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<u>Thm 1</u> (Creutzig - G. - Nakatsuka)

Gaiotto - Rapčák 's $Y_{L,M,N}[\Psi]$ - alg coincides w/

Rapčák - Soibelman - Yang - Zhao 's $W_{L,M,N}[\Psi]$ - alg

if one of L, M, N is equal to 0

& one of L, M, N is equal to 1. //

ex.

$$\begin{cases} Y_{0,1,N}[k+N] = \mathrm{Com}(\mathrm{Heis}, W^k(sl_N, \mathrm{sub})) \otimes \mathrm{Heis} \\ Y_{1,N,0}[\ell+N-1] = \mathrm{Com}(\mathrm{Heis}, W^{\ell}(sl_{N|1})) \otimes \mathrm{Heis} \\ Y_{N,0,1}[\ell'+N-1] = \mathrm{Com}(V^{-\ell'+2}(gl_N), V^{\ell'}(sl_{N|1})) \otimes \mathrm{Heis} \end{cases}$$

where $V^{\ell'}(\mathfrak{g})$ = affine Vertex superalg of $\mathfrak{g}$ at level ℓ'

$W_+^k := W^k(sl_N, \mathrm{sub})$ = subregular W-alg of sl_N at level k.

$= H^0_{DS, f_{sub}}(V^k(sl_N))$

$W_-^{\ell} := W^{\ell}(sl_{N|1})$ = principal W-superalg of $sl_{N|1}$ at level k

$= H^0_{DS, f_{prin}}(V^k(sl_{N|1}))$

Thm 2 (CGN)

$$\mathcal{W}_+^k = \mathrm{Com}(\mathrm{diag}(\mathrm{Heis}), \mathcal{W}_-^\ell \otimes V_{\sqrt{-1}\mathbb{Z}})$$

$$\mathcal{W}_-^\ell = \mathrm{Com}(\mathrm{diag}(\mathrm{Heis}), \mathcal{W}_+^k \otimes V_{\mathbb{Z}})$$

w/ $(k+N)(\ell+N-1) = 1$. //

$\rightsquigarrow$ duality between $\mathcal{W}_+^k$ & $\mathcal{W}_-^\ell$.

$$\begin{cases} \mathcal{W}_+^k\text{-mod} = \bigoplus_{\lambda \in \mathbb{C}/\mathbb{Z}} \mathcal{W}_+^k\text{-mod}_\lambda \\ \mathcal{W}_-^\ell\text{-mod} = \bigoplus_{\lambda \in \mathbb{C}/\mathbb{Z}} \mathcal{W}_-^\ell\text{-mod} \end{cases}$$

$\rightsquigarrow$ Thm 2 for modules of $\mathcal{W}_\pm$.

Thm 3 (Creutzig – G. – Nakatsuka – Sato)

$$\mathcal{W}_+^k\text{-mod}_\lambda \underset{\Omega_\lambda^-}{\overset{\Omega_\lambda^+}{\rightleftarrows}} \mathcal{W}_-^\ell\text{-mod}_{\check{\lambda}}$$

gives a block-wise equiv. of categories.

For $M_\pm \in W_\pm\text{-mod}_\lambda$.

$H^{0,\pm}_{rel,\lambda}(M_\pm) :=$ (0-th) relative semi-infinite $\hat{gl}_1$-cohomology of $M_\pm \otimes V_{\sqrt{\pm1}\mathbb{Z}} \otimes \mathrm{Heis}_{\varepsilon_\pm\lambda}$ relative to gl_1-action.

<u>Thm 4</u> (CGNS)

1) $H^{0,\pm}_{rel,\lambda}(M_\pm) \cong \Omega^\pm_\lambda(M_\pm)$

2) $H^{0,\pm}_{rel,\lambda}$ gives an isom between.

superspaces of logarithmic intertwining op.'s :

$$\begin{pmatrix} & M^3_\pm & \\ M^1_\pm & . & M^2_\pm \end{pmatrix} \simeq \begin{pmatrix} & H^{0,\pm}_{rel,\lambda_3}(M^3_\pm) & \\ H^{0,\pm}_{rel,\lambda_1}(M^1_\pm) & , & H^{0,\pm}_{rel,\lambda_2}(M^2_\pm) \end{pmatrix}$$

where $M^i_\pm \in W_\pm\text{-mod}_{\lambda_i}$.

