Speculations on Physical Discretization and Arithmetic Geometry

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Based On

hep-th/xxxx.xxxxx

A "working paper" version is available at:

`www.jjheckman.com/research`

Disclaimers

What I will be talking about is ***speculative***:

- Both on the physics side
- And on the math side

However, the structures are suggestive enough that I decided to write up some notes

Underlying Question:
How to Define QFT?

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(in such a way that it can be coupled to gravity)

Physical Discretization

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In quantum mechanics (QM),
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QFT: $$\langle \phi_f(\overrightarrow{x}) | \phi_i(\overrightarrow{x}) \rangle \sim \int_{\phi_i(\overrightarrow{x})}^{\phi_f(\overrightarrow{x})} [d\phi] \exp(iS[\phi]/\hbar)$$

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"Transition Amplitude"

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"Path Integral", e.g.

$$\int d\phi(N) \int d\phi(N-1) ... \int d\phi(2) \int d\phi(1)$$

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$$i = \sqrt{-1} \in \mathbb{C}$$

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$$\langle \phi_f | \phi_i \rangle \sim \int_{\phi_i}^{\phi_f} [d\phi] \exp(iS[\phi]/\hbar)$$

"Action" $S[\phi] = \int L[\phi]\, dt$

"Lagrangian" $L[\phi]$, e.g. $\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi^2$

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In quantum mechanics and quantum field theory,

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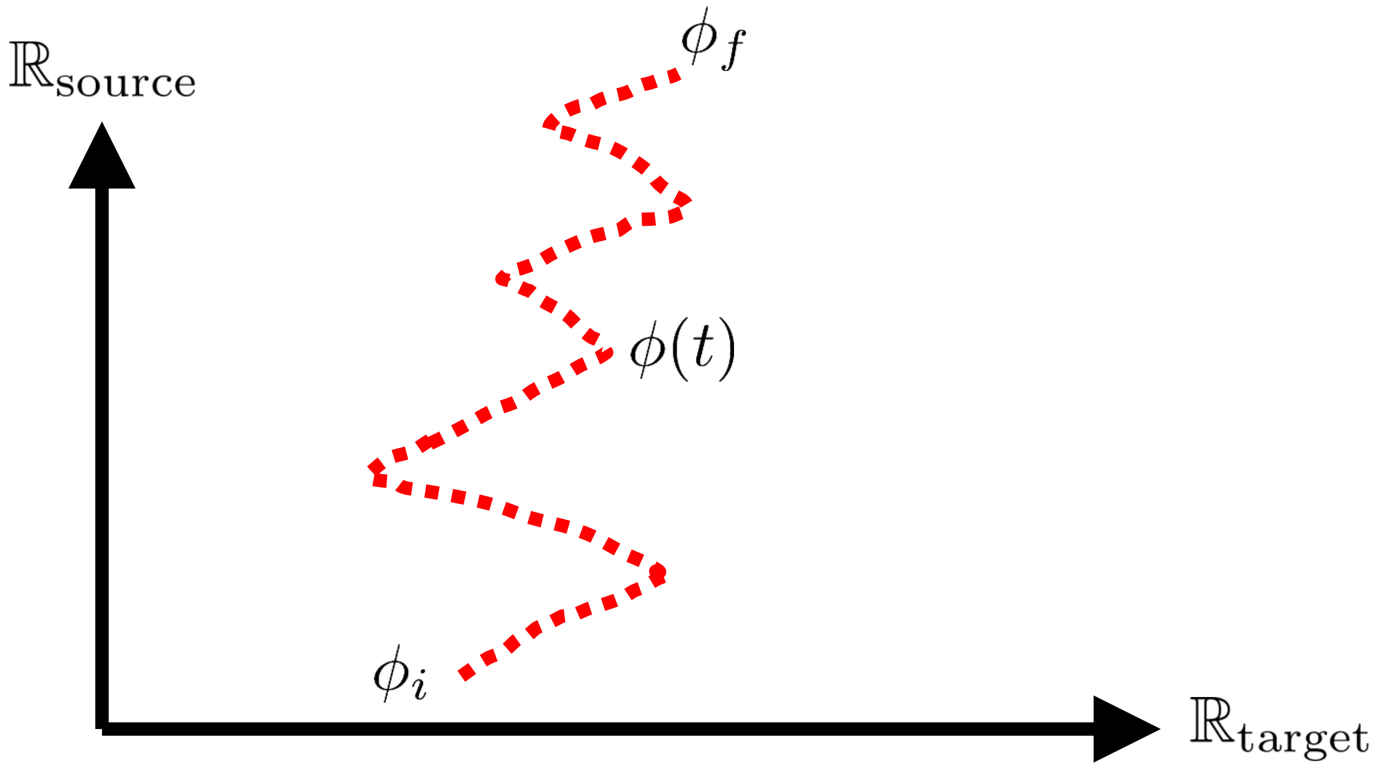
$$\langle \phi_f | \phi_i \rangle \sim \int_{\phi_i}^{\phi_f} [d\phi] \exp(iS[\phi]/\hbar)$$

Planck Constant $\hbar \sim 4.2 \times 10^{-15}$ eV s

High Energy Units Set $\hbar = 1$

The "Paths"

