Obstructions to Gapped Boundaries

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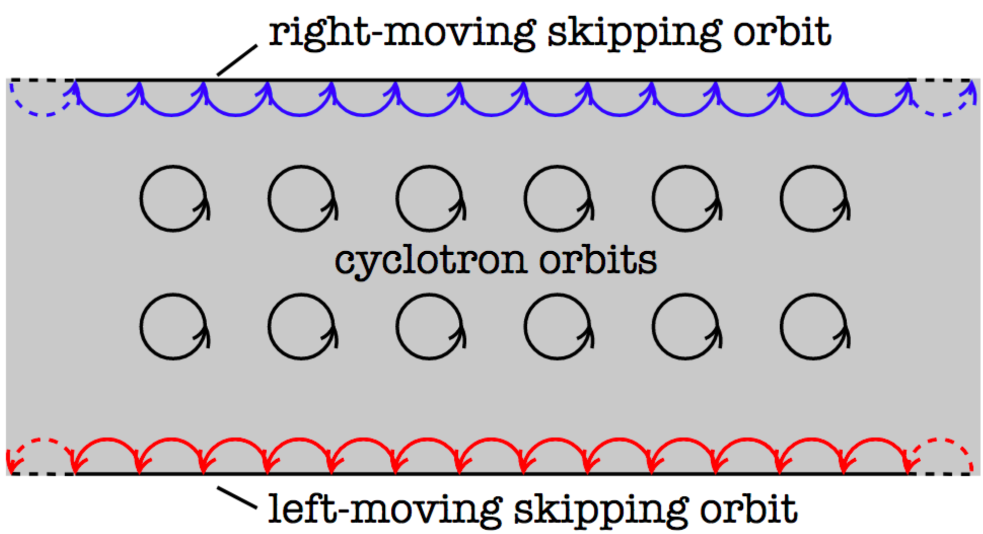
Gapped Boundaries

- **An interesting phenomenon in QFT is that certain theories which are gapped in the bulk must be gapless on the boundary.**

- **In fact, this is *generically* the case, e.g. $U(1)_{2N}$ CS and the IQHE.**



Source: topcondmat.org

- **In this case, can be explain via *anomalies*.**
  - In particular, $c_- \neq 0$ so there is a gravitational anomaly on the boundary.

Gapped Boundaries

- **But as we will see, the presence of gapless edge modes** *cannot* **always be explained by anomalies.**

- **Q: When does a** *bosonic* **(2+1)d TQFT admit a** *gapped* **boundary?**

  – Equivalently, when do two (2+1)d TQFTs have a gapped interface?

- **Many results already in the literature:** [Kapustin, Saulina '10; Davydov, Müger, Nikshych, Ostrik '10; Kitaev, Kong '11; Wang, Wen '12; Fuchs, Schweigert, Valentino '12; Levin '13; Lan,Wang, Wen '14; Freed,Teleman '20, . . .]

- **Today I will discuss:**

  – Geometric reinterpretation of previous algebraic results

  – New necessary and sufficient conditions for Abelian TQFTs to admit gapped boundaries

  – An infinite class of new obstructions labelled by 3-manifolds

Abelian TQFT

- **We begin with Abelian TQFTs. All such theories can be written as**

$$S = \frac{1}{4\pi}\int K_{ij}\, A_i \wedge dA_j \qquad\qquad K_{ij} \in \mathbb{Z}$$

  – We restrict to bosonic theories, i.e. $K_{ii} \in 2\mathbb{Z}$.

