm{i}(\vartheta - \frac{\pi}{2})} t$$

$t \in \mathbb{R}$, $\vartheta = \pi/2$: Euclidean metric, $\vartheta = 0^+$: Minkowski metric

Continuation of Dirac operator:

$$\begin{aligned} \not{D}^{\mathrm{E}} =& (\not{\partial}^{\mathrm{E}}_m + \gamma^{\mathrm{E}}_m A^{\mathrm{E}}_m) \\ &\to \left(-\mathrm{i}\frac{\partial}{\partial x^0}\gamma^{\mathrm{E}}_4 + \vec{\gamma}^{\mathrm{E}} \cdot \nabla + \gamma^{\mathrm{E}}_4 A^{\mathrm{E}}_4(\vec{x}, x_4 = \mathrm{i}x^0) + \vec{\gamma}^{\mathrm{E}} \cdot \vec{A}^{\mathrm{E}}(\vec{x}, x_4 = \mathrm{i}x^0) \right) \\ &= -\mathrm{i}\left(\frac{\partial}{\partial x^0}\gamma^0 + \vec{\gamma} \cdot \nabla + \gamma^0 A_0(x^0, \vec{x}) + \vec{\gamma} \cdot \vec{A}(x^0, \vec{x}) \right) = -\mathrm{i}\not{D}\,, \end{aligned}$$

where $\vec{\gamma} \cdot \nabla \equiv \sum_i \gamma^i \partial_i$ and accordingly for $\vec{\gamma} \cdot \vec{A}$, $\gamma^0 = \gamma^{\mathrm{E}}_4$ and $\gamma^i = \mathrm{i}\gamma^{\mathrm{E}}_i$ for $i = 1, 2, 3$

Green's functions, as they are inverse Dirac operators, transform straightforwardly.

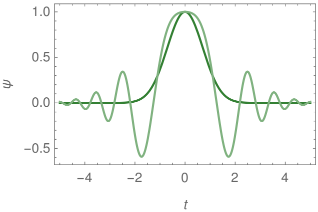
Continuation of the eigensystem

Issues of spectral representation in Minkowski spacetime:

- The operator $\mathrm{i}\not{D}\gamma^0$ in Minkowski spacetime is not of definite Hermiticity because of the complex gauge-field configuration in the analytically continued soliton background.
- The inner product $\int \mathrm{d}^4x\, \bar{\psi}_\xi(x)\psi_{\xi'}(x)$ is not positive definite.
- Zero-modes in anti-instanton (instanton) background are purely left (right)-handed. An operator breaking chiral symmetry—such as the effective instanton vertex—mixes left-and right chiral degrees of freedom. How does this play out in Minkowski spacetime?

Determine continuation by behaviour in asymptotic, homogeneous spacetime region where the solutions go to either damped or oscillatory exponentials [Ai, BG, Tamarit (2019)]

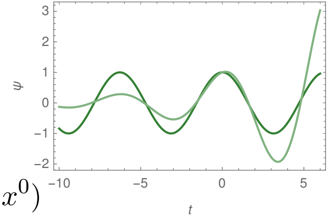
Discrete modes—straightforward continuation as modes remain properly normalizable for $0 < \vartheta \leq \pi/2$:



$$\psi_n^\vartheta(x) = \psi_n^\vartheta(x^0, \vec{x}) = \sqrt{\mathrm{i}e^{-\mathrm{i}\vartheta}}\,\psi_n^{\mathrm{E}}(\vec{x}, x_4 = \mathrm{i}e^{-\mathrm{i}\vartheta}x^0)$$

$$\xi_n^\vartheta = -\,\xi_n^{\mathrm{E}}$$

Continuum modes—continue time **and** asymptotic k^0 (avoid blowup on one side, use asymptotic plane waves to label eigensystem)



$$\psi_{\{k\}}^\vartheta(x) = \psi_{\{k^0,\vec{k}\}}^\vartheta(x^0, \vec{x}) = \psi_{\{\vec{k}, -\mathrm{i}e^{\mathrm{i}\vartheta}k_0\}}^{\mathrm{E}}(\vec{x}, x_4 = \mathrm{i}e^{-\mathrm{i}\vartheta}x^0)$$

$$\xi_{\{k^0,\vec{k}\}}^\vartheta = -\,\xi_{\{\vec{k}, -\mathrm{i}e^{\mathrm{i}\vartheta}k_0\}}^{\mathrm{E}}$$