$\phi : \mathbb{R}_{\text{source}} \to \mathbb{R}_{\text{target}}$ is usually not smooth



Source Discretization

Lattice Approximation of Source, e.g.:

$$\phi : \mathbb{Z}_{\text{source}} \to \mathbb{R}_{\text{target}}$$

$$S[\phi] \approx \sum_{j \in \mathbb{Z}} \varepsilon \left(\frac{1}{2} \left(\frac{\phi(j+1) - \phi(j)}{\varepsilon} \right)^2 - \frac{1}{2} \phi(j)^2 \right) + \ldots$$

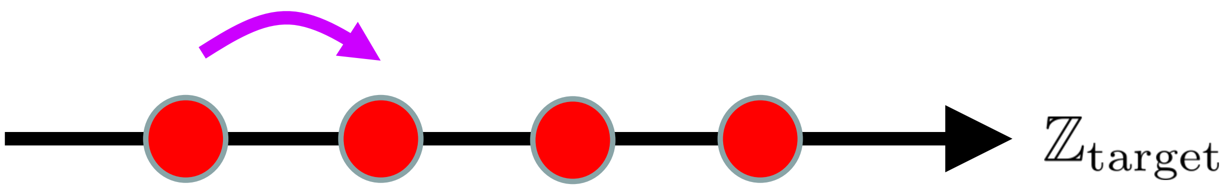
The "..." are often very hard to control!

Also Problematic: Chiral Fermions / Supersymmetry

Target Discretization

Discretization on Target Also Natural

Example 1: Hopping on a Crystal



View as $q : \mathbb{R}_{\text{source}} \to \mathbb{R}_{\text{target}}$ with strong potential

Target Discretization

Discretization on Target Also Natural

Example 2: Quantum Gravity (?!)

Continuum QFT breaks down when:

$\phi_{\text{max}} \sim M_{\text{Planck}} \sim 10^{19}$ GeV (in 4D)

Discretize Field Values: $\phi = \phi_{\text{small}} \times n$?

(UV / IR mixing)

Parameter Discretization

In string theory / quantum gravity, physical parameters *also* descend from quantum fields ("moduli").

This suggests they should also be discretized...

Suggestive Arithmetic Structures:

(Moore '98; Kachru Nally Yang '20; Hay Lam '20)

(see also Candelas de la Ossa Rodriguez-Villegas '00)

Main Physical Question

How much can we discretize?

Maps $\phi : \mathbb{Z}_{\text{source}} \to \mathbb{Z}_{\text{target}}$?

Parameters?

A First Attempt

A First Attempt
(Which will be Unsatisfactory)

$\phi : \mathbb{Z}_{\text{source}} \to \mathbb{Z}_{\text{target}}$

Lattice Approximation of Source, e.g.:

$$\phi : \mathbb{Z}_{\text{source}} \to \mathbb{Z}_{\text{target}}$$

$$S[\phi] \equiv \tfrac{2\pi}{N} \sum_{j \in \mathbb{Z}} \left(\alpha \left(\phi(j+1) - \phi(j) \right)^2 - \beta \phi(j)^2 \right) + \ldots$$

Assume $\alpha, \beta, N \in \mathbb{Z}$,

i.e. parameters are also “naturally quantized”.

¿Path Integral?

$$S[\phi] \equiv \frac{2\pi}{N}\sum_{j\in\mathbb{Z}} \left(\alpha\left(\phi(j+1)-\phi(j)\right)^2 - \beta\phi(j)^2\right) + \ldots$$

$$\langle\phi_f|\phi_i\rangle \sim \sum_{\phi:\mathbb{Z}\to\mathbb{Z}} \exp\left(\frac{2\pi i}{N}\sum_{j\in\mathbb{Z}} \left(\alpha\left(\phi(j+1)-\phi(j)\right)^2 - \beta\phi(j)^2\right) + \ldots\right)$$

Note: $e^{iS/\hbar}$ is now a character of $\mathbb{Z}/N\mathbb{Z}$.

¿Operators?

If we want $\phi(t) \in \mathbb{Z}/N\mathbb{Z}$, more natural to consider:

$\widehat{\mathcal{O}} = \exp\left(\frac{2\pi i}{N} a\phi\right)$ for $a \in \mathbb{Z}/N\mathbb{Z}$

DEFINE / Compute Correlators via:

$$\langle \widehat{\mathcal{O}}_1 ... \widehat{\mathcal{O}}_m \rangle = \frac{1}{N} \sum_{\phi:\mathbb{Z}\to\mathbb{Z}} \exp\left(\frac{2\pi i}{N} S[\phi]\right) \mathcal{O}_1 ... \mathcal{O}_m$$

Summary So Far...

Summarizing, we have:

$$\langle \phi_f | \phi_i \rangle \sim \sum_{\phi:\mathbb{Z}\to\mathbb{Z}} \exp\left(\tfrac{2\pi i}{N} S[\phi]\right)$$

Quick Answer: "$\hbar = \frac{N}{2\pi}$"

Difficulties

"Lattice Artifacts" still very problematic

Symmetries Explicitly Broken...