- **These theories have anyons**

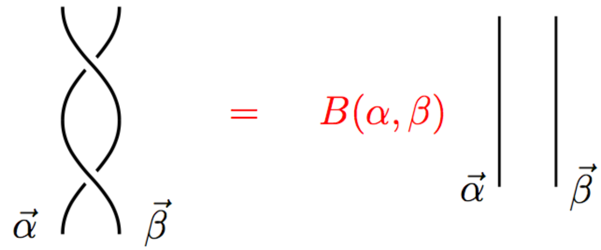
$$W_{\vec{\alpha}}(\gamma) = e^{i\int_\gamma \alpha_j A_j} \qquad\qquad \vec{\alpha} \in \mathbb{Z}^{|K|}$$

- **QM-ly, $\vec{\alpha} \sim \vec{\alpha} + K\vec{\ell}$ for any $\vec{\ell} \in \mathbb{Z}^{|K|}$, so the anyons belong to**

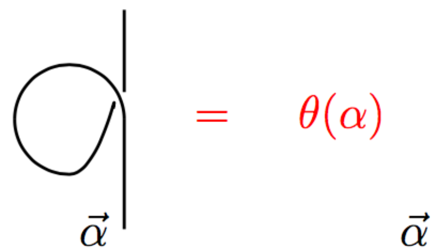
$$G := \frac{\mathbb{Z}^{|K|}}{K \cdot \mathbb{Z}^{|K|}}$$

Abelian TQFT

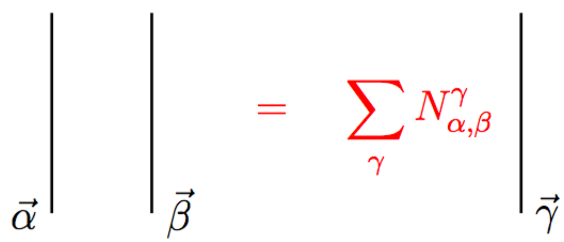
- **In addition to G, we must specify the following data**
  - 1) Braiding



  - 2) Spin (not independent from B)



  - 3) Fusion (trivial for Abelian theories)



$$N^{\gamma}_{\alpha,\beta} = \delta_{\alpha\beta,\gamma}$$

- **Note $B(\alpha,\beta) = \frac{\theta(\alpha+\beta)}{\theta(\alpha)\theta(\beta)}$, so Abelian TQFTs are specified by (G,θ).**

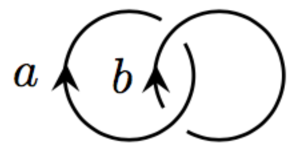
Abelian TQFT

- **This data can be used to construct matrices S and T,**
  - T-matrix:

$$T_{ab} = \theta(a)\delta_{ab}$$

  - S-matrix:

$$S_{ab} = \frac{1}{\sqrt{|G|}}$$



$$= \frac{1}{\sqrt{|G|}} \frac{\theta(a\bar{b})}{\theta(a)\theta(b)}$$

which have properties similar to 2d S- and T-matrices.

- **This is true for both Abelian and non-Abelian (with slight adjustments to the formula for S)**

Lagrangian subgroups

- **A complete characterization of gapped boundaries for Abelian TQFT was given by:** [Kapustin, Saulina '10; Levin '13]

- **Theorem:** (G, θ) **has a gapped boundary** *if and only if* **there is a Lagrangian subgroup** $L \subset G$**.**

  – A *Lagrangian subgroup* is a subset of lines such that:
    1) $\theta(\alpha) = 1 \quad \forall \;\; \alpha \in L$
    2) $|L| = \sqrt{|G|}$

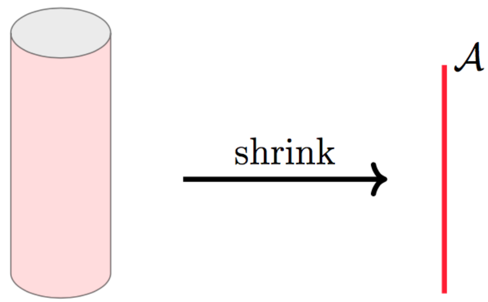
  – Physically, L is a maximal non-anomalous 1-form symmetry. If we gauge L, the theory becomes trivial.