This eigensystem is orthonormal with respect to the following inner product:

$$(\psi^\vartheta_\xi, \psi^\vartheta_{\xi'})_\vartheta = \int \mathrm{d}^4x\, \tilde{\psi}^\vartheta_\xi(x)\, \psi^\vartheta_{\xi'}(x)$$

$$\tilde{\psi}^\vartheta_n(x^0,\vec{x}) = \sqrt{\mathrm{i}\mathrm{e}^{-\mathrm{i}\vartheta}}\,(\psi^\mathrm{E}_n(\vec{x},x_4))^\dagger\Big|_{x_4=\mathrm{i}\mathrm{e}^{-\mathrm{i}\vartheta}x^0} = \mathrm{i}\mathrm{e}^{-\mathrm{i}\vartheta}(\psi^\vartheta_n(x^0,\vec{x}))^\dagger\Big|_{x^0\to -\mathrm{e}^{-2\mathrm{i}\vartheta}x^0}$$

$$\tilde{\psi}^\vartheta_{\{k^0,\vec{k}\}}(x^0,\vec{x}) = \Big(\psi^\mathrm{E}_{\{\vec{k},k_4\}}(\vec{x},x_4)\Big)^\dagger\Bigg|_{\substack{x_4=\mathrm{i}\mathrm{e}^{-\mathrm{i}\vartheta}x^0\\ k_4=-\mathrm{i}\mathrm{e}^{\mathrm{i}\vartheta}k^0}} = \psi^\vartheta_{\{k^0,\vec{k}\}}(x^0,\vec{x})^\dagger\Bigg|_{\substack{x^0\to -\mathrm{e}^{-2\mathrm{i}\vartheta}x^0\\ k^0\to -\mathrm{e}^{2\mathrm{i}\vartheta}k^0}}$$

Spectral representation in Minkowski spacetime

$$S^\vartheta(x,x') \equiv (\mathrm{i}\not{D}^\vartheta - m\mathrm{e}^{\mathrm{i}\alpha\gamma_5})^{-1}(x,x') = \sum\!\!\!\!\!\!\!\int_{\xi^\vartheta} \frac{1}{\xi^\vartheta}\,\psi^\vartheta_\xi(x)\tilde{\psi}^\vartheta_\xi(x')$$

$$= \sum_n \frac{1}{\xi^\vartheta_n}\,\psi^\vartheta_n(x)\tilde{\psi}^\vartheta_n(x') + \int \mathrm{d}^4k\,\frac{1}{\xi^\vartheta_{\{k\}}}\psi^\vartheta_{\{k\}}(x)\tilde{\psi}^\vartheta_{\{k\}}(x')$$

Green's function in Minkowski spacetime

Application to zero mode in the $\eta = -1$ background gives

$$\psi_{0\mathrm{L}}(x^0, \vec{x}) \equiv \sqrt{\mathrm{i}}\,\varphi_{0\mathrm{L}}(x^0, \vec{x}) = \sqrt{\mathrm{i}}\,\psi^{\mathrm{E}}_{0\mathrm{L}}(\vec{x}, \mathrm{i}x^0)$$

where

$$\varphi_{0\mathrm{L}}(x) = \begin{pmatrix} \chi_0(x) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}, \qquad \chi_0(x) = \frac{\varrho u}{\pi(\varrho^2 - x^2)^{\frac{3}{2}}}$$