Example: $SO(3,1)$

QFT: $\phi : \mathbb{R}^{3,1} \to \mathbb{R}$ versus $\phi : \mathbb{Z}^4 \to \mathbb{Z}$ (in 4D)

Very hard to incorporate chiral fermions

A Second Attempt:
"QFT in Char p"

Reminders on Char p

$\mathbb{R}$ and $\mathbb{C} = \mathbb{R}(\sqrt{-1})$ are examples of "fields"

$\{$Integers modulo p $\} = \mathbb{F}_p$ *also* a field!

Finite fields $\mathbb{F}_q$? Find roots of $f(x) = 0$ for $f(x) \in \mathbb{F}_p[x]$

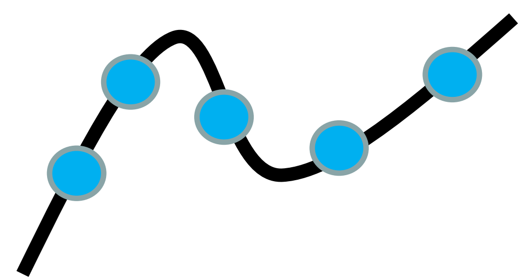
Galois Group: $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$: permutes the roots
(turns out it is always cyclic)

Geometry in Char p

Algebraic Geometry Still Makes Sense...

Main Difference: Over $\mathbb{F}_q$, finite number of points!

Discretization from the start...



¿¿¿"QFT in Char p"???

Main Idea (via Example)

Fix $N = p$ an odd prime

Consider $\phi(x) \in \mathbb{F}_p[x]$

evaluation of a polynomial

Action: $S[\phi] = \sum_{t \in \mathbb{F}_p} \mathrm{ev}_{x=t}(\alpha \partial_x \phi \partial_x \phi - V(\phi)) \in \mathbb{F}_p$

Path Integral Phase: $\exp\left(\frac{2\pi i}{p} S[\phi]\right)$ a character $\in \mathbb{C}$

Correlators

$$\widehat{\mathcal{O}}(t) = \exp\left(\frac{2\pi i}{p} a\phi(t)\right) \text{ for } a \in \mathbb{F}_p$$

Compute Correlators via:

$$\langle \widehat{\mathcal{O}}_1 ... \widehat{\mathcal{O}}_m \rangle \equiv \frac{1}{\mathcal{N}} \sum_{\phi \in \mathbb{F}_p[x]} \exp\left(\frac{2\pi i}{p} S[\phi]\right) \mathcal{O}_1 ... \mathcal{O}_m$$

More Geometrically...

Fix X_{source} and Y_{target} varieties / $\mathbb{F}_p$

Consider $\phi : X \dashrightarrow Y$ rational (allow poles)

$$\langle \widehat{\mathcal{O}}_1 ... \widehat{\mathcal{O}}_m \rangle \equiv \frac{1}{\mathcal{N}} \sum_{\phi : X \dashrightarrow Y} \exp\left(\frac{2\pi i}{p} S[\phi] \right) \mathcal{O}_1 ... \mathcal{O}_m$$

Mode Expansions

Example

Consider $\phi : \mathbb{A}^\times \to \mathbb{A}^\times$

Local Expansion: $\phi = \sum_m \phi_m x^m \in \mathbb{F}_p[x, x^{-1}]$

Action: $S[\phi] = \sum_{t \in \mathbb{F}_p^\times} \mathrm{ev}_{x=t}(x\partial_x \phi x \partial_x \phi) \in \mathbb{F}_p$

$S[\phi] = \sum_{m,n} \widehat{\delta}_{m-n} \phi_m \phi_n$ with: $\widehat{\delta}_l = \delta_{l \bmod p-1}$

What About / $\mathbb{F}_q$?

Clear interpretation for $Y_{\text{target}}/\mathbb{F}_q$: just adding more fields! ($\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ a "global symmetry")

Example: $\mathbb{F}_q = \mathbb{F}_p(\omega)$ with $q = p^2$.

Locally, $\phi = \phi_1\omega + \phi_2\omega^p \in \mathbb{F}_q[x]$

Note: X_{source} can often also be viewed as a target space

Path Integrals / $\mathbb{F}_q$

Fix X_{source} and Y_{target} varieties / $\mathbb{F}_q$

Consider $\phi : X \dashrightarrow Y$ rational (allow poles)

Still demand $S[\phi] \in \mathbb{F}_p$ (Unitarity)

(Can enforce by adding Frobenius conjugates)

$$x \mapsto x^p$$

$$\langle \widehat{\mathcal{O}}_1 ... \widehat{\mathcal{O}}_m \rangle \equiv \frac{1}{\mathcal{N}} \sum_{\phi : X \dashrightarrow Y} \exp\left(\frac{2\pi i}{p} S[\phi]\right) \mathcal{O}_1 ... \mathcal{O}_m$$

Formal limits available $\mathbb{F}_p \subset \mathbb{F}_{p^2} \subset ... \subset \overline{\mathbb{F}_p}$

Hilbert Space

Subtleties with States

Consider $\phi : \mathbb{A}^1_{\text{time}} \to \mathbb{A}^1_{\text{target}}$ (over $\mathbb{F}_q$)

