Calabi-Yau categories and topological field theories

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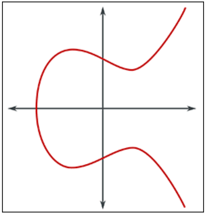
Calabi-Yau varieties

Algebraic geometry $\approx$ the study of varieties (spaces defined by polynomial equations).

Example of a variety: elliptic curve



$$y^2 = x^3 - x + 1$$

This diagram shows the real slice of an elliptic curve.

Calabi-Yau varieties

There are three basic building blocks in algebraic geometry:

1. Fano varieties ("positive" case, curvature > 0)
2. canonically polarized varieties ("negative" case, curvature < 0)
3. Calabi-Yau varieties ("neutral" case, curvature $= 0$)

In this talk, we will focus on the "neutral", or Calabi-Yau case.

A variety is said to be Calabi-Yau if its canonical line bundle is trivial.

Derived category

$$\text{variety } X \quad \Rightarrow \quad \text{derived category } \mathcal{D}(X)$$

The theory of derived categories:

- beautiful in itself
- has applications in topology, representation theory, etc.
- homological mirror symmetry

How does the derived category $\mathcal{D}(X)$ reflect the geometry of X?

The study of this question is (one branch of) noncommutative algebraic geometry.

Calabi-Yau categories

There are three basic building blocks in algebraic geometry:

1. Fano varieties ("positive" case)
2. canonically polarized varieties ("negative" case)
3. **Calabi-Yau varieties** ("neutral" case) $\leftarrow$ we focus on this.

What properties does $\mathcal{D}(X)$ have that reflects the Calabi-Yau-ness of X?

Because a Calabi-Yau variety is "neutral", its derived category has a self-dual structure.

Calabi-Yau categories

Definition

Calabi-Yau category = triangulated category with a self-dual structure

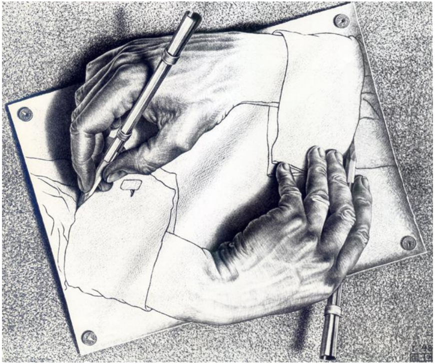
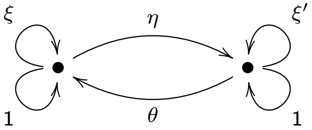


Figure: M.C. Escher, “Drawing Hands”

Some examples

$X =$ elliptic curve. Its derived category is described in

A. Polishchuk, "A_∞-algebra of an elliptic curve and Eisenstein series"



$$m_2(\theta, \eta) = \xi\,, \qquad m_2(\eta, \theta) = \xi'$$

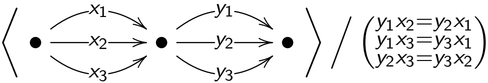
(with higher A_∞-structures $m_3, m_4, \ldots$)

The arrows come in pairs, pointing in opposite directions.
$\Rightarrow$ Self-dual, hence it is a Calabi-Yau category.

Some (non)examples

The projective space $\mathbb{P}^2$. (Fano variety, or "positive" case)

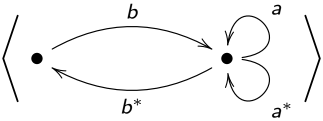
Beilinson's description of its derived category



All arrows point to the right. Not self dual.
This is **not** a Calabi-Yau category.

Some examples

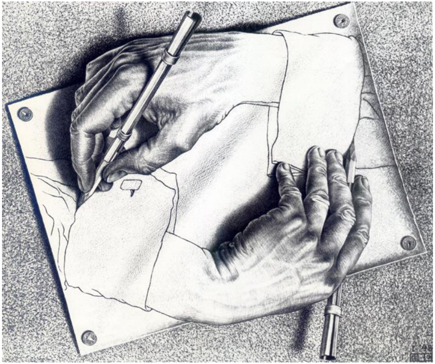
The deformed preprojective algebra:



$$\Big/ \begin{pmatrix} aa^* - a^*a - b^*b = t_1 \\ bb^* = t_2 \end{pmatrix}$$

Again, the arrows come in dual pairs.
This is a Calabi-Yau category.

Self-dual phenomena



Where do these self-dual phenomena come from?

Self-dual phenomena

Algebraic geometry	CY variety X	Serre duality: $H^i(X,\mathcal{E}) \cong H^{n-i}(X,\mathcal{E}^\vee)^*$ for holomorphic vector bundle $\mathcal{E}$
Topology	Closed oriented manifold M	Poincare duality: $H^i(M,L) \cong H^{n-i}(M,L^*)^*$ for local system L
Algebra, or Rep. theory	CY algebra A	Van den Bergh duality: $HH^i(A,M) \cong HH^{n-i}(A,M^*)^*$ for finite dim bimodule M

All these are instances of CY categories of dimension n.

