Descendent series for Hilbert schemes of points

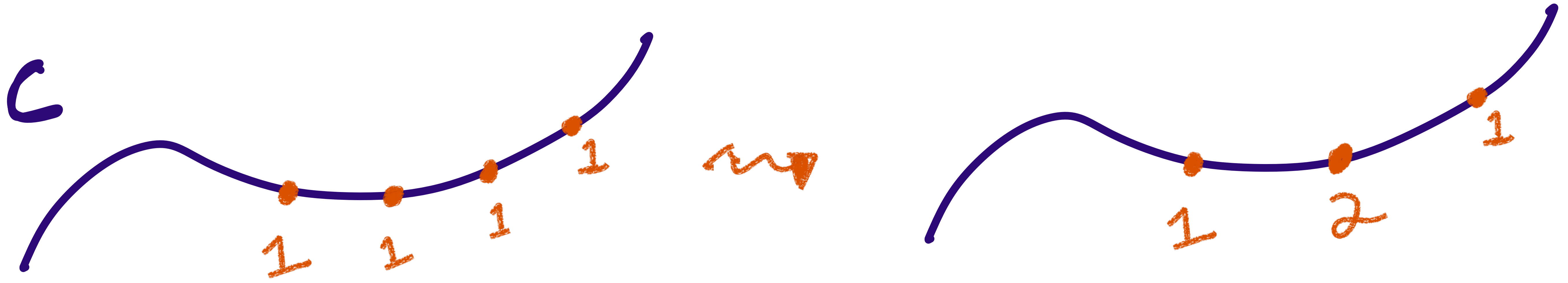
Noah Arbesfeld
(starting @ IPMU in mid-October)

Y a smooth quasi-projective variety.

$$Y^{[n]} = \left\{ Z \subset Y \;\middle|\; \begin{matrix} \dim(Z) = 0 \\ \dim_{\mathbb{C}} H^0(\mathcal{O}_Z) = n \end{matrix} \right\}$$

eg/ C a curve

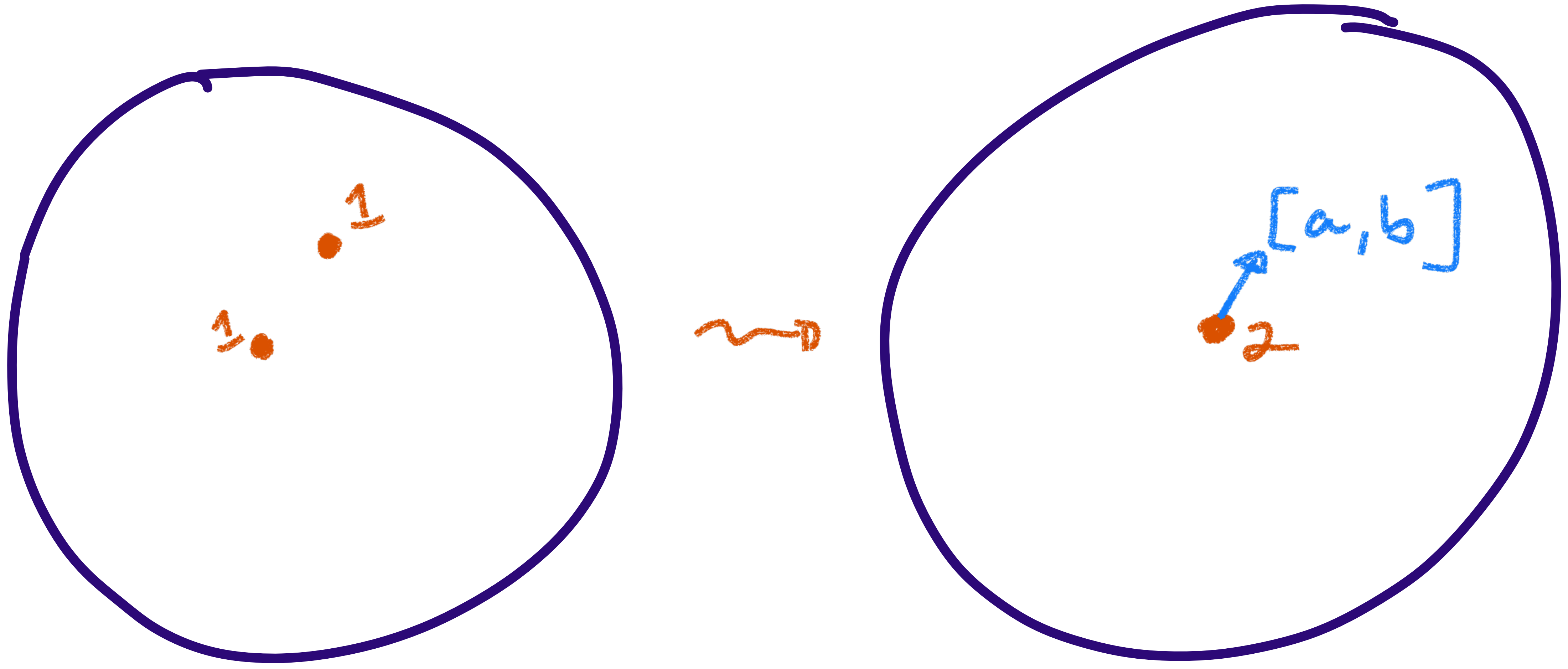
$$C^{[n]} = \left\{ \begin{matrix} \text{effective divisors} \\ \text{of degree } n \end{matrix} \right\} = \mathrm{Sym}^n C$$