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We can define $|\phi : \mathbb{A}^1 \to \mathbb{A}^1\rangle \in \mathcal{H}_{\text{big}}$ a Hilbert space / $\mathbb{C}$

"Explicit Time Dependence"

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Consider $\phi : \mathbb{A}^1_{\text{time}} \to \mathbb{A}^1_{\text{target}}$ (over $\mathbb{F}_q$)

We can define $|\phi : \mathbb{A}^1 \to \mathbb{A}^1\rangle \in \mathcal{H}_{\text{big}}$ a Hilbert space / $\mathbb{C}$

"Explicit Time Dependence"

We can also define $|\phi\rangle \in \mathcal{H}_{\text{small}}$ a Hilbert space / $\mathbb{C}$

"Lattice Approximation" projects $\mathbb{A}^1 \to \mathbb{F}_q$

Focus for now: Small Hilbert space

Qudits

Consider $|\phi\rangle \in \mathcal{H}_{\mathrm{small}}$

Qudit Operations:

$$\exp\left(\frac{2\pi i}{p}\mathrm{Tr}_{\mathrm{Fr}} a\widehat{\phi}\right)|\phi\rangle = \exp\left(\frac{2\pi i}{p}\mathrm{Tr}_{\mathrm{Fr}} a\phi\right)|\phi\rangle$$

$$\exp\left(\frac{2\pi i}{p}\mathrm{Tr}_{\mathrm{Fr}} b\widehat{\pi}\right)|\phi\rangle = |\phi + b\rangle$$

Error Operations

Qudit Operations:

$$\exp\left(\frac{2\pi i}{p}\mathrm{Tr}_{\mathrm{Fr}} a\widehat{\phi}\right)|\phi\rangle = R_a|\phi\rangle$$

$$\exp\left(\frac{2\pi i}{p}\mathrm{Tr}_{\mathrm{Fr}} b\widehat{\pi}\right)|\phi\rangle = T_b|\phi\rangle$$

$$E_{ab} = R_a T_b = \text{“Quantum Error”}$$

Codes

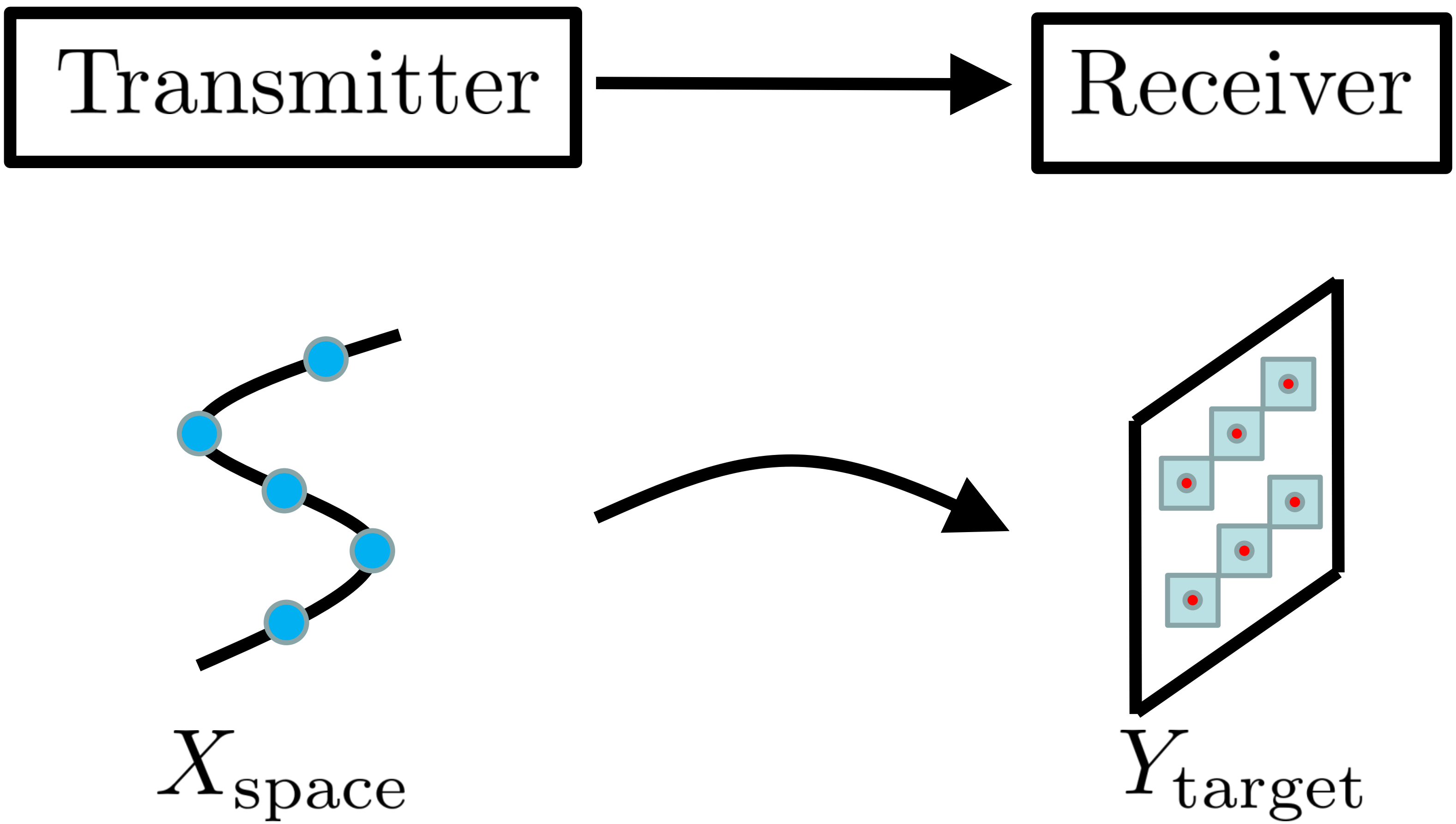
Consider $\phi : \mathbb{A}^1 \times X_{\text{space}} \to \mathbb{A}^1$