- **Example:** $U(1)_{2N_1} \times U(1)_{-2N_2}$
  – One-form symmetry is $G = \mathbb{Z}_{2N_1} \times \mathbb{Z}_{2N_2}$, with $|G| = 4N_1N_2$.
  – Lagrangian subgroup must have order $|L| = 2\sqrt{N_1N_2} \Rightarrow N_1N_2$ is a perfect square.
  – Turns out that N_1N_2 a perfect square is also sufficient to have such an L

Geometric point of view

- **We now give a geometric interpretation of this theorem (in fact, what we say here will be applicable to Abelian and non-Abelian)**

- **Say that our theory has a gapped boundary. Then we can hollow out a tube with the gapped bc on it:**

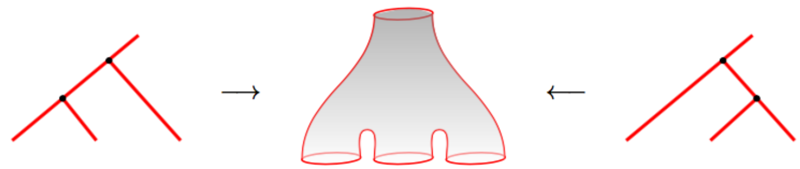


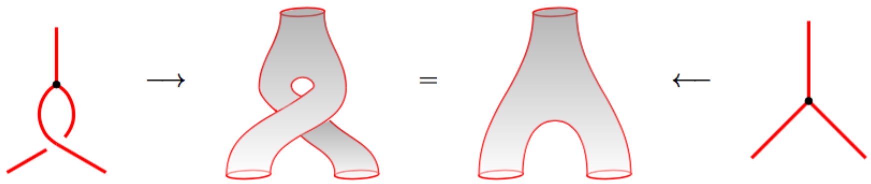
- **Because the configuration is topological, we can shrink it to a line. This defines a (non-simple) anyon,**

$$\mathcal{A} = \bigoplus_a Z_{0a}\, a$$

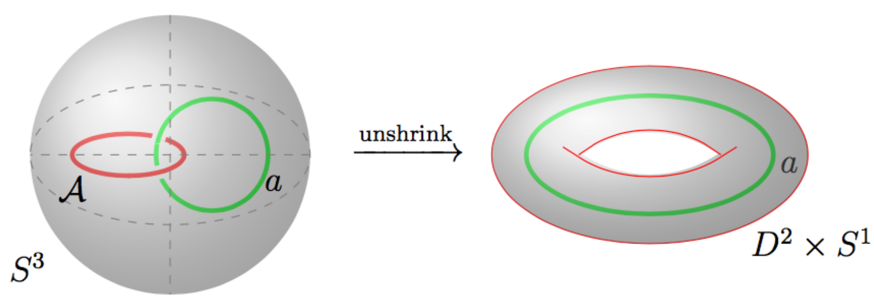
Geometric point of view

- **Properties of this anyon:**
  - Fusion and braiding are trivial:





  - Twisting it does nothing, i.e. $\theta(a) = 1$ for all a with $Z_{0a} \neq 0$. Thus $TZ = Z$.
  - By considering the Hopf link in S^3, we can also prove that $SZ = Z$.



Geometric point of view

- **If we consider the $a = 0$ component of $\sum_b S_{ab} Z_{0b} = Z_{0a}$, we get**

$$\dim(\mathcal{A}) = \sqrt{\sum d_a^2}$$

where $\dim(\mathcal{A}) := \sum_a Z_{0a} d_a$ (here $d_a = \frac{S_{0a}}{S_{00}}$ is the "quantum dimension").