Add contributions from far from the instanton [cf. Diakonov, Petrov (1986)]

$$\begin{aligned} \mathrm{i}S(x,x') &= \mathrm{i}S_{\mathrm{cont}}(x,x') + \frac{\varphi_{0\mathrm{L}}(x-x_0)\,\varphi^{\dagger}_{0\mathrm{L}}(x'-x_0)}{m\mathrm{e}^{-\mathrm{i}\alpha}} \\ &\approx \mathrm{i}S_{0\mathrm{inst}}(x,x') + \frac{\varphi_{0\mathrm{L}}(x-x_0)\,\varphi^{\dagger}_{0\mathrm{L}}(x'-x_0)}{m\mathrm{e}^{-\mathrm{i}\alpha}} \end{aligned}$$

$$\mathrm{i}S_{0\mathrm{inst}}(x,x') = (-\gamma^\mu\partial_\mu + \mathrm{i}m\mathrm{e}^{-\mathrm{i}\alpha\gamma^5}) \int \frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p(x-x')}\frac{1}{p^2 - m^2 + \mathrm{i}\epsilon}$$

Green's function in n-instanton, $\bar{n}$-anti-instanton background

$$\mathrm{i}S_{n,\bar{n}}(x,x') \approx \mathrm{i}S_{0\mathrm{inst}}(x,x') + \sum_{\bar{\nu}=1}^{\bar{n}} \frac{\varphi_{0\mathrm{L}}(x-x_{0,\bar{\nu}})\varphi_{0\mathrm{L}}^{\dagger}(x'-x_{0,\bar{\nu}})}{m\mathrm{e}^{-\mathrm{i}\alpha}} + \sum_{\nu=1}^{n} \frac{\varphi_{0\mathrm{R}}(x-x_{0,\nu})\varphi_{0\mathrm{R}}^{\dagger}(x'-x_{0,\nu})}{m\mathrm{e}^{\mathrm{i}\alpha}}$$

Comments:

- For small masses, zero-modes dominate close to the cores of the instantons, far away from the instantons the solution goes to the zero-instanton configuration [Diakonov, Petrov (1986)]
- Analytic continuation from Euclidean to Minkowski space verifies chiral block-structure [Ai, Cruz, BG, Tamarit (2020)]
- **Alignment** of phase α between Lagrangian mass and instanton-induced χSB $\longrightarrow$ No indication of CP-violation here
- Perhaps expected—θ-phase has not entered calculation thus far

IV. Interferences within the topological sectors

Within a topological sector, interfere/sum/integrate over

- all instanton/anti-instanton numbers $n + \bar{n}$ with $\Delta n = n - \bar{n}$ fixed
- locations of all instantons/anti-instantons
- remaining collective coordinates

$\longrightarrow$ Dilute instanton gas approximation

Choose θ-vacuum in Minkowski spacetime as $|\text{vac}\rangle = \sum_{n_{\text{CS}}} |n_{\text{CS}}\rangle$

Absorb CP-odd phase in topological term/fermion mass

Evaluate correlation and partition function first for **fixed** Δn

$$
\begin{aligned}
&\langle \psi(x)\bar{\psi}(x')\rangle_{\Delta n} \\
&= \sum_m {}_{\text{out}}\langle m+\Delta n|\psi(x)\bar{\psi}(x')|m\rangle_{\text{in}} = \sum_{\substack{\bar{n},n\geq 0\\ n-\bar{n}=\Delta n}} \int \mathcal{D}A_{\bar{n},n}\mathcal{D}\bar{\psi}\mathcal{D}\psi\, \psi(x)\bar{\psi}(x')\mathrm{e}^{\mathrm{i}S_{\bar{n},n}} \\
&= \sum_{\substack{\bar{n},n\geq 0\\ n-\bar{n}=\Delta n}} \frac{1}{\bar{n}!n!} \left(\prod_{\bar{\nu}=1}^{\bar{n}} \int_{VT} \mathrm{d}^4x_{0,\bar{\nu}}\mathrm{d}\Omega_{\bar{\nu}} J_{\bar{\nu}}\right) \left(\prod_{\nu=1}^{n} \int_{VT} \mathrm{d}^4x_{0,\nu}\mathrm{d}\Omega_{\nu} J_{\nu}\right) \mathrm{i}S(x,x') \\
&\qquad \times\ \mathrm{e}^{-S_{\text{E}}(\bar{n}+n)}\mathrm{e}^{-\mathrm{i}(\bar{n}-n)(\alpha+\theta)}(-\Theta\varpi)^{(\bar{n}+n)}
\end{aligned}
$$