$|\phi(x_s)\rangle$ is several qudits in $\mathcal{H}_{\text{small}}$

Can define E_{ab}'s acting on $\mathcal{H}_{\text{small}} \simeq \mathbb{C}^n$

$\Rightarrow$ Implicitly Building a quantum stabilizer code

(Gottesman '97)


Transmitter
Receiver
X_{space}
Y_{target}

Swampland Comments

Vafa '05; Ooguri Vafa '06

Consider $L = (\partial\phi)^2 - V(\phi) + \ldots$

$V(\phi)$ *truncates*: (since $x^q = x$)

Note: $\partial^m \phi$ *does* not!

Roughly in accord with "bounding the # of EFTs"

c.f. Heckman Vafa '19

Other Physical Fields?

Other Physical Fields?

- Vector Bosons?
- Gravitons?
- Fermions?

(see also Schmidt '08)

(focus for today)

Fermions

Returning to $\mathbb{F}_p$, extend by Grassmann numbers

Define: χ, ψ as $\mathbb{F}_p$ Grassman variables when:

- $\chi\psi = -\psi\chi$

- Extend Frobenius: $F(\chi) = \chi$, $F(\psi) = \psi$, $F(\chi\psi) = \psi\chi$

  $x \mapsto x^p$

Fermionic Action (Example)

Example Action: $S[\chi,\psi] = \sum_{t\in\mathbb{F}_p} \mathrm{ev}_{x=t}\,\widehat{i}\chi\partial_x\psi + ...$

Need Field Extension with $\widehat{i} \in \mathbb{F}_q$ and $F(\widehat{i}) = -\widehat{i}$

Grassmann path integral no different than char 0

Supersymmetry & Cohomology

A Supersymmetric Theory

Introduce $W(\phi)$ a "superpotential" (poly in ϕ)

$$L = \frac{1}{2}(\partial_t \phi)^2 + \widehat{i}\chi\partial_t\psi - \frac{\widehat{i}^2}{2} f^2 + W' f + \widehat{i} W'' \chi\psi$$

$\delta_{\mathrm{SUSY}} L =$ exact differential

$$Q_+ = \widehat{i}\psi \left(\frac{\partial}{\partial\phi} + \widehat{i}^{-1} \frac{\partial W}{\partial \phi} \right) = - \left(\partial_t \phi + \widehat{i}^{-1} \frac{\partial W}{\partial \phi} \right) \frac{\partial}{\partial \chi}$$

$$Q_- = \widehat{i}\chi \left(\frac{\partial}{\partial\phi} - \widehat{i}^{-1} \frac{\partial W}{\partial \phi} \right) = - \left(\partial_t \phi - \widehat{i}^{-1} \frac{\partial W}{\partial \phi} \right) \frac{\partial}{\partial \psi}$$

A Cohomology Theory (I)

$Q^2 = 0 \Rightarrow$ cohomology theory

Question: Is $H_Q^\bullet$ related to known $H^\bullet$'s?

Answer (1): In char 0, we would say de Rham cohomology

Question: What about in char p?

A Cohomology Theory (II)

Note: Our construction also makes sense over $\mathbb{Z}/p^n\mathbb{Z}$

$\Rightarrow H^\bullet_Q$ computed via inverse limit

Proposal: $H^\bullet_Q \simeq H^\bullet_{\text{cris}}$ (in smooth case)

(Grothendieck '66; Berthelot '74;...)

Proposal: $H^\bullet_Q \simeq H^\bullet_{\text{rig}}$ (more generally)

(Berthelot '86; Kedlaya '06;...)

An Index

Witten Index $\mathrm{Tr}(-1)^{\mathbf{F}} = \mathrm{Tr}_{\mathrm{Fr}}(\mathrm{dim}\mathrm{ker}Q - \mathrm{dim}\mathrm{coker}Q)$
(Witten '82)

Same setup as Hasse-Weil Zeta Function:

$$\log Z_{V,\mathbb{F}_q}(z) = \sum_n \#V(\mathbb{F}_{q^n})\frac{z^n}{n} = \sum_n \mathrm{Tr}_n(-1)^{\mathbf{F}}\frac{z^n}{n}$$

(can use étale ℓ-adic or rigid cohomology)

(see Kedlaya '06)

Geometric Engineering (Characteristic p Case)

Philosophical Idea

Much of what we can reliably compute in string compactification involves algebraic geometry / $\mathbb{C}$:

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"Metric is superfluous"

Algebraic geometry also makes sense over $\mathbb{F}_p$, $\mathbb{Q}_p$, ...