$\in C^{[4]}$

$\dim Y = 2$:

$\mathbb{C}^2$



$\in \mathbb{C}^{2^{[2]}}$

$$\operatorname{Spec} \frac{\mathbb{C}[x,y]}{(x^2, xy, y^2, ax+by)}$$

Thm [Fogarty] If S is a smooth quasiprojective surface, then $S^{[n]}$ is smooth of dim $2n$.

When $\dim Y > 2$:

$Y^{[n]}$ generally highly singular of unknown dimension.

But if $\dim Y = 3$ or $\dim Y = 4$ and Y is Calabi-Yau,

then $Y^{[n]}$ can be equipped with a virtual class.

([Thomas], [Borisov-Joyce, Oh-Thomas])

Tautological bundles:

V a vector bundle of rank r on Y $\rightsquigarrow$ $V^{[n]}$ a vector bundle of rank rn on $Y^{[n]}$

$$V^{[n]}\big|_{Z\in Y^{[n]}} = H^0(Y, \mathcal{O}_Z \otimes V)$$

Examples of tautological/descendent integrals:

Fix a projective surface S, and $V \in K(S)$.

$$\int_{S^{[n]}} \mathrm{ch}_k(V^{[n]})\, c_{tot}(TS^{[n]}) \quad , \quad \chi(S^{[n]}, \Lambda^k V^{[n]})$$

$$\int_{S^{[n]}} \mathrm{segre}_{2n}(V^{[n]}) \quad , \quad \chi(S^{[n]}, \det(V^{[n]}))$$

When packaged together, tautological integrals enjoy nice properties:

eg [Ellingsrud-Göttsche-Lehn]: Fix r.

There exist "universal series" $A_1^r, \ldots, A_5^r \in \mathbb{Q}[[q,y]]$ such that:

for any projective surface S and vector bundle V on S of rank r,

one has

$$\sum_{n,k} q^n y^k \chi\left(S^{[n]}, \Lambda^k V^{[n]}\right)$$

$$\|$$

$$\left(A_1^r\right)^{c_2(S)} \left(A_2^r\right)^{c_1(S)^2} \left(A_3^r\right)^{c_1(S)c_1(V)} \left(A_4^r\right)^{c_2(V)} \left(A_5^r\right)^{c_1(V)^2}$$

$$\sum_{n,k} q^n y^k \chi(S^{[n]}, \Lambda^k V^{[n]})$$

$$\| $$

$$(A_1^r)^{c_2(S)} (A_2^r)^{c_1(S)^2} (A_3^r)^{c_1(S)c_1(V)} (A_4^r)^{c_2(V)} (A_5^r)^{c_1(V)^2}$$

Q: What properties do these universal series exhibit?

First, we look at contributions of fixed y-degree:

eg. $\sum_n q^n \chi(S^{[n]}, \mathcal{O}_{S^{[n]}}) = 1/(1-q)^{\chi(\mathcal{O}_S)}$ [Ellingsrud-Göttsche-Lehn]

• $\operatorname{rk} \mathcal{L} = 1$:

$$\sum_n q^n \chi(S^{[n]}, \Lambda^3 \mathcal{L}^{[n]}) = q^3 \binom{\chi(\mathcal{L})}{3} \Big/ (1-q)^{\chi(\mathcal{O}_S)}$$ [Scala] [Wang-Zhou]

• $\operatorname{rk} V = 2$:

$$\sum_n q^n \chi(S^{[n]}, \Lambda^3 V^{[n]}) = \frac{q^3 \binom{\chi(V)}{3} + (q^3 - q^2)\big(\chi(V \otimes \Lambda^2 V) - \chi(V)\cdot\chi(\Lambda^2 V)\big)}{(1-q)^{\chi(\mathcal{O}_S)}}$$ [A.]