3) We have a natural isom

$$H^{0,\pm}_{rel,\lambda_1+\lambda_2}(M^1_\pm \boxtimes M^2_\pm) \simeq H^{0,\pm}_{rel,\lambda_1}(M^1_\pm) \boxtimes H^{0,\pm}_{rel,\lambda_2}(M^2_\pm),$$

provided $\boxtimes$ is well-defined. //

Plan (0. Main Thm)

1. GR's $Y_{L,M,N}[\Psi]$.

2. RSYZ's $W_{L,M,N}[\Psi]$.

3. Dualies on W_+^k & W_-^ℓ.

4. Applications

5. Future works.

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1. GR's $Y_{L,M,N}[\Psi]$ - alg.

$L, M, N \in \mathbb{Z}_{\geq 0}$. $\Psi \in \mathbb{C}$.

$Y_{L,M,N}[\Psi] \leftarrow$ GL-twisted $\mathcal{N}=4$ SYM gauge thy.

trivial duality : $Y_{L,M,N}[\Psi] = Y_{L,N,M}[1-\Psi]$.

"S-duality" : $Y_{L,M,N}[\Psi] = Y_{M,L,N}[\Psi^{-1}]$

$\rightsquigarrow$ Conj

$$Y_{L,M,N}[\Psi] = Y_{N,L,M}\left[\frac{1}{1-\Psi}\right] = Y_{M,N,L}\left[1-\frac{1}{\Psi}\right]. //$$

Thm (Creutzig - Linshaw)

one of L, M, N is equal to 0

$\Rightarrow$ conj is true //

Notations $\quad \mathfrak{gl}_N \ni$ nilpotent elements f

$$= \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{matrix}} & & \\ & \boxed{\begin{matrix} 0 & 1 \\ & 0 \end{matrix}} & \\ & & \boxed{0} \end{pmatrix} \quad \text{(up to conjugations)}$$

$=$ Jordan blocks w/ diag $= 0$

$=$ Partitions of N.

$W^k(\mathfrak{gl}_N, p)$ $\quad p$: partition of N

$=$ W-alg asso. to $\mathfrak{gl}_N$ and nilp. ele. f_p corr. to p. at level k

$= H^0_{DS, f_p}(V^k(\mathfrak{gl}_N))$

$W^\ell(\mathfrak{gl}_{M|N}, p|q)$ $\quad p$: partition of M, q : partition of N

$=$ W-alg asso. to $\mathfrak{gl}_{M|N}$ and nilp ele $f_p + f_q$ corr. to p in $\mathfrak{gl}_M$, q in $\mathfrak{gl}_N$ at level ℓ

$= H^0_{DS, f_p+f_q}(V^\ell(\mathfrak{gl}_{M|N}))$

$\rightsquigarrow$

$$Y_{0,M,N} = \begin{cases} \mathrm{Com}(V^{\psi-M-1}(\mathfrak{gl}_M), W^\psi(\mathfrak{gl}_N, (N-M, 1^M))) & (M \leq N) \\ \mathrm{Com}(V^{-\psi-N}(\mathfrak{gl}_N), W^{-\psi+1}(\mathfrak{gl}_M, (M-N, 1^N))) & (M > N) \end{cases}$$

$$Y_{L,0,N} = \mathrm{Com}(V^{-\psi-L+1}(\mathfrak{gl}_L), W^{\psi-N+L}(\mathfrak{gl}_{N|L}, (N|1^L)))$$

$$Y_{L,M,0} = \mathrm{Com}(V^{\psi-L}(\mathfrak{gl}_L), W^{-\psi-N+L+1}(\mathfrak{gl}_{M|L}, (M|1^L)))$$

examples two of L, M, N are equal to 0

$\rightsquigarrow$ principal W-alg of $\mathfrak{gl}_N$. simple quot of $V^1(\mathfrak{gl}_N)$

$$Y_{N,0,0} = \mathrm{Com}\left(V^{\Psi-N+1}(\mathfrak{gl}_N),\ V^{\Psi-N}(\mathfrak{gl}_N) \otimes L_1(\mathfrak{gl}_N) \right)$$

$$Y_{0,N,0} = W^{-\Psi+1}(\mathfrak{gl}_N)$$

$$Y_{0,0,N} = W^{\Psi}(\mathfrak{gl}_N)$$

$\rightsquigarrow$ S-duality on $Y_{0,N,0}$, $Y_{0,0,N}$

= Feigin - Frenkel duality

S-duality on $Y_{N,0,0}$

= Coset construction of $W^{-\Psi^{-1}+1}(\mathfrak{gl}_N)$

(Arakawa - Creutzig - Linshaw

$N=2$: GKO coset of Virasoro alg.)