$\mathrm{d}\Omega_\nu J_\nu$:	Zero modes & pertaining Jacobians
Θ,ϖ :	Reduced fermion & gauge/ghost determinants in instanton background
$\mathrm{i}S(x,x')$:	Fermion propagator in n instantons, $\bar{n}$ anti-instanton background
$S_{\bar{n},n}$:	Action for saddle with n instantons, $\bar{n}$ anti-instantons
S_{E}:	Euclidean action for one (anti-)instanton

Likewise, partition function:

$$Z_{\Delta n} = \sum_m {}_{\mathrm{out}}\langle m + \Delta n | m \rangle_{\mathrm{in}} = \sum_{\substack{\bar{n}, n \geq 0 \\ n - \bar{n} = \Delta n}} \int \mathcal{D}A_{\bar{n},n} \mathcal{D}\bar{\psi} \mathcal{D}\psi \, \mathrm{e}^{\mathrm{i}S_{\bar{n},n}}$$

$$= \sum_{\substack{\bar{n}, n \geq 0 \\ n - \bar{n} = \Delta n}} \frac{1}{\bar{n}! n!} \left(-\textstyle\int \mathrm{d}\Omega \, J \, V T \, \Theta \, \varpi \, \mathrm{e}^{-S_{\mathrm{E}}} \right)^{(\bar{n}+n)} \mathrm{e}^{-\mathrm{i}(\bar{n}-n)(\alpha+\theta)}$$

Integrate out locations of the instanton

$$
\begin{aligned}
&\int\limits_{VT} \mathrm{d}^4 x_{0,\bar\nu}\; \mathrm{i}S(x,x') \\
&\approx \int\limits_{VT} \mathrm{d}^4 x_{0,\bar\nu} \left[\mathrm{i}S_{0\mathrm{inst}}(x,x') + \frac{\varphi_{0\mathrm{L}}(x-x_{0,\bar\nu})\varphi^\dagger_{0\mathrm{L}}(x'-x_{0,\bar\nu})}{m\mathrm{e}^{-\mathrm{i}\alpha}} + \cdots \right] \\
&= VT\left(\mathrm{i}S_{0\mathrm{inst}}(x,x') + \cdots\right) + m^{-1}\mathrm{e}^{\mathrm{i}\alpha} h(x,x') P_\mathrm{L}
\end{aligned}
$$

Dots represent contributions from the zero modes of the (anti)-instantons whose centres were not integrated over

$h(x,x')$ is defined as a block-diagonal matrix (with two identical blocks):

$$
h(x,x')P_\mathrm{L} = \int\limits_{VT} \mathrm{d}^4 x_{0,\bar\nu}\, \varphi_{0\mathrm{L}}(x-x_{0,\bar\nu})\varphi^\dagger_{0\mathrm{L}}(x'-x_{0,\bar\nu})
$$

$$
h(x,x')P_\mathrm{R} = \int\limits_{VT} \mathrm{d}^4 x_{0,\bar\nu}\, \varphi_{0\mathrm{R}}(x-x_{0,\bar\nu})\varphi^\dagger_{0\mathrm{R}}(x'-x_{0,\bar\nu})
$$

$$
\bar h(x,x') \equiv \frac{\int d\Omega\, h(x,x')}{\int d\Omega}
$$

Integrating over all locations $\longrightarrow$ Correlation function for fixed Δn:

