Elliptic stable envelopes of type A quiver varieties

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Main Result

Theorem (Dinkins 2021), arxiv:2107.09569

There exists an explicit formula for the elliptic stable envelopes of (finite or affine) type A quiver varieties.

Outline

1. Cohomological stable envelopes
2. Elliptic stable envelopes
3. Main result
4. Computer examples

Fundamental papers

- (Maulik and Okounkov 2012) equivariant cohomology, constructed Yangian action, described quantum product.
- (Okounkov 2015) equivariant K-theory, quantum difference equations.
- (Aganagic and Okounkov 2016) equivariant elliptic cohomology.

Setup

- Let Q be a type A quiver with vertex set I.
- Let $\mathrm{v}, \mathrm{w} \in \mathbb{Z}^I_{\geqslant 0}$.
- $Rep(\mathrm{v}, \mathrm{w}) := \bigoplus_{i \to j} Hom(V_i, V_j) \oplus \bigoplus_{i \in I} Hom(W_i, V_i)$, where $\dim(V_i) = \mathrm{v}_i$ and $\dim(W) = \mathrm{w}_i$.
- The group $G := \prod_i GL(V_i)$ acts on $Rep(\mathrm{v}, \mathrm{w})$.
- Moment map: $\mu : T^* Rep(\mathrm{v}, \mathrm{w}) \to \mathrm{Lie}(G)^*$

The Nakajima quiver variety is defined as a GIT quotient:

$$X(\mathrm{v}, \mathrm{w}) := \mu^{-1}(0) /\!/_{\theta} G$$

We will always use the character $\theta : (g_i)_{i \in I} \mapsto \prod_{i \in I} \det(g_i)$.

Setup

The vector spaces V_i descend to bundles $\mathscr{V}_i$ on $X(\mathsf{v}, \mathsf{w})$.

Theorem (McGerty and Nevins 2018)

The ring $K_{\mathsf{T}}(X(\mathsf{v}, \mathsf{w}))$ is generated by the tautological bundles.

Torus Action

The variety $X(\mathsf{v}, \mathsf{w})$ has a natural action of a torus

$$\mathsf{T} \cong (\mathbb{C}^\times)^{\sum_i \mathsf{w}_i} \times \mathbb{C}^\times_{t_1} \times \mathbb{C}^\times_{t_2}$$

where

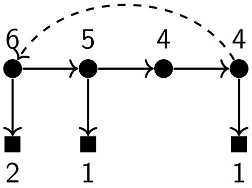
- $\mathbb{C}^\times_{t_1}$ scales the maps $V_{i+1} \to V_i$ and $W_i \to V_i$.
- $\mathbb{C}^\times_{t_2}$ scales the maps $V_i \to V_{i+1}$ and $W_i \to V_i$.

Let $\hbar = t_1 t_2$ and $a = \sqrt{\frac{t_1}{t_2}}$.

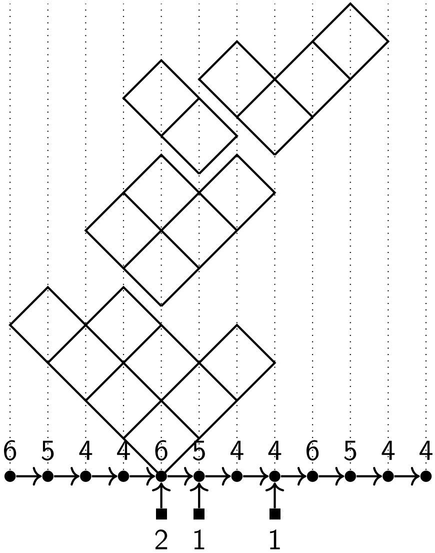
The symplectic form on $X(\mathsf{v}, \mathsf{w})$ is scaled with T-weight $\hbar^{-1}$. It is preserved by the other factors.

The A action has finitely many fixed points, indexed by $\sum_{i \in I} \mathsf{w}_i$-tuples of partitions that respect v.

Fixed point



$$\vec{\lambda} = ((4,3,2),(2,2,1),(2),(2,1,1))$$



Cohomological stable envelopes

Let $X := X(\mathsf{v}, \mathsf{w})$, $\mathsf{A} := \ker(\hbar) \subset \mathsf{T}$, $p \in X^{\mathsf{A}}$.
Each A-weight w of T_pX gives a hyperplane

$$H_w = \{w^\perp\} \subset \mathrm{Lie}_{\mathbb{R}}(\mathsf{A}) = \mathsf{cochar}(\mathsf{A}) \otimes_{\mathbb{Z}} \mathbb{R}$$

Definition

A *chamber* is a connect component of $\mathrm{Lie}_{\mathbb{R}}(\mathsf{A}) \setminus \bigcup_w H_w$, where w runs over all A-weights of the tangent space at all fixed points.