Consequence of self-dual phenomena

	Moduli space $\mathfrak{M}$	Tangent space of $\mathfrak{M}$
Algebraic geometry	Moduli space of sheaves on X	$T_{\mathcal{E}}\mathfrak{M} \cong \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \cong H^1(X,\mathcal{E}\otimes\mathcal{E}^\vee)$
Topology	Moduli space of local systems on M	$T_L\mathfrak{M} \cong H^1(M, L\otimes L^*)$
Algebra, or Rep. theory	Moduli space of representations of A	$T_V\mathfrak{M} \cong HH^1(A, V\otimes V^*)$

Last page: $H^i \cong (H^{n-i})^*$.
Hence if $n = 2$ then $H^1 \cong (H^1)^*$.
i.e., tangent space $\cong$ cotangent space.

Given a CY category $\mathcal{C}$ of dimension 2, then the moduli space of objects on $\mathcal{C}$ has a (holomorphic) symplectic structure.

Consequence of self-dual phenomena

Theorem [Pantev-Toën-Vaquié-Vezzosi, Brav-Dyckerhoff, Y.]

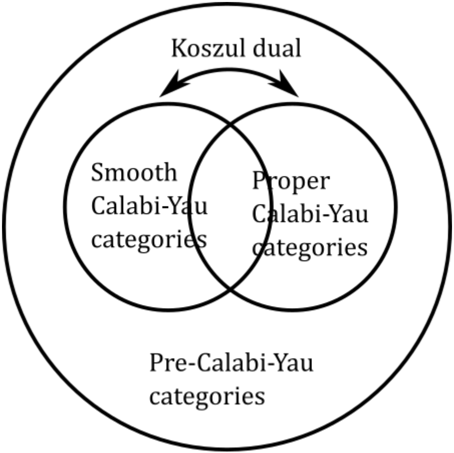
Any n-Calabi-Yau structure on A induces a $(2-n)$-shifted symplectic structure on the derived moduli stack of representations of A.

Theorem [Y.]

Any n-pre-Calabi-Yau structure on A induces a $(2-n)$-shifted Poisson structure on the derived moduli stack of representations of A.

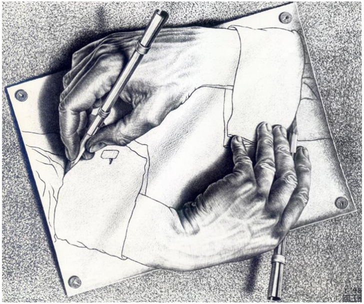
Question: how to quantize this Poisson structure?

Three types of Calabi-Yau categories



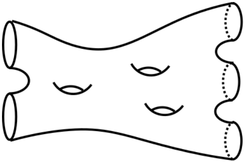
Summary

- Calabi-Yau category = triangulated category with a self-dual structure
- We conduct a general study of things with a self-dual structure
- These arise naturally in many other areas of mathematics not related to CY varieties



Topological field theory

2-dimensional topological field theories are theories with interactions controlled by Riemann surfaces:



$$\mathscr{H}^{\otimes 2} \longrightarrow \mathscr{H}^{\otimes 3}$$

Topological field theory

Hilbert space of states for the (closed) string: $\mathscr{E}$.
It is a chain complex with the BRST operator $Q : \mathscr{E}_i \to \mathscr{E}_{i-1}$.
Physical Hilbert space: $\mathscr{H} := H_\bullet(\mathscr{E}, Q)$

Let $\hat{\mathscr{M}}_{g,m,n}$ be the moduli space of Riemann surfaces of genus g and $m+n$ parametrized S^1-boundaries.

In topological string theory, functional integral gives a differential form

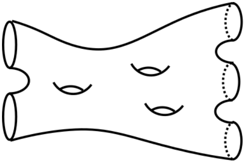
$$\omega \in \Omega^*(\hat{\mathscr{M}}_{g,m,n}, \mathrm{Hom}(\mathscr{E}^{\otimes m}, \mathscr{E}^{\otimes n}))$$

This form is closed under $d+Q$, so that it determines a map

$$H_*(\hat{\mathscr{M}}_{g,m,n}) \to \mathrm{Hom}(\mathscr{H}^{\otimes m}, \mathscr{H}^{\otimes n})$$

Topological field theory

2-dimensional topological field theory:



For each $\alpha \in H_*(\hat{\mathscr{M}}_{g,m,n})$, it gives a map

$$\alpha_* : \mathscr{H}^{\otimes m} \longrightarrow \mathscr{H}^{\otimes n}$$

Moreover, we require:

- Gluing of surfaces $\Rightarrow$ composition of maps
- Disjoint union $\Rightarrow$ tensor products
- Everything work at the level of chain complexes

Topological field theory

Proper CY category $\xRightarrow{\text{Kontsevich, Costello}}$ TFT with #output ≥ 1

Smooth CY category $\xRightarrow{\text{Lurie}}$ TFT with #input ≥ 1

Pre-CY category $\xRightarrow{\text{expected(?)}}$ TFT with #input ≥ 1, #output ≥ 1

The TFT associated to pre-Calabi-Yau categories was announced by Kontsevich. I am developing tools to obtain this (and related) results.

Remark: In fact, a CY category determine an open-closed TFT.

Two classes of examples of CY categories:
(1) Derived category $\mathcal{D}^b_{\mathrm{coh}}(X)$; (2) Fukaya category $\mathrm{Fuk}(X)$.
Input $\mathrm{Fuk}(X) \Rightarrow$ get the A-model TFT.
Input $\mathcal{D}^b_{\mathrm{coh}}(X) \Rightarrow$ get the B-model TFT.