$$
\begin{aligned}
&\langle\psi(x)\bar\psi(x')\rangle_{\Delta n}\\
=&\sum_{\substack{\bar n,n\geq 0\\ n-\bar n=\Delta n}}\frac{1}{\bar n! n!}\left[\bar h(x,x')\left(\frac{\bar n}{m\mathrm{e}^{-\mathrm{i}\alpha}}P_{\mathrm{L}}+\frac{n}{m\mathrm{e}^{\mathrm{i}\alpha}}P_{\mathrm{R}}\right)(VT)^{\bar n+n-1}+\mathrm{i}S_{0\mathrm{inst}}(x,x')\,(VT)^{\bar n+n}\right]\\
&\qquad\times(\mathrm{i}\kappa)^{\bar n+n}(-1)^{n+\bar n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}\\
=&\left[\left(\mathrm{e}^{\mathrm{i}\alpha}I_{\Delta n+1}(2\mathrm{i}\kappa VT)P_{\mathrm{L}}+\mathrm{e}^{-\mathrm{i}\alpha}I_{\Delta n-1}(2\mathrm{i}\kappa VT)P_{\mathrm{R}}\right)\frac{\mathrm{i}\kappa}{m}\,\bar h(x,x')+I_{\Delta n}(2\mathrm{i}\kappa VT)\mathrm{i}S_{0\mathrm{inst}}(x,x')\right]\\
&\qquad\times(-1)^{\Delta n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}
\end{aligned}
$$

where $\mathrm{i}\kappa=\int\mathrm{d}\Omega\,J\,\Theta\,\varpi\,\mathrm{e}^{-S_{\mathrm{E}}}$, $I_\nu(x)$ is the modified Bessel function and $\bar h$ the spacetime-averaged correlation

Sum is dominated by particular value of $n\approx\bar n$: [Diakonov, Petrov (1986)]

$$
\langle n\rangle=\frac{\sum_{n=0}^{\infty}n\frac{(\alpha VT)^n}{n!}}{\sum_{n=0}^{\infty}\frac{(\alpha VT)^n}{n!}}=\alpha VT\,,\qquad\frac{\sqrt{\langle(n-\langle n\rangle)^2\rangle}}{\langle n\rangle}=\frac{1}{\sqrt{\alpha VT}}
$$

Cf. $\lim_{x\to\infty}I_{\Delta n}(\mathrm{i}x\,\mathrm{e}^{-\mathrm{i}0^+})/I_{\Delta n'}(\mathrm{i}x\,\mathrm{e}^{-\mathrm{i}0^+})=1$

$\longrightarrow$ No relative CP phase between mass and instanton induced breaking of χral symmetry—**alignment** in infinite-volume limit

Correspondingly, partition function for fixed Δn: [cf. Leutwyler, Smilga (1992)]

$$Z_{\Delta n} = I_{\Delta n}(2\mathrm{i}\kappa VT)\,(-1)^{\Delta n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

Note: The topological phase $\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$ multiplies $\langle\psi(x)\bar{\psi}(x')\rangle_{\Delta n}$ and $Z_{\Delta n}$ entirely—not just the contributions induced by instantons.

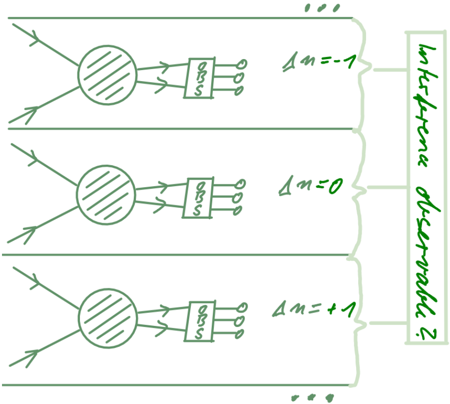
Other correlation functions (n point, stress-energy,...) are calculated from the Feynman diagram with the Green's function in the n instanton, $\bar{n}$ anti-instanton background.
Then it remains to average over n, $\bar{n}$, locations and remaining collective coordinates.

There is no CP violation/misalignment of phases to this end. It remains to consider the interference between the topological sectors.

Note: The sectors are separated by barrieres of infinite action.

Can interference between topological sectors be observed?

- Effective action well-defined for each sector separately—no indication that perturbative expansion requires/prefers summation over the Δn
- Possible to observe the interference between the topological sectors of different Δn? **Superobserver?**



Topological phases $e^{i\Delta n(\alpha+\theta)}$ appear globally for each topological sector. It is not clear how an observer made up of local quantum fields can access separate sectors.