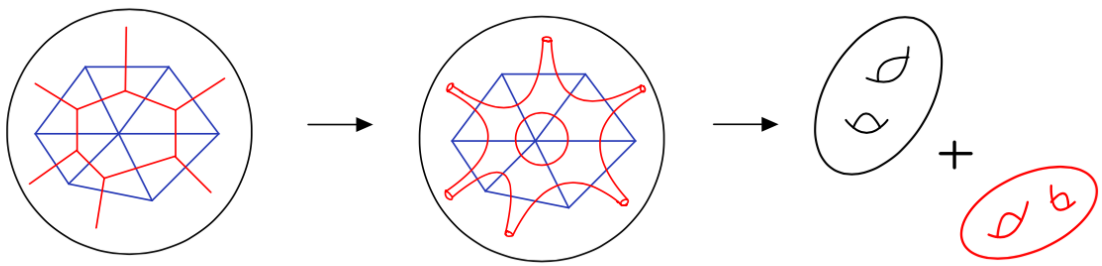
  - For Abelian TQFTs $d_a = 1$ and so $|\dim(\mathcal{A})| = \sqrt{|G|}$.
  - In this case we can think of $\mathcal{A}$ as a sum over elements of the Lagrangian subgroup!
  - But $\mathcal{A}$ also exists in *non-Abelian* theories with gapped boundaries

- **Any anyon $\mathcal{A}$ with the above properties is known as a *Lagrangian algebra anyon*.**
  - *Every* TQFT with a gapped boundary has such an $\mathcal{A}$.
  - We can obtain the trivial TQFT by *gauging* $\mathcal{A}$.

Gauging a Lagrangian algebra

- **By *gauging* a Lagrangian algebra, we mean inserting a fine mesh of it (fine mesh = dual to triangulation)** [Frölich, Fuchs, Runkel, Schweigert '09]

- **Replacing $\mathcal{A}$ by empty tubes, the spacetime becomes a handlebody of some genus g ("Heegaard splitting")**



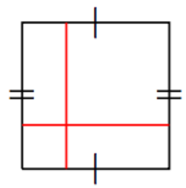
- **Partition function is then given by $Z_{\text{gauged}} = Z(D^3)^{1-g} = 1$**
  - So gauging $\mathcal{A}$ gives a trivial theory!

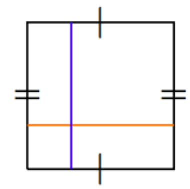
- **By putting "Dirichlet boundary conditions" at the interface between gauged and ungauged theory, we get the promised gapped boundary.**

Boundaries and Dijkgraaf-Witten theory

- **In the Abelian case, since $\mathcal{A} = \bigoplus_{a \in L} a$, "gauging" of $\mathcal{A}$ corresponds to usual gauging of the Lagrangian subgroup L. Schematically,**

$$Z\left(\quad \right) = \sum_{a,b \in L} Z\left(\quad \right)$$





- **Upon gauging a 1-form symmetry L, we get a *dual* 0-form symmetry $\widehat{L}$.**

- **Conversely, gauging $\widehat{L}$ takes us back to the original theory.**

- **Since after gauging L we get the trivial theory, we conclude that an Abelian TQFT admits a gapped boundary if and only if it is an $\widehat{L}$ gauge theory! (i.e. Dijkgraaf-Witten theory)**

Alternative to Lagrangian subgroup (Abelian)

- **Given a TQFT, actually *finding* a Lagrangian algebra anyon is not straightforward.**
  - Requires knowledge of F matrices etc. . .
- **We will now give an alternative set of *necessary and sufficient* conditions for the existence of a gapped boundary for *Abelian* TQFTs**
- **First a reminder: chiral central charge c_- is the simplest obstruction to gapped b.c.**
  - c_- captures the perturbative gravitational anomaly of the boundary theory, which must be matched by gapless edge modes.
- **The chiral central charge can be written in terms of defining data of the Abelian TQFT as**

$$e^{2\pi i c_-/8} = \frac{\sum_{\alpha\in G}\theta(\alpha)}{\left|\sum_{\alpha\in G}\theta(\alpha)\right|}$$