How to prove ?

Maulik - Okounkov, Schiffmann - Vasserot :

affine Yangian $Y_\hbar(\hat{\mathfrak{gl}}_1) \curvearrowright \bigoplus_{n=0}^{\infty} H^{D}(M_{r,n})_{loc}$

$M_r = \bigsqcup_{n=0}^{\infty} M_{r,n}$: moduli sp

of rk r torsion free sheaves

on $\mathbb{P}^2$ w/ framing $\mathbb{P}^1_\infty \subset \mathbb{P}^2$

$\rightsquigarrow$ $Y_\hbar(\hat{\mathfrak{gl}}_1) \twoheadrightarrow \mathcal{U}(W^{\Psi}(\mathfrak{gl}_r))$

$\uparrow$

$\downarrow$

vertex alg. counter part

$=$ $W(c, \lambda)$: Linshaw's universal two-para. W_∞-alg.

$= \langle L, W_3, W_4, W_5, \cdots \rangle_{VA} / \mathbb{C}[c, \lambda]$.

w/ rel.

$\uparrow$

$$\left\{ \begin{array}{l} L : \text{Virasoro of c.c.} = c \\ W_3 : \text{primary of conf. wt.} = 3 \\ W_i = W_{3\,(1)} W_{i-1} \text{ for } i \geq 4. \\ W_{3\,(3)} W_4 = (31 - 16(\lambda + c)) W_3. \\ \text{Jacobi rel :} \\ [W_{i\,(r)}, W_{j\,(s)}] W_k = \sum_{\ell=0}^{r} \binom{r}{\ell} (W_{i\,(\ell)} W_j)_{(r+s-\ell)} W_k \\ \vdots \\ \text{and so on.} \end{array} \right.$$

<u>Claim</u> All $Y_{L,0,N}$, $Y_{L,M,0}$, $Y_{0,M,N}$ are simple quot of $W(c, \lambda)$ over $\mathbb{C}[c, \lambda] / I$. ($\otimes$ Heis.)

$\exists$ ideal $\swarrow$

$\rightsquigarrow$ Uniqueness forces the triality.

2. M-O, Braverman - Finkelberg - Nakajima

$G = ADE$.

$\mathcal{U}_G^d$ = Uhlenbeck sp

= cpt of moduli sp of G-bdl over $\mathbb{P}^2$

w/ instonton # = d w/ trivialization $\mathbb{P}_\infty$

$\mathbb{G} = G \times (\mathbb{C}^\times)^2$. $\mathbb{T} = T \times (\mathbb{C}^\times)^2$.

$$\bigoplus_{d=0}^{\infty} IH^\bullet_{\mathbb{G}}(\mathcal{U}_G^d)_{loc} = \bigoplus_{d=0}^{\infty} IH^\bullet_{\mathbb{T}}(\mathcal{U}_G^d)_{loc}^W \subset \bigoplus_{d=0}^{\infty} IH^\bullet_{\mathbb{T}}(\mathcal{U}_G^d)_{loc}$$

$$\updownarrow$$

$$W^k(\mathfrak{g}) = \bigcap_i \mathrm{Ker} \int e^{-\frac{\alpha_i}{\Phi}}(z)\,dz \subset \mathrm{Heis}^{\otimes \mathrm{rk}\,\mathfrak{g}}.$$

RSYZ : Using moduli sp of spiked instantons,

$$W_{L,M,N} = \bigcap_i \mathrm{Ker} \int e^{-\frac{\alpha_i}{\Phi}}(z)\,dz \subset \mathrm{Heis}^{\otimes L+M+N}$$

w/ $Y(\widehat{\mathfrak{gl}}_1) \twoheadrightarrow \mathcal{U}(W_{L,M,N})$

Q. $W_{L,M,N} \overset{?}{=} Y_{L,M,N}$.

Thm 1 One of L, M, N is 0

$\quad\quad\quad\quad\quad\quad$ " $\quad\quad\quad\quad$ is 1

$\Rightarrow$ this is true.

3.