Common features:
• rational functions in q
• denominator $(1-q)^{\chi(\mathcal{O}_S)}$
• numerator is polynomial of degree ≤ 3.

Thm [A.]: Let S be a projective surface, $V_1, \dots, V_\ell$ vector bundles on S, and $k_1, \dots, k_\ell \geq 0$. Then

$$\sum_n q^n \chi\left(S^{[n]}, \Lambda^{k_1} V_1^{[n]} \otimes \dots \otimes \Lambda^{k_\ell} V_\ell^{[n]}\right)$$

"K-theoretic descendent" series

is a rational function of q of the form $f(q) / (1-q)^{\chi(\mathcal{O}_S)}$

with $\deg f \leq k_1 + \dots + k_\ell$.

Particular examples shown by [Danila], [Ellingsrud-Göttsche-Lehn], [Krug], [Scala], [Wang-Zhou], [Zhou].

Cohomology?

Tautological integrals still governed by universal series, but, eg.

$$\sum_n q^n \int_{S^{[n]}} e(S^{[n]}) = \prod_{m>0} \frac{1}{(1-q^m)^{e(S)}}$$

$$\sum_n q^n \int_{S^{[n]}} ch_{k_1}(\mathcal{L}^{[n]}) \cdots ch_{k_\ell}(\mathcal{L}^{[n]})\, c_{tot}(TS^{[n]})$$

conjectured to be related to multiple q-zeta values [Okounkov]

when $S = \mathbb{C}^2$, is a "quasimodular form" [Carlsson] [Zhou]

Curves?

Fix proj. curve C, bundles $V_1, \dots, V_\ell$ and $k_1, \dots, k_\ell$.

Both

$$\sum_n q^n \chi\left(C^{[n]}, \Lambda^{k_1} V_1^{[n]} \otimes \dots \otimes \Lambda^{k_\ell} V_\ell^{[n]}\right)$$

and $$\sum_n q^n \int_{C^{[n]}} ch_{k_1}(V_1^{[n]}) \cdot \ldots \cdot ch_{k_\ell}(V_\ell^{[n]})\, c_{tot}(T C^{[n]})$$

[Johnson-Oprea-Pandharipande]

are rational functions in q.

Generalizes to higher rank: Quot schemes.

S a surface:

$$S^{[n]} = \left\{ 0 \to \mathcal{I}_Z \to \mathcal{O}_S \twoheadrightarrow \mathcal{O}_Z \to 0 \;\middle|\; \begin{array}{l} \dim(Z)=0 \\ \dim_{\mathbb{C}} H^0(\mathcal{O}_Z) = n \end{array} \right\}$$

$$TS^{[n]}\big|_Z = \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z)$$

can also be equipped with perfect obstruction theory arising from standard deformation theory of Quot schemes:

$$T^{vir} S^{[n]}\big|_Z = \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \ominus \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z).$$

$$\mathcal{O}^{vir}_{S^{[n]}} = \bigoplus_i (-1)^i \Lambda^i K_S^{[n]} \qquad\qquad [S^{[n]}]^{vir} = e\left(\left(K_S^{[n]}\right)^\vee\right) \cap [S^{[n]}]$$

Virtual K-theoretic descendent series:

Fix projective surface S, vector bundles $V_1, \ldots, V_\ell$, and $k_1, \ldots, k_\ell$.

Thm [A-Johnson-Lim-Oprea-Pandharipande]:

The series

$$\sum_n q^n \chi^{vir}\left(S^{[n]}, \Lambda^{k_1} V_1^{[n]} \otimes \cdots \otimes \Lambda^{k_\ell} V_\ell^{[n]}\right)$$

$$=$$

$$\sum_n q^n \chi\left(S^{[n]}, \Lambda^{k_1} V_1^{[n]} \otimes \cdots \otimes \Lambda^{k_\ell} V_\ell^{[n]} \otimes \mathcal{O}^{vir}\right)$$

is a rational function of q.

Virtual cohomological descendent series:

Fix a projective surface S, vector bundles $V_1, \dots, V_\ell$, and $k_1, \dots k_\ell$.

Thm [Johnson-Oprea-Pandharipande]:

The series

$$\sum_n q^n \int_{[S^{[n]}]^{vir}} ch_{k_1}(V_1^{[n]}) \cdots ch_{k_\ell}(V_\ell^{[n]})\, c_{tot}(T^{vir} S^{[n]})$$

is a rational function of q.

$$\sum_n q^n \int_{S^{[n]}} \mathrm{segre}_{2n}(V^{[n]}) \qquad\qquad \sum_n q^n \chi(S^{[n]}, \det(W^{[n]}))$$

"Segre series" "Verlinde series"

(expected to be algebraic functions)

When $\mathrm{rk}\, V = -1 - \mathrm{rk}\, W$,

[Johnson] and [Marian-Oprea-Pandharipande].

predict a relationship under change of variables

Virtual Segre-Verlinde correspondence:

Thm [A-Johnson-Lim-Oprea-Pandharipande]

$$(-1)^{n}\int_{[S^{[n]}]^{\mathrm{vir}}} \mathrm{segre}_{n}(V^{[n]}) = \chi^{\mathrm{vir}}(S^{[n]}, \det(V^{[n]}))$$

Q: Does this equality have a geometric explanation?