A Correspondence (over $\mathbb{C}$)

Fix Σ a genus g curve (smooth, proj...)

Non-trivial correspondence between:

Curve of ADE Singularities (assume 3-fold is Calabi-Yau)	ADE Hitchin System
e.g. $y^2 = x^2 + z^N + f_{N-2}(z)$	$\det(z\mathbb{I} - \Phi) = 0$ in K_Σ
Intermediate Jacobian	$\widetilde{\Sigma} \xrightarrow{\pi} \Sigma$ $\quad$ $\widetilde{\mathcal{L}} \xrightarrow{\pi_*} \mathcal{E}$

Katz Vafa '96; Donagi Markman '95; Diaconescu Donagi Pantev '05;...

A Correspondence (over $\mathbb{C}$)

Purely Algebraic!

Curve of ADE Singularities (assume 3-fold is Calabi-Yau)	ADE Hitchin System
e.g. $y^2 = x^2 + z^N + f_{N-2}(z)$	$\det(z\mathbb{I} - \Phi) = 0$ in K_Σ
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Katz Vafa '96; Donagi Markman '95; Diaconescu Donagi Pantev '05;...

A Correspondence (over $\overline{\mathbb{F}_p}$)

Fix Σ a genus g curve (smooth, proj...)

Non-trivial correspondence between:

Curve of ADE Singularities	Characteristic p
(assume 3-fold is Calabi-Yau)	ADE Hitchin System
	(Simpson '92,...)
e.g. $y^2 = x^2 + z^N + f_{N-2}(z)$	$\det(z\mathbb{I} - \Phi) = 0$ in K_Σ
Intermediate Jacobian	$\widetilde{\Sigma} \xrightarrow{\pi} \Sigma$ $\qquad$ $\widetilde{\mathcal{L}} \xrightarrow{\pi_*} \mathcal{E}$

More Generally, (over $\mathbb{C}$)

Fix S a dim n variety (smooth, proj...)

Non-trivial correspondence between:

S fibered by ADE Singularities (assume (n+2)-fold is CY)	Partial Twist $(n,0)$-form Higgs Bundle
e.g. $y^2 = x^2 + z^N + f_{N-2}(z)$	$\det(z\mathbb{I} - \Phi) = 0$ in K_S
Deligne Cohomology	$\widetilde{S} \xrightarrow{\pi} S$ $\quad$ $\widetilde{\mathcal{L}} \xrightarrow{\pi_*} \mathcal{E}$

JJH et al. '08,'16; many more

More Generally, (over $\overline{\mathbb{F}_p}$)

Fix S a dim n variety (smooth, proj...)

Non-trivial correspondence between:

S fibered by ADE Singularities (assume (n+2)-fold is CY)	Characteristic p n-form Higgs Bundle
e.g. $y^2 = x^2 + z^N + f_{N-2}(z)$	$\det(z\mathbb{I} - \Phi) = 0$ in K_S
Syntomic Cohomology	$\widetilde{S} \xrightarrow{\pi} S$ $\quad$ $\widetilde{\mathcal{L}} \xrightarrow{\pi_*} \mathcal{E}$

How About $\hbar = \frac{p^a}{2\pi}$?

Path Integrals?

Phase Factor: $\exp\left(\frac{2\pi i}{p^a} S\right)$

$S = S_0 + S_1 p^1 + ... + S_{a-1} p^{a-1} + ...$

View $S \in \mathbb{Z}_p \subset \mathbb{Q}_p \subset \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$

$\phi : X \dashrightarrow Y$ (viewed as $\mathbb{Z}_p$ schemes)

(Recall $|p|_p = p^{-1}$, $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \text{ s.t. } |x|_p \leq 1\}$)

Order of Limits

Consider $\hbar = \frac{p^a}{2\pi}$, as $a \to \infty$

"Convergence of action": Demand $S = S_0 + ... + S_{a-1}p^{a-1}$

Converges in $\mathbb{C}_p$

"Convergence of phase": Demand $\lim \exp\left(\frac{2\pi i}{p^a} S\right)$

Converges in $\mathbb{C}$

Convergence of Action:

$$\phi : X(\mathbb{C}_p) \to Y(\mathbb{C}_p)$$

Convergence of Action

Consider the family of actions:

Action: $S[\phi] = \sum_{t \in \mathbb{Z}/p^a\mathbb{Z}} \mathrm{ev}_{x=t}(\alpha \partial_x \phi \partial_x \phi - V(\phi)) \in \mathbb{Z}_p$

Assume $a \to \infty$ limit makes sense

$\Rightarrow$ Classical Equations of Motion Makes Sense!