Higher Central Charges

- **One may generalize to *higher* central charges** [Ng, Schopieray, Wang '18; Ng, Rowell, Wang, Zhang '20]

$$\xi_n := \frac{\sum_{\alpha\in G}\theta(\alpha)^n}{\left|\sum_{\alpha\in G}\theta(\alpha)^n\right|}$$

- **Theorem:** (G,θ) **has a gapped boundary *only if* $\xi_n = 1$ for all n such that** $\gcd(n,|G|)=1$**.** [math papers above]

- **By extending the range of n, we can get necessary *and sufficient* conditions:**

- **Theorem:** (G,θ) **has a gapped boundary *if and only if* $\xi_n = 1$ for all n such that** $\gcd(n,\frac{2|G|}{\gcd(n,2|G|)})=1$**.** **[KKOSS]**

Examples

- **Example:** $U(1)_2 \times U(1)_{-4}$
  - $U(1)_2$ has $\theta = \{1,\, i\}$
  - $U(1)_{-4}$ has $\theta = \{1,\, -1,\, 2 \times e^{-2\pi i/8}\}$
  - Putting them together, $U(1)_2 \times U(1)_{-4}$ has $\theta = \{\pm 1,\, \pm i,\, 2 \times e^{\pm 2\pi i/8}\}$
  - Note that $|G| = |\mathbb{Z}_2 \times \mathbb{Z}_4| = 8$, so the relevant set of n is

$$\gcd\left(n, \frac{16}{\gcd(n,16)}\right) = 1 \qquad \Rightarrow \qquad n = 1,3,5,7$$

  - We compute $\xi_1 = \xi_7 = 1$, but $\xi_3 = \xi_5 = -1$. So *no* gapped bdy!

- **By similar means, we can show that $U(1)_2 \times U(1)_{-8}$ *does* admit a gapped boundary!**

Obstructions from 3-manifolds

- **Now let us give a geometric interpretation of the higher central charges**
- **Recall that c_- is the phase of the S^3 partition function (in some scheme).**
- **It turns out that ξ_n are the phases of lens space $L(n,1)$ partition functions!**
  - Can be shown via surgery presentation (note $S^3 \cong L(1,1)$)

- **In fact, we have the following general result:**
- **Theorem:** **An Abelian TQFT (G,θ) has a gapped boundary *only if* $Z(G,M) > 0$ for every closed 3-manifold M such that $\gcd(|G|,|H_1(M)|) = 1$.** [KKOSS]
  - The condition $\gcd(|G|,|H_1(M)|) = 1$ generalizes $\gcd(|G|,n) = 1$

Obstructions from 3-manifolds

- **Theorem:** **An Abelian TQFT (G, θ) has a gapped boundary *only if* $Z(G, M) > 0$ for every closed 3-manifold M such that $\gcd(|G|, |H_1(M)|) = 1$.** [KKOSS]

- **Proof:** **Abelian TQFT has gapped boundary iff there exists a Lagrangian subgroup L that we can gauge to get the trivial theory,**

$$\frac{|H^0(M,L)|}{|H^1(M,L)|} \sum_{a \in H^2(M,L)} Z(G, M, a) = 1$$

But it turns out that

$$\gcd(|G|, |H_1(M)|) = 1 \Rightarrow \gcd(|L|, |H_1(M)|) = 1 \Rightarrow H_1(M, L) = 0 \Rightarrow H^2(M, L) = 0$$

Hence the gauging is trivial, and

$$Z(G, M) = \frac{1}{|L|} > 0$$

Conclusions

1. **We have given a geometric reinterpretation of various algebraic properties of Lagrangian algebras.**

2. **We have found new necessary and sufficient conditions for Abelian TQFT to admit a gapped boundary.**
   – Highly computable; easier than looking for a Lagrangian subgroup!

3. **We have identified an infinite number of obstructions for Abelian TQFT, labelled by 3-manifolds.**
   – Again easily computable in terms of S and T matrices.

- **Can we generalize 2. and 3. to the non-Abelian case??**

The End (for now)

Thank you!