Universal series:

eg [Ellingsrud-Göttsche-Lehn, AJLOP]

Fix r. There exist "universal series" $B_1^r, B_2^r \in \mathbb{Q}[[q,y]]$ such that: for any projective surface S and vector bundle V on S of rank r, one has

$$\sum_{n,k} q^n y^k \chi^{vir}(S^{[n]}, \Lambda^k V^{[n]})$$

$$\|$$

$$\left(B_1^r\right)^{c_1(S)c_1(V)} \left(B_2^r\right)^{c_1(S)^2}.$$

$$\sum_{n,k} q^n y^k \chi^{vir}(S^{[n]}, \Lambda^k V^{[n]}) = (B_1^r)^{c_1(S)c_1(V)} \quad (B_2^r)^{c_1(S)^2}$$

Using a conjectural framework of [Gross-Joyce-Tanaka], work of [Bojko] shows how the series

$$B_1^r$$

is expected to control corresponding virtual descendents for Hilbert schemes of points on Calabi-Yau 4-folds!

Virtual invariants for Hilbert schemes on 4-folds:

X - a proj. Calabi-Yau 4-fold

$$X^{[n]} = \left\{ \text{torsion-free rank 1 sheaves } \mathcal{I} \;\middle|\; \begin{array}{l} \bullet \det \mathcal{I} \cong \mathcal{O}_X \\ \bullet\ \mathcal{I} \text{ cuts out 0-dim'l,} \\ \text{length } n \text{ subscheme.} \end{array} \right\}$$

Standard deformation theory: $\mathrm{Ext}^1(\mathcal{I},\mathcal{I}) - \mathrm{Ext}^2(\mathcal{I},\mathcal{I}) + \mathrm{Ext}^3(\mathcal{I},\mathcal{I})$

[Borisov-Joyce], [Oh-Thomas] produce: $[X^{[n]}]^{vir}$, $\hat{\mathcal{O}}^{vir}_{X^{[n]}}$

(very roughly, behave like virtual structures arising from a p.o.t. with

$$T^{vir}|_{\mathcal{I}} = \mathrm{Ext}^1(\mathcal{I},\mathcal{I}) - \tfrac{1}{2} \text{ of } \mathrm{Ext}^2(\mathcal{I},\mathcal{I})$$
)

Descendents for 4-folds [Bojko]:

For a vector bundle W on X, define:

$$\sum_{n,k} q^n y^k \chi^{vir}\left(X^{[n]}, \Lambda^k W^{[n]}\right)$$

$$\vdots\vdots$$

$$\sum_{n,k} q^n y^k \chi\left(X^{[n]}, \Lambda^k W^{[n]} \otimes \hat{\mathcal{O}}^{vir} \otimes \sqrt{\det(\mathcal{O}_X^{[n]})}\right)$$

Also, define a transformation $\mathcal{U}$: $\quad \mathcal{U}(f(q)) := \prod_{m>0} \prod_{i=1}^{m} f\left(-e^{\frac{2\pi i}{m}} q\right)^{-m}$

Thm [Bojko] Assuming conjectures of [Gross-Joyce-Tanaka]:

For any rank r v. bundle W on X, one has

$$\sum_{n,k} q^n y^k \chi^{vir}(X^{[n]}, \Lambda^k W^{[n]}) = \mathcal{U}(B_1^r)^{c_1(W)c_3(X)}.$$

(Recall that B_1^r is the universal series appearing in the virtual theory for surfaces:

$$\sum_{n,k} q^n y^k \chi^{vir}(S^{[n]}, \Lambda^k V^{[n]}) = (B_1^r)^{c_1(S)c_2(V)} (B_2^r)^{c_1(S)^2}$$
)

$$\sum_{n,k} q^n y^k \chi^{vir}\left(X^{[n]}, \Lambda^k W^{[n]}\right) = \mathcal{U}\left(B_1^r\right)^{c_1(W)\, c_3(X)}$$

Q: Is there a further geometric explanation for the relationship between virtual descendents for Hilbert schemes on surfaces and virtual descendents for Hilbert schemes on fourfolds?