# 3d mirror symmetry and quantum $K$-theory

Zijun Zhou

Kavli IPMU

Postdoc colloquium, Sep. 2021

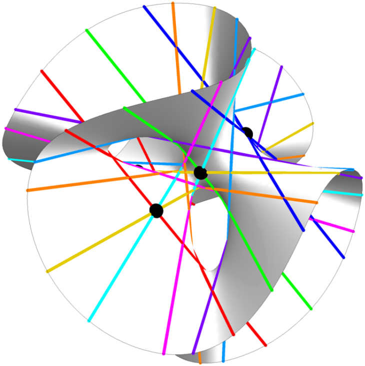
# Outline

- Introduction to enumerative geometry
- 3d mirror symmetry
- Conjectures and results

# I. Introduction to enumerative geometry

# Enumerative geometry

Example: there are 27 lines on a cubic surface (generic smooth surface in $\mathbb{CP}^3$ defined by a polynomial of degree 3).



(https://blogs.ams.org/visualinsight/2016/02/15/27-lines-on-a-cubic-surface/)

# Two viewpoints

- Count subvarieties.
  e.g. 1-dimensional subvarieties of degree 1 in a cubic surface.
  $\rightsquigarrow$ count *subschemes in a ambient space*
  or *sections of a vector bundle*.

- Count maps.
  e.g. maps from $\mathbb{P}^1$ to a cubic surface, whose image is of degree 1.
  $\rightsquigarrow$ count *maps from a domain curve to a target space*.

# Example: how about curves on $\mathbb{P}^2$?

$N_d$: number of degree $d$ rational curves on $\mathbb{P}^2$ passing through $3d-1$ points in general positions.

$N_1 = 1$.

Kontsevich–Manin:

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} N_{d_1} N_{d_2} \left( d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

For example, $N_2 = 1$, $N_3 = 12$, $N_4 = 620$, $N_5 = 87304$, $\cdots$

idea of proof: use of WDVV equation

$\rightsquigarrow$ quantum cohomology

(deformation of the usual cohomology ring, via counting of curves)

# Geometric formulation

Consider the *moduli space*

$$\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2, d) = \{f : (C, p_1, \cdots, p_{3d-1}) \to \mathbb{P}^2\},$$

where $C$ is a nodal curve, $g(C) = 0$, $f_*[C] = d[\mathbb{P}^1]$.

There are *evaluation maps* $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2, d) \to \mathbb{P}^2$, $\mathrm{ev}_i(f) = f(p_i)$.

$$N_d = \int_{\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2,d)} \prod_{i=1}^{3d-1} \mathrm{ev}_i^* P = \deg[\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2, d)] \cap \prod_{i=1}^{3d-1} \mathrm{ev}_i^* P,$$

where $P \in H^*(\mathbb{P}^2)$ is the point class.

# Modern example: genus 0 Gromov–Witten theory of quintic 3-folds

Let $Q \subset \mathbb{P}^4$ be a smooth hypersurface of degree 5.

Consider the moduli space of stable maps from genus 0 nodal curves to $Q$, with 1 marked point

$$\overline{\mathcal{M}}_{0,1}(Q,\beta) = \{f : (C,p) \to Q \mid g(C) = 0, f_*[C] = \beta \in H_2(Q)\}$$

This is a proper Delinge–Mumford stack, with evaluation map $\mathrm{ev}_1 : \overline{\mathcal{M}}_{0,1}(Q,\beta) \to Q$, $\mathrm{ev}_1(f) = f(p) \in X$.

It admits a virtual fundamental class $[\overline{\mathcal{M}}_{0,1}(Q,\beta)]^{\mathrm{vir}} \in H_*(Q)$, which allows us to define GW invariants such as (where $\gamma \in H^*(Q)$)

$\deg\left([\overline{\mathcal{M}}_{0,1}(Q,\beta)]^{\mathrm{vir}} \cap \mathrm{ev}_1^*\gamma\right) \in \mathbb{Q}$, or simply $\mathrm{ev}_{1*}[\overline{\mathcal{M}}_{0,1}(Q,\beta)]^{\mathrm{vir}} \in H_*(Q)$

# Givental's $J$-function

One can collecting enumerative invariants into generating functions.

Consider $\mathrm{ev}_1 : \overline{\mathcal{M}}_{0,1}(Q,\beta) \to Q$.

$$J(z) = \sum_\beta \mathrm{ev}_{1*} \frac{1}{-h-\psi} z^\beta \qquad \in H^*(Q)[[z]]$$

encodes the information of $g=0$ GW invariants.