Resolution: Turns out interferences are immaterial in the limit $VT \to \infty$

V. Interferences among topological sectors (are immaterial)

Partition function in θ-vacuum (recall phase resides in topological term):

$$Z =_{\text{out}}\langle \text{vac}|\text{vac}\rangle_{\text{in}} = \sum_{m,n} {}_{\text{out}}\langle m|n\rangle_{\text{in}} = \sum_{\Delta n=-\infty}^{\infty}\sum_{m} {}_{\text{out}}\langle m+\Delta n|m\rangle_{\text{in}} = \sum_{\Delta n=-\infty}^{\infty} Z_{\Delta n}$$

Fermion correlator

$$\langle\psi(x)\bar\psi(x')\rangle = \lim_{\substack{N\to\infty\\ N\in\mathbb{N}}}\lim_{VT\to\infty}\frac{\sum_{\Delta n=-N}^{N}\langle\psi(x)\bar\psi(x')\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}}$$

$$= \mathrm{i}S_{0\text{inst}}(x,x') + \mathrm{i}\kappa\bar h(x,x')m^{-1}\mathrm{e}^{-\mathrm{i}\alpha\gamma^5} \quad \text{(same as for fixed } \Delta n\text{)}$$

Recall: $\mathrm{i}S_{0\text{inst}}(x,x') = (-\gamma^\mu\partial_\mu + \mathrm{i}m\mathrm{e}^{-\mathrm{i}\alpha\gamma^5})\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p(x-x')}\frac{1}{p^2-m^2+\mathrm{i}\epsilon}$

$\longrightarrow$ No relative CP-phase between mass and instanton term

Can also obtain this for coincident correlators without dilute-gas approximation, using the index theorem

Limits ordered the other way around

First sum over all Δn as well:

$$\sum_{\bar{n},n\geq 0}\frac{1}{\bar{n}!n!}\Big[\bar{h}(x,x')(\bar{n}\,m^{-1}\mathrm{e}^{\mathrm{i}\alpha}P_{\mathrm{L}}+n\,m^{-1}\mathrm{e}^{-\mathrm{i}\alpha}P_{\mathrm{R}})\,(VT)^{\bar{n}+n-1}+\mathrm{i}S_{0\mathrm{inst}}(x,x')\,(VT)^{\bar{n}+n}\Big]$$
$$\times\,(-mi\kappa)^{\bar{n}+n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$
$$=\left[-\left(\mathrm{e}^{-\mathrm{i}\theta}P_{\mathrm{L}}+\mathrm{e}^{\mathrm{i}\theta}P_{\mathrm{R}}\right)\frac{\mathrm{i}\kappa}{m}\bar{h}(x,x')+\mathrm{i}S_{0\mathrm{inst}}(x,x')\right]\mathrm{e}^{-2\mathrm{i}\kappa VT\cos(\alpha+\theta)}$$

$$Z\to\sum_{n,\bar{n}}\frac{1}{n!\bar{n}!}(-\mathrm{i}\kappa VT)^{\bar{n}+n}\mathrm{e}^{-\mathrm{i}(\bar{n}-n)(\alpha+\theta)}=\mathrm{e}^{-2\mathrm{i}\kappa VT\cos(\alpha+\theta)}$$

Then, $VT\to\infty$ trivial as VT-dependence cancels
$\longrightarrow$ Relative CP phase leading to CP-violating observables

However: The order of limits is not a choice but dictated by the fact that boundary conditions for the topological sectors are imposed at infinity.