Definition

A *polarization* is a class $T^{1/2} \in K_{\mathsf{T}}(X)$ so that
$TX = T^{1/2} + \hbar \otimes (T^{1/2})^\vee$.

Cohomological stable envelopes

Fix a chamber $\mathfrak{C}$ with cocharacter σ. The chamber provides the following data:

- $\mathrm{Attr}_{\mathfrak{C}}(p) = \{x \in X \mid \lim_{z \to 0} \sigma(z) \cdot x = p\}$.
- $q < p \iff q \in \mathrm{Attr}_{\mathfrak{C}}(p)$.
- $\mathrm{Attr}^f_{\mathfrak{C}}(p) = \bigcup_{q < p} \mathrm{Attr}_{\mathfrak{C}}(q)$.
- $T_p X = N_p^+ \oplus N_p^-$

Cohomological stable envelopes

Theorem (Maulik and Okounkov 2012)

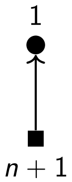
There exists a unique map of $H_{\mathsf{T}}(pt)$-modules

$$\mathrm{Stab}_{\mathfrak{C},T^{1/2}} : H_{\mathsf{T}}(X^{\mathsf{A}}) \to H_{\mathsf{T}}(X)$$

such that

- $\text{supp}\left(\mathrm{Stab}_{\mathfrak{C},T^{1/2}}(p)\right) \subset \text{Attr}^f_{\mathfrak{C}}(p)$
- $\mathrm{Stab}_{\mathfrak{C},T^{1/2}}(p)|_p = \pm\text{Euler}(N_p^-)$, where the sign depends on $T^{1/2}$.
- $\deg_{\mathsf{A}}\left(\mathrm{Stab}_{\mathfrak{C},T^{1/2}}(p)|_q\right) < \frac{1}{2}\text{codim}(q)$ for all $q \prec p$.

Example



$X = T^*\mathbb{P}^n$
$\mathsf{T} = (\mathbb{C}^\times)^{n+1} \times \mathbb{C}^\times_\hbar,$
$\mathsf{A} = (\mathbb{C}^\times)^{n+1}$
$X^\mathsf{A} = \{p_1, p_2, \ldots, p_{n+1}\}$
$H_\mathsf{T}(X) =$
$\mathbb{C}[c, u_1, \ldots, u_{n+1}, \hbar] / \left(\prod_{i=1}^{n+1}(c - u_i)\right)$

Choose $\mathfrak{C}$ such that $p_1 \prec p_2 \prec \ldots \prec p_{n+1}$.

Let

$$F_k := \prod_{i<k}(u_i - c - \hbar) \prod_{i>k}(u_i - c)$$

Check:

- $F_k|_{p_i} = 0$ if $i > k$.
- $F_k|_{p_k} = \prod_{i<k}(u_i - u_k - \hbar) \prod_{i>k}(u_i - u_k) = \mathrm{Euler}(N_p^-)$
- $\deg_\mathsf{A} F_k|_{p_i} < n$ if $i < k$.

By uniqueness of stable envelopes, $\mathrm{Stab}_{\mathfrak{C}, T^{1/2}}(p_k) = F_k$.

R-matrix

Let $\mathfrak{C}, \mathfrak{C}'$ be two chambers.

Definition

$R_{\mathfrak{C}',\mathfrak{C}} := \mathrm{Stab}^{-1}_{\mathfrak{C}', T^{1/2}} \circ \mathrm{Stab}_{\mathfrak{C}, T^{1/2}} \in End(H_{\mathsf{T}}(X^{\mathsf{A}}))_{loc}$.

Using these R-matrices, (Maulik and Okounkov 2012) construct a Yangian $Y(\mathfrak{g}_Q)$, which acts on

$$X(\mathsf{w}) := \bigoplus_{v} H_{\mathsf{T}}(X(\mathsf{v}, \mathsf{w}))$$

In K-theory, stable envelopes depend additionally on a choice of fractional line bundle. They provide a geometric construction of $U_{\hbar}(\widehat{\mathfrak{g}}_Q)$.

Applications of stable envelopes

- For $\bigsqcup_n \mathsf{Hilb}^n(\mathbb{C}^2)$, $\mathrm{Stab}(\lambda) = \mathsf{Schur}_\lambda$, (Bezrukavnikov and Okounkov).
- Used to compute the 2-leg DT vertex (Kononov, Okounkov, and Osinenko 2019).
- Puzzle rules for multiplication by Schubert classes (Knutson and Zinn-Justin 2021)
- Canonical bases of Lusztig (Hikita 2020).
- 3d mirror symmetry (Rimányi et al. 2019), (Dinkins 2020), (Kononov and Smirnov 2020).