$z$: Kähler parameter.

Much more difficult to compute directly!

# Insights from physics: 2d (2,2) mirror symmetry

GW theory of $Q \rightsquigarrow$ nonlinear sigma model with target space $Q$

(A model)

- Candelas–de la Ossa–Green–Parkes 90's:
  2d (2,2) mirror symmetry produces a B-model, which is mirror to A-model.
  A family of *mirror* CY 3-orbifolds. Variation of Hodge structures yields Gauss–Manin connection / Picard–Fuchs equation.
- Givental, Lian–Liu–Yau:

$$J(z) \approx I(z),$$

  $I(z)$ is the $I$-function, solution of an explicit differential equation, called Picard–Fuchs equation.

# Enumerative geometry and physics

- Supersymmetric gauge theory.

  $\rightsquigarrow$ count connections on principal bundles, or sections of vector bundles.

  Moduli of sheaves, Donaldson–Thomas theory, Pandharipande–Thomas theory.

- Topological string theory.

  $\rightsquigarrow$ count maps from algebraic curves (e.g. $\mathbb{P}^1$) to a target space.

  Gromov–Witten theory, quantum $K$-theory ($K$-theoretic GW theory)

## II. 3d mirror symmetry

# 3d $\mathcal{N} = 4$ SUSY gauge theory

Mathematically, consider a reductive group $G$, and a $G$-representation $N$.

The physics theory is associated to the pair $(G, T^*N)$, where $T^*N$ is given the canonical symplectic structure.

The *space of vacua* of such theory has two distinguished components, called *Higgs branch* and *Coulomb branch*.

Such theories are parameterized by two set of parameters, called *FI parameters* and *mass parameters*.

# 3d $\mathcal{N} = 4$ SUSY gauge theory

| Physics | Math |
|---|---|
| 3d $\mathcal{N} = 4$ SUSY gauge theory | symp. representation $(G, T^*N)$ |
| Higgs branch | holo. symp. quotient $\mu^{-1}(0)//_\theta G$ |
| Coulomb branch | BFN construction |
| FI parameter | Kähler parameter |
| mass parameter | equivariant parameter |
| R-symmetry $SU(2)_H \times SU(2)_C$ | ? |

# Example: $G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$, Higgs branch

$G$ acts on $T^*N$ by $t \cdot (x_1, \cdots, x_{n+1}, y_1, \cdots, y_{n+1}) = (tx_1, \cdots, tx_{n+1}, t^{-1}y_1, \cdots, t^{-1}y_{n+1})$.

Moment map $\mu : T^*N \to \mathbb{C}$, $\mu(\vec{x}, \vec{y}) = \sum_{i=1}^{n+1} x_i y_i$.

Choose $\theta > 0$, the Higgs branch is the GIT quotient

$$X = \mu^{-1}(0)//_\theta \mathbb{C}^* = T^*\mathbb{P}^n.$$

The action of $(\mathbb{C}^*)^{n+1}$ on $T^*\mathbb{C}^{n+1}$ descends to the quotient.

The actual torus acting on $X$ is $(\mathbb{C}^*)^{n+1}/\mathbb{C}^* \cong (\mathbb{C}^*)^n$.

# Example: $G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$, Coulomb branch

By the construction of Bravermann–Finkelberg–Nakajima, the *quantized Coulomb branch* is the noncommutative algebra $\mathcal{A}_h$ generated by $r_1, r_{-1}, H, h$ over the ring of equivariant parameters $\mathbb{C}[\lambda_1, \cdots, \lambda_{n+1}]$, up to relations

$$r_1 r_{-1} = \prod_{i=1}^{n+1} (\lambda_i + H), \qquad r_{-1} r_1 = \prod_{i=1}^{n+1} (\lambda_i + H - h).$$

Note that $r_1 H = (H - h) r_1$, $r_{-1} H = (H + h) r_{-1}$.

The *classical Coulomb branch* is $\operatorname{Spec} \mathcal{A}_0$, where $\mathcal{A}_0$ is the commutative ring obtained by setting $h = 0$. The relation becomes

$$\bar{r}_1 \bar{r}_{-1} = \prod_{i=1}^{n+1} (\lambda_i + H),$$

which is an $\mathcal{A}_n$ singularity.

One may also refer to its resolution – the $\mathcal{A}_n$ surface, as the Coulomb branch.