Quantum mechanical systems: For a finite number of degenerate minima, there is only a finite number of distinct tunneling transitions.
$\rightarrow$ Order of limits not an issue

Effective operators

Effective interactions in the theory with fermions (present analysis)
$\longrightarrow$ Effective operators in χral perturbation theory
$\longrightarrow$ Observables such as neutron EDM, $\eta' \to \pi\pi$

$VT \to \infty$ **before** $\sum_{\Delta n}$	$\sum_{\Delta n}$ **before** $VT \to \infty$
$\mathcal{L} \to \mathcal{L} - \bar{\psi}(x)\Gamma e^{i\alpha\gamma^5}\psi(x)$	$\mathcal{L} \to \mathcal{L} + \bar{\psi}(x)\Gamma e^{-i\theta\gamma^5}\psi(x)$
Alignment with $\bar{\psi}m\exp(i\alpha\gamma^5)\psi$	Misaligned with $\bar{\psi}m\exp(i\alpha\gamma^5)\psi$
No CP-violating observables	CP-violating observables

Note: both operators transform in compliance with χral anomaly:
$\psi \to e^{i\beta\gamma_5}\psi, \qquad \bar{\psi} \to \bar{\psi}\, e^{i\beta\gamma_5}, \qquad \alpha \to \alpha - 2\beta, \qquad \theta \to \theta + 2\beta$

N_f flavours: $\mathcal{L} \to \mathcal{L} - \Gamma_{N_f} e^{-i\bar{\alpha}} \prod_{j=1}^{N_f}(\bar{\psi}_j P_{\rm L}\psi_j) - \Gamma_{N_f} e^{i\bar{\alpha}} \prod_{j=1}^{N_f}(\bar{\psi}_j P_{\rm R}\psi_j)$

Can also compute more general correlation functions, where again factors $\sum_{\Delta n} (-1)^{\Delta n} e^{i\Delta n(\alpha+\theta)}$ cancel when taking $VT \to \infty$ first

VI. Finite sub(volumes)

Boundary conditions at infinity crucial for alignement of the CP phases

Calculations in finite spacetime volumes should also be possible

- for subvolumes of Minkowski spacetime ($\rightarrow$ open boundary conditions),
- for periodic boundary conditions, e.g. on a torus as in lattice field simulations.

Note: On a torus, Δn is topologically conserved.

Lattice simulations sample over Δn because of finite lattice spacing.

Topology “freezes” in the continuum limit.

Finite vs infinite spacetime volume—cluster decomposition

Consider expectation value of an operator $\mathcal{O}$ in spacetime volume Ω, interfere different topological sectors Δn: [Weinberg QFT]

$$\langle \mathcal{O} \rangle_\Omega = \lim_{\substack{N\to\infty \\ N\in\mathbb{N}}} \frac{\sum_{\Delta n=-N}^{N} f(\Delta n) \int_{\Delta n} \mathcal{D}\phi\, \mathcal{O}\, \mathrm{e}^{-S_\Omega[\phi]}}{\sum_{\Delta n=-N}^{N} f(\Delta n) \int_{\Delta n} \mathcal{D}\phi\, \mathrm{e}^{-S_\Omega[\phi]}}$$

Factorize path integral into volume contributions, $\Omega = \Omega_1 \cup \Omega_2$:

(Assume $\Delta n(\Omega) = \Delta n_1(\Omega_1) + \Delta n_2(\Omega_2)$)

$$\langle \mathcal{O}_1 \rangle_\Omega = \lim_{\substack{N_2\to\infty \\ N_2\in\mathbb{N}}} \lim_{\substack{N_1\to\infty \\ N_1\in\mathbb{N}}} \frac{\sum_{\Delta n_1=-N_1}^{N_1} \sum_{\Delta n_2=-N_2}^{N_2} f(\Delta n_1 + \Delta n_2) \int_{\Delta n_1} \mathcal{D}\phi\, \mathcal{O}_1\, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_2}[\phi]}}{\sum_{\Delta n_1=-N_1}^{N_1} \sum_{\Delta n_2=-N_2}^{N_2} f(\Delta n_1 + \Delta n_2) \int_{\Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_2}[\phi]}}$$

Independence of $\langle \mathcal{O}_1 \rangle_\Omega$ from the fluctuations in Ω_2 is achieved if the contributions from Ω_2 cancel (absorb determinant phases in f):

$$f(\Delta n_1 + \Delta n_2) = f(\Delta n_1) f(\Delta n_2) \Rightarrow f(\Delta n) = \mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