$W^k(\mathfrak{g})$: $\{\alpha_i\}_i \longleftrightarrow$ Dynkin diag of $\mathfrak{g}$

$W_{0,1,N}$: $\{\alpha_i\}_i \longleftrightarrow$ Dynkin diag of $\mathfrak{sl}_{N|1}$

$$W_{0,1,N} = \mathrm{Com}(\mathrm{Heis}, W_+^k) \otimes \mathrm{Heis} = \mathrm{Com}(\mathrm{Heis}, W_-^\ell) \otimes \mathrm{Heis}$$

$$= \bigcap_i \mathrm{Ker} \int e^{-\frac{\alpha_i}{\Psi}}(z)\, dz \subset \mathrm{Heis}^{\otimes N+1}$$

$\rightsquigarrow$ Extend domains of screening operators to lattice

$$W_+^k = \bigcap_i \mathrm{Ker} \int e^{-\frac{\alpha_i}{\Psi}}(z)\, dz \Big|_{\mathrm{Heis}^{\otimes N-1} \otimes V_{0,\mathbb{Z}}}$$

$$W_-^\ell = \bigcap_i \mathrm{Ker} \int e^{-\frac{\alpha_i}{\Psi}}(z)\, dz \Big|_{\mathrm{Heis}^{\otimes N-1} \otimes V_{\mathbb{Z}}}$$

Thm 2

$$W_+^k = \mathrm{Com}(\mathrm{diag}(\mathrm{Heis}), W_-^\ell \otimes V_{\sqrt{-1}\mathbb{Z}})$$

$$W_-^\ell = \mathrm{Com}(\mathrm{diag}(\mathrm{Heis}), W_+^k \otimes V_{\mathbb{Z}})$$

//

special case : $\{L, M, N\} = \{0, 1, 2\}$

$W_+^k = \widehat{\mathfrak{sl}}_2$ at level k.

$W_-^\ell = \mathcal{N} = 2$ Super Conformal Alg of c.c. $= -3(2\ell + 1)$

$\rightsquigarrow$ Duality between $\widehat{\mathfrak{sl}}_2$ & $N=2$ SCA

(Di Vacchia – Peterson – Yu – Zheng
Adamović, Kazama – Suzuki. Creutzig – Linshaw
... etc.)

Feigin – Semikhatov – Tipunin, Sato:

$$\begin{cases} \widehat{\mathfrak{sl}}_2\text{-mod} = \bigoplus_\lambda \widehat{\mathfrak{sl}}_2\text{-mod}_\lambda \ni M_+ \\ N=2\ \text{SCA-mod} = \bigoplus_\lambda N=2\ \text{SCA-mod}_\lambda \ni M_- \end{cases}$$

$$\Rightarrow M_\pm \otimes V_{\sqrt{\pm 1}\mathbb{Z}} = \bigoplus_{\mu,\nu} \Omega^\pm(M_\pm)_{\mu\mp\nu} \otimes \mathrm{Heis}_{\mu+\tilde{\varepsilon}_\mp\nu} \otimes \mathrm{Heis}_\nu$$

$$\Omega^\pm_\lambda(M_\pm) := \bigoplus_\mu \Omega^\pm(M_\pm)_{\mu\mp\lambda} \otimes \mathrm{Heis}_{\tilde{\mu}}$$

$$\Rightarrow \quad \widehat{\mathfrak{sl}}_2\text{-mod}_\lambda \underset{\Omega^-_{\lambda^\vee}}{\overset{\Omega^+_\lambda}{\rightleftarrows}} N=2\ \text{SCA-mod}_{\lambda^\vee}$$

: block-wise equiv of categories.

$\wr$ generalization to $W_\pm$
$\downarrow$

Thm 3.
$$W^k_{+}\text{-mod}_{\lambda} \underset{\Omega^{-}_{\lambda^\vee}}{\overset{\Omega^{+}_{\lambda}}{\rightleftarrows}} W^{\ell}_{-}\text{-mod}_{\lambda^\vee}$$

: block-wise equiv of categories. //

Cohomological approach.