# 3d mirror symmetry

- Intriligator–Seiberg '96, Hanany–Witten '96, Boer–Hori–Ooguri–Oz–Yin '96, ...
- There exists mirror pairs of 3d $\mathcal{N} = 4$ theories, with
  1. Higgs branch $\leftrightarrow$ Coulomb branch
  2. FI parameters $\leftrightarrow$ mass parameters
  3. $SU(2)_H \leftrightarrow SU(2)_C$

# Example

- $G = \mathbb{C}^*$, $N = \mathbb{C}^{n+1}$.

  Higgs branch $X = T^*\mathbb{P}^n$, Coulomb branch $\approx \mathcal{A}_n$,

  Kähler parameter $z$, equivariant parameters $a_1/a_2, \cdots, a_n/a_{n+1}$.

- Mirror theory $G^! = (\mathbb{C}^*)^n$, $N^! = \mathbb{C}^{n+1}$.

  Higgs branch $X^! = \mathcal{A}_n$, Coulomb branch $\approx T^*\mathbb{P}^n$,

  Kähler parameters $z_1^!/z_2^1, \cdots, z_n^!/z_{n+1}^!$, equivariant parameter $a^!$.

- There is a natural bijection between the fixed points of $X$ and $X^!$.

# Vertex function from Higgs branch

Given a 3d $\mathcal{N} = 4$ theory $(G, T^*N)$, consider *quasimaps* into the Higgs branch $X = \mu^{-1}(0)//_\theta G$.

They are analogues of stable maps, but different:

**Definition**

- A quasimap from $\mathbb{P}^1$ to the GIT quotient $X = \mu^{-1}(0)//_\theta G$ is a map to the stacky quotient

$$f : \mathbb{P}^1 \to \mathfrak{X} = [\mu^{-1}(0)/G]$$

  which maps generically into the stable locus $X$.
- Alternatively, a principal $G$-bundle $\mathcal{P}$ over $\mathbb{P}^1$, together with a $G$-equivariant morphism $\mathcal{P} \to \mu^{-1}(0) \subset T^*M$, which maps generically into the stable locus $\mu^{-1}(0)^s$.

The *vertex function* is defined via equivariant $K$-theoretic counting of quasimaps.

$z$: Kähler parameters; $a$: equivariant parameters.

**Definition (Givental–Lee, A. Okounkov)**

Vertex function/$I$-function

$$V(z,a) := \sum_{\beta} z^{\beta} \, \mathrm{ev}_{\infty *} \, \widehat{\mathcal{O}}_{\mathrm{vir}} \in K_{T \times \mathbb{C}^*_{\hbar} \times \mathbb{C}^*_{q}}(X)_{loc}[[z]],$$

where "loc" means to pass to fraction field of $K_{T \times \mathbb{C}^*_{\hbar} \times \mathbb{C}^*_{q}}(\mathrm{pt})$.

Physics meaning of $V(z,a)$: partition function of the 3d $\mathcal{N} = 4$ theory $(G, T^*N)$ on the domain $S^1 \times_q D^2$; vortex partition function.

# Computations

Assume that $X$ admits a group action $T$ which admits isolated fixed points.

Equivariant localization $\rightsquigarrow K_T(X)_{loc} \cong K_T(X^T)_{loc}$.

For each fixed point, $V(z,a)|_p$ is easy to compute by localization methods.

# Example: $T^*\mathbb{P}^n$

We have $K_T(T^*\mathbb{P}^n) = \mathbb{C}[\hbar^{\pm 1}, a_1^{\pm 1}, \cdots, a_{n+1}^{\pm 1}, s^{\pm 1}] / \prod_{i=1}^{n+1}(1 - a_i s)$, where $s = [\mathcal{O}(-1)]$. For the fixed point $p_k$, $s|_{p_k} = a_k^{-1}$

$$\boxed{V(z,a) = \sum_{d=0}^{\infty} (-q\hbar^{-1/2})^{(n+1)d} \prod_{i=1}^{n+1} \frac{(\hbar a_i s)_d}{(q a_i s)_d} \cdot z^d}$$

which lies in $K_{T\times\mathbb{C}^*_\hbar\times\mathbb{C}^*_q}(T^*\mathbb{P}^n)_{loc}[[z]]$, where $q \in \mathbb{C}$, $|q| < 1$,

$$(y)_d := \frac{(y)_\infty}{(q^d y)_\infty} = (1-y)\cdots(1-q^{d-1}y), \qquad (y)_\infty := \prod_{l=0}^{\infty}(1-q^l y).$$

# Properties

- Vertex functions can be uniquely characterized as solutions of $q$-difference equations;
- There are two sets of $q$-difference equations, with respect to $z$ and $a$.
- One can define the quantum $K$-theory ring, which is a deformation of the usual $K$-theory ring of $X$.