$\partial_t^2 \phi = -V'(\phi)$

Classical Solutions

p-adic Differential Equations Still Make Sense

Example: $\partial_t^2 \phi = -\Omega^2 \phi$ for $\Omega \in \mathbb{Q}_p$

Power Series: $\exp(\sqrt{-1}\Omega t) \equiv \sum_{n \geq 0} \frac{(\sqrt{-1}\Omega t)^n}{n!}$

Radius of Convergence: $|\Omega t|_p < p^{-1/(p-1)}$

Quantum Version

Recall: Our states are $|\phi : X \dashrightarrow Y\rangle \in \mathcal{H}_{\mathbb{C}}^{\text{big}}$

"p-adic Operators" $\widehat{O} : \mathbb{C}_p[[x]] \to \mathbb{C}_p[[x]]$

Note: $\exp\left(\frac{2\pi i}{p^a}\{\widehat{O}\}\right) : \mathcal{H}_{\mathbb{C}}^{\text{big}} \to \mathcal{H}_{\mathbb{C}}^{\text{big}}$

Example: $\widehat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2$

Time Evolution

Practical Definition: $U(T) = \exp\left(\frac{2\pi i}{p^a} T \widehat{H}\right)$

(assume $\widehat{H} = \mathrm{diag}(E_1, ...,)$)

Minimal Timestep: $t_{\min} = \frac{2\pi}{p^a} \in \mathbb{R}$

Convergence of Phase:

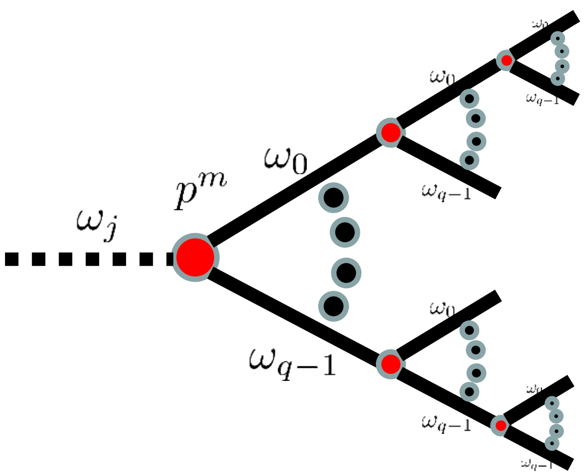
$$\phi : X(\mathbb{C}_p) \to Y(\mathbb{C})$$

"p-adic Physics"

Standard Case: $\phi : \mathbb{Q}_p \to \mathbb{R}$

p-adic string, p-adic AdS/CFT, ...

Volovich et al. '80's; Gubser et al. '16 + ...



$$M = \sum_i \omega_i p^i$$

p-adic expansion $\simeq$ tensor networks

Gubser et al. '16 ;

Heydeman et al. '16

Swingle '09

"p-adic Physics"

Standard Case: $\phi : \mathbb{Q}_p \to \mathbb{R}$

p-adic string, p-adic AdS/CFT, ...

Volovich et al. '80's; Gubser et al. '16 + ...

Awkward Features: $\mathbb{Q}_p$ and $\mathbb{C}_p$ have coarse topology:

Not Path Connected!

Example: No obvious $T_{\mu\nu}$...

Operator Algebras

Proposal: to make sense of limits of $\langle O(t_1)...O(t_m)\rangle$

have to add "additional points" to $\mathbb{C}_p$

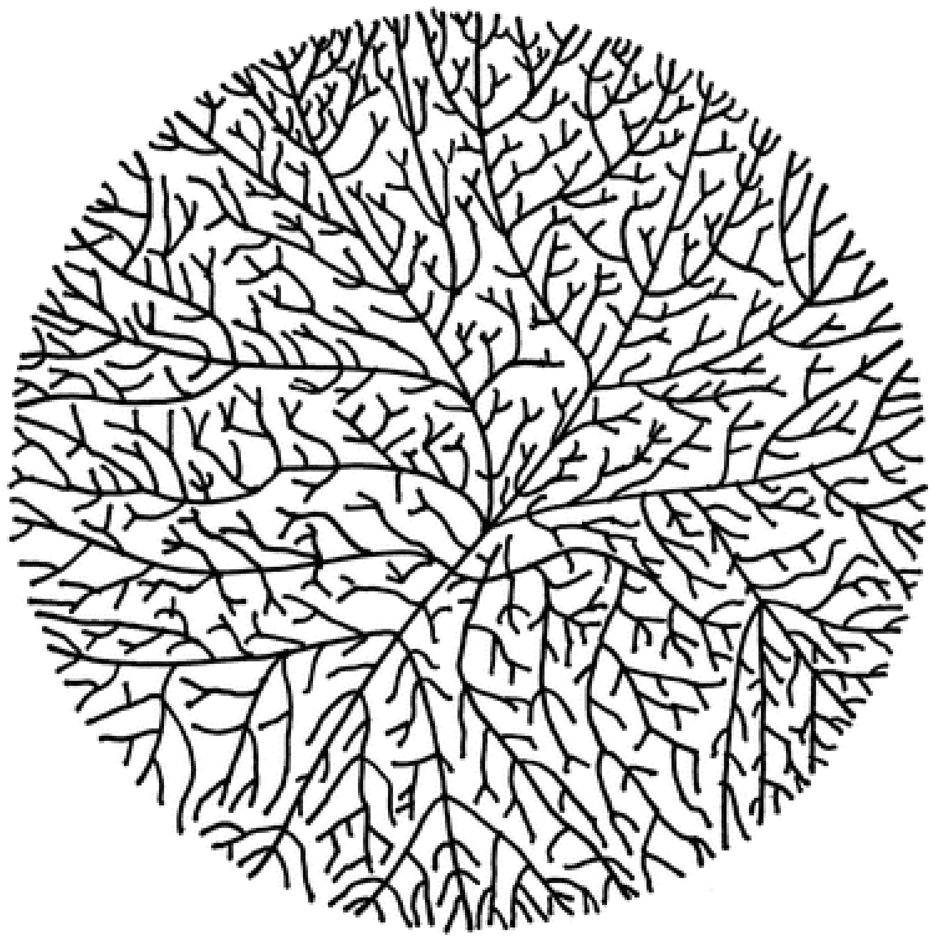
Minimally: "Rigid Analytic Geometry"