$$* \quad W_{\mp} = H^0_{\mathrm{rel}}(\widehat{gl}_1, gl_1 ; W_{\pm} \otimes V_{\sqrt{\pm 1}\mathbb{Z}} \otimes \mathrm{Heis})$$

$$\rightsquigarrow \quad M_{\pm} \in W_{\pm}\text{-mod}_{\lambda}$$

$$H^{0,\pm}_{\mathrm{rel},\lambda}(M_{\pm}) := H^0_{\mathrm{rel}}(\widehat{gl}_1, gl_1 ; M_{\pm} \otimes V_{\sqrt{\pm 1}\mathbb{Z}} \otimes \mathrm{Heis}_{\varepsilon \pm \lambda})$$

$$\in W_{\mp}\text{-mod}_{\lambda^\vee}$$

Thm 4 $\quad M^i_{\pm} \in W_{\pm}\text{-mod}_{\lambda_i}$

1) $\Omega^{\pm}_{\lambda}(M_{\pm}) \simeq H^{0,\pm}_{\mathrm{rel},\lambda}(M_{\pm})$

2) $$\begin{pmatrix} M^3_{\pm} \\ M^1_{\pm}, M^2_{\pm} \end{pmatrix} \simeq \begin{pmatrix} H^{0,\pm}_{\mathrm{rel},\lambda_3}(M^3_{\pm}) \\ H^{0,\pm}_{\mathrm{rel},\lambda_1}(M^1_{\pm}), \; H^{0,\pm}_{\mathrm{rel},\lambda_2}(M^2_{\pm}) \end{pmatrix}$$

3) $$H^{0,\pm}_{\mathrm{rel},\lambda_1+\lambda_2}(M^1_{\pm} \otimes M^2_{\pm})$$
$$\simeq H^{0,\pm}_{\mathrm{rel},\lambda_1}(M^1_{\pm}) \boxtimes H^{0,\pm}_{\mathrm{rel},\lambda_2}(M^2_{\pm}) \quad //$$

2) 3) are induced from quasi-isom between complexes

$\Rightarrow$ naturality.

4. $\mathcal{W}_{+,k}$ = simple quot of $\mathcal{W}_+^k$

$\mathcal{W}_{-,\ell}$ = simple quot of $\mathcal{W}_-^\ell$

<u>Cor</u>

$$\mathcal{W}_{+,k} = \mathrm{Com}\left(\mathrm{diag}(\mathrm{Heis}),\ \mathcal{W}_{-,\ell} \otimes V_{\sqrt{-1}\mathbb{Z}}\right)$$

$$\mathcal{W}_{-,\ell} = \mathrm{Com}\left(\mathrm{diag}(\mathrm{Heis}),\ \mathcal{W}_{+,k} \otimes V_{\mathbb{Z}}\right).$$

Arakawa - van Ekeren :

$$k + N = \frac{N+r}{N-1} \quad \text{w/} \quad r \geq 0,\ (N+r, N-1) = 1$$

$\Rightarrow$ $\mathcal{W}_{+,k}$ is C_2 - cofinite & rational

↑ · finiteness of simple mods · modularity of characters

↑ · semisimplicity of module category

<u>Cor</u> $\ell + N - 1 = \dfrac{N-1}{N+r}$ w/ $r \geq 0,\ (N+r, N-1) = 1$

$\Rightarrow$ $\mathcal{W}_{-,\ell}$ is C_2 - cofinite & rational //

Creutzig - Linshaw : $k + N = \dfrac{N+r}{N-1}$ w/ $r \geq 2,\ (N+r, N-1) = 1.$

$\Rightarrow$ $\mathrm{Com}(\mathrm{Heis}, \mathcal{W}_{+,k}) = \mathcal{W}_{k'}(\mathfrak{sl}_r)$: level-rank duality //

w/ $k' + r = \dfrac{r+N}{r+1}$ ($\Rightarrow$ rational by Arakawa)

. $W_{k'}(\mathfrak{sl}_r)$ = simple quot of principal W-alg of $\mathfrak{sl}_r$ at level k'

cf. $N=2$. Arakawa – Lam – Yamada

3. Arakawa – Creutzig – Linshaw

4. Creutzig – Linshaw

Frenkel – Kac – Wakimoto, Arakawa, Arakawa – van Ekeren, Creutzig :

$$\{\text{Simple of } L_N(\mathfrak{sl}_r)\} \to \{\text{Simple of } W_{k'}(\mathfrak{sl}_r)\}$$

$$\{L(\lambda)\}_{\lambda \in P_+^N} \mapsto \{\mathbb{L}_W(\lambda)\}_{\lambda \in P_+^N}$$

gives an isom of fusion rings.

$\rightsquigarrow$ simple current ext + duality.

<u>Thm</u> k, ℓ, k' are given as above.