# Example: $T^*\mathbb{P}^n$

$z_\sharp := (-\hbar^{1/2})^{n+1} z.$

A modification $\widetilde{V}$ of the vertex function is annihilated by the following $q$-difference operators

$$\boxed{(1 - q^{-z\partial_z})^{n+1} - z_\sharp (1 - \hbar^{-1} q^{-z\partial_z})^{n+1}.}$$

$$\boxed{(1 - q^{-a_i\partial_{a_i}})(1 - q\hbar^{-1} q^{-a_j\partial_{a_j}}) - \frac{a_j}{a_i}(1 - q\hbar^{-1} q^{-a_i\partial_{a_i}})(1 - q^{-a_j\partial_{a_j}})}$$

for any $i \neq j$.

# III. Conjectures and results

# 3d mirror symmetry for vertex functions

Let $(X, X^!)$ be Higgs branches of a 3d mirror pair. Let $V(q,z,a)$, $V^!(q,z^!,a^!)$ be their vertex functions, and $\widetilde{V}(q,z,a)$, $\widetilde{V}^!(q,z^!,a^!)$ be the renormalized vertex functions.

**Conjecture (Aganagic–Okounov, for vertex functions)**

Under the change of variables $z \mapsto a^!$, $a \mapsto z^!$, there is a (nontrivial) transition matrix $\mathcal{P} \in \mathrm{End}(K(X^T))$, such that

$$\widetilde{V}^! = \mathcal{P} \cdot \widetilde{V}.$$

Moreover, $\mathcal{P}$ is given in terms of the *elliptic stable envelope* Stab,

$$\mathcal{P}_{\mathsf{p},\mathsf{q}} \propto \frac{\mathrm{Stab}(\mathsf{p})|_{\mathsf{q}}}{\mathrm{Stab}(\mathsf{q})|_{\mathsf{q}}}.$$

This conjecture also implies a 3d mirror conjecture for Stab's.

One can also form a conjecture for the quantum $K$-theory rings.

# Current understanding of 3d mirror symmetry

- Hypertoric varieties (3d $\mathcal{N} = 4$ abelian theories) [Smirnov–Z, '19], for vertex functions and Stab;
- $T^*Fl(1, \cdots, n)$ [Dinkins, '20], for vertex functions;
- $T^*Gr$ [Rimányi–Smirnov–Varchenko–Z, '18], for Stab;
- $T^*Fl(1, \cdots, n)$ [Rimányi–Smirnov–Varchenko–Z, '18], for Stab;
- $T^*G/B$ [Rimányi–Weber, '20], for Stab;
- $T^*Fl(1, \cdots, n)$ [Koroteev–Zeitlin, '21], for quantum $K$-theory rings.

# Vertex function from Coulomb branch

[Z, '21]: introduce a generalization of BFN's construction of Coulomb branch, called the *virtual Coulomb branch*.

For $T^*\mathbb{P}^n$, relations in usual quantized $K$-theoretic Coulomb branch look like

$$r_{-1} \cdot r_1 = \prod_{i=1}^{n+1} (1 - q a_i s)$$

In virtual Coulomb branch they look like

$$r_{-1} \cdot r_1 = (-q^{1/2}\hbar^{-1/2})^{-n-1} \prod_{i=1}^{n+1} \frac{1 - q a_i s}{1 - \hbar a_i s}$$

# Vertex function from Coulomb branch

Results of [Z, '21], for an abelian theory $(G, T^*N)$ (for some of the following, nonabelian cases can also be treated via abelianization):

- $K$-theory of the moduli space of quasimaps into fixed points of $X$ can be realized as Verma modules of the virtual Coulomb branch;
  (inspired by physics work
  [Bullimore–Dimofte–Gaiotto–Hilburn–Kim, '16])
- Vertex functions can be realized as Whittaker functions;
- Quantum $q$-difference modules, quantum $K$-theory rings can be expressed in terms of the virtual Coulomb branch;
- variation of GIT of quantum $q$-difference modules.

# Interesting future directions

- Understand 3d mirror symmetry for virtual Coulomb branches;
- Understand elliptic stable envelopes from the perspective of virtual Coulomb branches;
- Relation to integrable systems via Whittaker functions.
- ...

Thank you!