Tate '71

Less Minimally: "Berkovich Space": $\mathrm{Berk}(\mathbb{C}_p)$

Berkovich '90; Huber '93

Berkovich Space



Berkovich Space

- Path Connected

- $\exists$ diff ops. d' and d'' such that:

  $d'd''\log\ell(x,y) = \delta(x,y)$

  Chambert-Loir Ducros '12

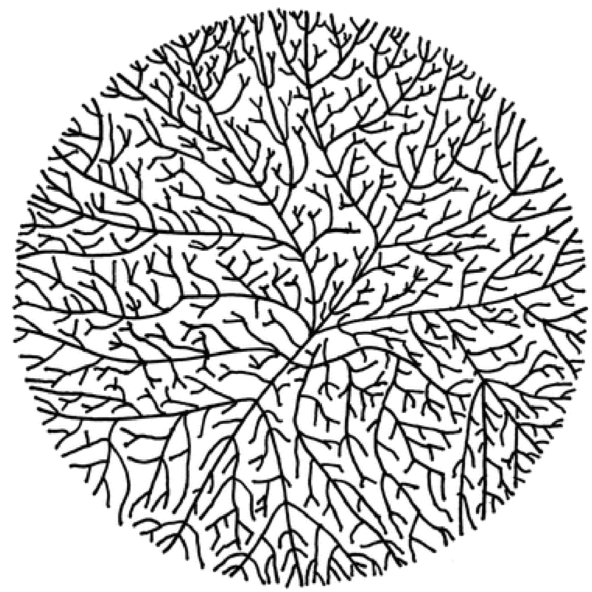
- Free Scalar Action:

  $\phi : \mathrm{Berk}(\mathbb{C}_p) \to \mathbb{R}$

  $S = \int -\phi d'd''\phi$

- Stress Tensor Makes Sense!

  $\Rightarrow$ Berkovich Strings!



How About $\hbar = \frac{p_1^{a_1} \ldots p_m^{a_m}}{2\pi}$?

Fibration Over $\operatorname{Spec}\mathbb{Z}$

$$
\begin{array}{ccc} X & \to & \widetilde{X} \\ & & \downarrow \\ & & \operatorname{Spec}\mathbb{Z} \end{array}
$$

$$2\pi\hbar : \operatorname{Spec}\mathbb{Z} \to \operatorname{Spec}\mathbb{Z}$$

$$x \mapsto x^n$$

$$\text{Phase: } \prod_{n\in\mathbb{N}} \prod_{p} \prod_{x\in X_p} e^{\left(\frac{2\pi i}{p^n} S_x\right)}$$

Geometric Engineering
(Arithmetic Case)

Engineering $\mathcal{N} = 4$ SYM

Working over $\mathbb{C}$

Consider Type II / $\mathbb{R}^{3,1} \times T^2 \times \mathbb{C}^2/\Gamma_{ADE}$

This engineers $\mathcal{N} = 4$ SYM on $\mathbb{R}^{3,1}$

Engineering $\mathcal{N} = 4$ SYM

Working over $K = \mathbb{C}$,

Consider $S \times \mathbb{E} \times M_{ADE}$

This engineers $\mathcal{N} = 4$ SYM on S

Arithmetic Version

Working over $K = \mathbb{Q}$,

Consider $S \times \mathbb{E} \times M_{ADE}$

$S \to \operatorname{Spec} \mathbb{Z}$ arithmetic surface

$S_p =$ mod p reduction of curve

Arithmetic Version

Working over K a number field

Consider $S \times \mathbb{E} \times M_{ADE}$

$S \to \mathcal{O}_K$ arithmetic surface

$S_{\mathfrak{p}}$ = mod $\mathfrak{p}$ reduction of curve

S-duality?

Working over $K \hookrightarrow \mathbb{C}$ a number field

Consider $S \times \mathbb{E} \times M_{ADE}$

$S \to \mathcal{O}_K$ arithmetic surface

$S_{\mathfrak{p}}$ = mod $\mathfrak{p}$ reduction of curve

$\tau \to -1/\tau$ still makes sense...

Summary

Conclusions (I / II)

- Proposal for physics in char p
- Proposal $H^\bullet_Q = H^\bullet_{\mathrm{rig}}$
- Geometric Engineering in char p
- Lift to p-adic analytic and arithmetic setting

Future (II / II)

- Explicit Computations?
- Gauge theory on arithmetic surfaces and S-duality / Langlands?
- Berkovich Strings?
- Numerical Simulations?