1)

$$\mathrm{Com}(W_{k'}(\mathfrak{sl}_r), W_{+,k}) = V_{\sqrt{Nr}\mathbb{Z}}$$

$$W_{+,k} \cong \bigoplus_{i=0}^{r-1} \mathbb{L}_W(N\varpi_i) \otimes V_{\sqrt{Nr}\mathbb{Z} + i\sqrt{\frac{N}{r}}}$$

2)

$$\mathrm{Com}(W_{k'}(\mathfrak{sl}_r), W_{-,\ell}) = V_{\sqrt{(N+r)r}\mathbb{Z}}$$

$$W_{-,\ell} \cong \bigoplus_{i=0}^{r-1} \mathbb{L}_W(N\varpi_i) \otimes V_{\sqrt{(N+r)r}\mathbb{Z} + i\sqrt{\frac{N+r}{r}}}$$

//

Cor

1) $\exists$ isom of fusion rings

$$K(L_r(\mathfrak{sl}_N)) \simeq K(W_{t,k})$$

2) $\{\text{Simple of } W_{-,\ell}\}$

$$\cong \{(\lambda, a) \in P_+^N \times \mathbb{Z}_{(N+r)N} \mid \pi(\lambda) = a\} / \mathbb{Z}_N$$

Moreover, $\exists$ isom of fusion rings

$$K(W_{-,\ell}) \cong \left(K(L_r(\mathfrak{sl}_N)) \otimes_{\mathbb{C}[\mathbb{Z}_N]} \mathbb{C}[\mathbb{Z}_{(N+r)N}] \right)^{\mathbb{Z}_N}$$

//

cf 1) N : even — Arakawa – van Ekeren

2) $N = 2$: Adamović, Wakimoto.

5.

- General case for relative semi-infinite cohomology.

Conj (Creutzig - Linshaw)

$$Y_{0,M,N} = H^{0}_{\mathrm{rel}}(\widehat{\mathfrak{gl}}_M, \mathfrak{gl}_1 ; Y_{M,0,N} \otimes A^{+}[\mathfrak{gl}_M, \Psi])$$

$A^{+}[\mathfrak{gl}_M, \Psi]$: kernel VOA

$$\frac{1}{k_1+2} + \frac{1}{k_2+2} = 1$$

$M = 1$: $V_{\mathbb{Z}} \otimes \mathrm{Heis}.$ $\quad k_1 = -\psi^{-1} - 1$

$M = 2$: $\displaystyle\bigoplus_{\lambda \in P_+} V^{k_1}(\lambda) \otimes V^{k_2}(\lambda^{t}) \otimes V_{\sqrt{2}\mathbb{Z} + s(\lambda)} \otimes \mathrm{Heis}$

$= L_1(D(2,1;-\psi^{-1}) \otimes \mathrm{Heis}.$ (Creutzig - Gaiotto)

$M \geq 3$: existence of VOA str is not known.

- Gaiotto - Rapčák's alg of type BCD Osp

Conj Triality for $Y^{\pm}_{L,M,N}[\Psi]$, $\tilde{Y}^{\pm}_{L,M,N}[\Psi]$.

Thm (Creutzig - Linshaw)
one of L, M, N is equal to 0

$\Rightarrow$ Conj is true

proof : simple quot of even-spin two-para. W_∞-alg
+
uniqueness.

- Screening operators. side

$$\widetilde{W}_+^k = W^k(so_{2N+1}, sub)$$
$$\widetilde{W}_-^\ell = W^\ell(osp_{2|2N})$$

Thm

$$\widetilde{W}_+^k = \mathrm{Com}(\mathrm{diag}(\mathrm{Heis}), \widetilde{W}_-^\ell \otimes V_{\sqrt{-1}\mathbb{Z}})$$
$$\widetilde{W}_-^\ell = \mathrm{Com}(\mathrm{diag}(\mathrm{Heis}), \widetilde{W}_+^k \otimes V_{\mathbb{Z}}) \quad //$$

$\exists$ conj for relative semi-infinite cohomology

by using kernel VOAs.

- Special case : $k+N = \dfrac{N}{N+1}$

$$\Rightarrow \quad W_{+,k} = \mathcal{B}^{(N+1)}\text{-alg}$$

$=$ the chiral alg of Argyres-Douglas thy of type (A_1, A_{2N-1})

$$= \left(\mathcal{A}^{(N+1)} \otimes V_{\sqrt{-\frac{N+1}{2}}\mathbb{Z}}\right)^{U(1)}$$

$\uparrow$ doublet alg.