Now keep Δn fixed, Ω_2 either finite or infinite:

$$\langle \mathcal{O}_1 \rangle_\Omega = \frac{\sum_{\Delta n_1=-\infty}^{\infty} f(\Delta n) \int_{\Delta n_1} \mathcal{D}\phi \, \mathcal{O}_1 \, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2 = \Delta n - \Delta n_1} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_2}[\phi]}}{\sum_{\Delta n_1=-\infty}^{\infty} f(\Delta n) \int_{\Delta n_1} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2 = \Delta n - \Delta n_1} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_2}[\phi]}}$$

$$= \frac{\sum_{\Delta n_1=-\infty}^{\infty} f(\Delta n) I_{\Delta n - \Delta n_1}(2\mathrm{i}\kappa\Omega_2) \int_{\Delta n_1} \mathcal{D}\phi \, \mathcal{O}_1 \, \mathrm{e}^{-S_{\Omega_1}[\phi]}}{\sum_{\Delta n_1=-\infty}^{\infty} f(\Delta n) I_{\Delta n - \Delta n_1}(2\mathrm{i}\kappa\Omega_2) \int_{\Delta n_1} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_1}[\phi]}}$$

$$\underset{\Omega_2 \gg \Omega_1}{\approx} \frac{\sum_{\Delta n_1=-\infty}^{\infty} \int_{\Delta n_1} \mathcal{D}\phi \, \mathcal{O}_1 \, \mathrm{e}^{-S_{\Omega_1}[\phi]}}{\sum_{\Delta n_1=-\infty}^{\infty} \int_{\Delta n_1} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_1}[\phi]}}$$

Result also holds for $\Omega_2 \to \infty$, Δn free

Path integral makes sense for finite subvolumes with open boundary conditions $\widehat{=}$ lattice simulations sampling over topological sectors Δn *without* phases

The theory with fixed Δn (gauge invariant) in large but finite volumes complies with cluster decomposition [cf. Leutwyler, Smilga (1992)]. Lattice results with frozen topology are therefore okay up to finite volume effects.

For $\Omega_2 \to \infty$, the result does not depend on whether Δn is fixed or free.

Observables in the dilute instanton gas for finite $\bar{\theta}$

	Δn free, $\Omega \to \infty$ first or Δn fixed, $\Omega \to \infty$	Δn free, $\Omega \to \infty$ last or Δn free, Ω finite	Δn free, $\Omega_1 \subset \Omega_2$ $\Omega_2 \to \infty$ first or Δn fixed, $\Omega_1 \subset \Omega_2 \leq \infty$	Δn fixed, Ω finite
$\chi \stackrel{\star}{=} -\frac{1}{\Omega}\,\frac{\partial^2 \ln Z}{\partial \bar{\theta}^2}$	0	2κ	2κ	$\Delta n^2/\Omega$
$\langle n \rangle/\Omega$	κ	$(-1)^{N_f}\,\kappa\,\mathrm{e}^{\mathrm{i}\bar{\theta}}$	κ	$\frac{\kappa I_{\Delta n-1}(2\kappa\Omega)}{I_{\Delta n}(2\kappa\Omega)}$
$\langle \Delta n \rangle/\Omega$	0	$2\mathrm{i}(-1)^{N_f}\,\kappa \sin\bar{\theta}$	0	$\Delta n/\Omega$

*for $\bar{\theta} = 1/2(1-(-1)^{N_f})\pi$

VII. Conclusions

In infinite spacetime volume, we derive observables from vanishing physical boundary conditions imposed on the fields at infinity.

The interferences between topological sectors that may lead to misaligned CP-odd phases between correlations from mass and instanton contributions then disappear.

Among finite volumes (e.g. a torus), the interference remains in principle. However, it should not be observable within local QFT.

There is agreement between the theories in infinite volumes (with Δn either free or fixed) and in large, finite volumes for fixed Δn.

In a subvolume Ω_1, we can work with free Δn_1 when accounting for the fluctuations in the volume complement and the topological conservation of the total Δn.

No CP-violating observables in QCD with massive quarks