Elliptic cohomology

Cohomology and K-theory provide affine schemes

$$H_{\mathsf{T}}(X) \rightsquigarrow \mathrm{Spec}(H_{\mathsf{T}}(X))$$
$$K_{\mathsf{T}}(X) \rightsquigarrow \mathrm{Spec}(K_{\mathsf{T}}(X))$$

The elliptic analog provides a non-affine scheme $(\mathrm{Ell}_{\mathsf{T}}(X), \mathscr{O}_{\mathrm{Ell}_{\mathsf{T}}(X)})$, (Grojnowski 1994), (Ginzburg, Kapranov, and Vasserot 1995). Global functions are replaced by sections of certain line bundles:

$$\mathrm{Stab}_{\mathfrak{C}, T^{1/2}} : \text{Line bundle on } E_{\mathsf{T}}(X^{\mathsf{A}}) \to \text{Line bundle on } E_{\mathsf{T}}(X)$$

Elliptic cohomology

Fix an elliptic curve $E = \mathbb{C}^\times / q^{\mathbb{Z}}$.
$\mathrm{Ell}_{\mathsf{T}}(X)$ is a scheme over $\mathrm{Ell}_{\mathsf{T}}(pt) = \mathrm{cochar}(\mathsf{T}) \otimes_{\mathbb{Z}} E \cong E^{\mathrm{rank}(\mathsf{T})}$.

Line bundles

Line bundles on E are determined by the transformation properties of their sections. For example, the function

$$\vartheta(x) := (x^{1/2} - x^{-1/2}) \prod_{i=1}^{\infty} (1 - q^i x)(1 - q^i / x)$$

defines a line bundle on E, whose sections satisfy the transformation property

$$f(qx) = -\frac{1}{\sqrt{q}x} f(x)$$

Line bundles

A rank r vector bundle $\mathscr{V}$ on X provides an elliptic Chern class map:

$$c : \mathsf{Ell}_\mathsf{T}(X) \to S^r E$$

Let $D = \{0\} + S^{r-1}E \subset S^r E$. "Coordinates" $x_1, \dots x_r$ on $S^r E$ are called *elliptic Chern roots*. Sections of $\mathscr{O}(D)$ transform like the function

$$\prod_{i=1}^{r} \vartheta(x_i)$$

under shifts $x_i \to q x_i$.

Definition

The elliptic Thom class of $\mathscr{V}$ is the line bundle

$$\Theta(\mathscr{V}) = \mathsf{c}^* \mathscr{O}(D)$$

on $\mathsf{Ell}_\mathsf{T}(X)$. It extends to a map $\Theta : K_\mathsf{T}(X) \to \mathrm{Pic}(\mathsf{Ell}_\mathsf{T}(X))$.

Extended elliptic cohomology

- We extend the elliptic cohomology scheme by taking the product of everything with $E^{|I|}$.
- We obtain a scheme $E_{\mathsf{T}}(X) := \mathrm{Ell}_{\mathsf{T}}(X) \times E^{|I|}$ over the base $B_X = E^{\mathrm{rank}(\mathsf{T})} \times E^{|I|}$.

Write z_i for the elliptic "coordinate" on the ith copy of E in $E^{|I|}$.

Universal line bundle

For $X = X(\mathsf{v},\mathsf{w})$, tautological generation of $K_\mathsf{T}(X)$ translates to the injectivity of the map

$$ch : \mathrm{E}_\mathsf{T}(X) \to E^{\mathrm{rank}(\mathsf{T})} \times E^{|I|} \times \prod_{i\in I} S^{\mathsf{v}_i} E$$

Let $\mathcal{U}$ be the line bundle $E^{\mathrm{rank}(\mathsf{T})} \times E^{|I|} \times \prod_{i\in I} S^{\mathsf{v}_i} E$ associated to the function

$$\prod_{i\in I} \phi\left(z_i, \prod_{j=1}^{\mathsf{v}_i} x_{i,j}\right), \quad \text{where} \quad \phi(a,b) := \frac{\vartheta(ab)}{\vartheta(a)\vartheta(b)}$$

Definition (Aganagic and Okounkov 2016)

The line bundle $\mathscr{U} := ch^*\mathcal{U}$ is called the universal line bundle on $E_\mathsf{T}(X)$.

Elliptic stable envelopes

Definition

The elliptic stable envelope of X is the unique map of $\mathscr{O}_{B_X}$-modules

$$\mathrm{Stab}_{\mathfrak{C},T^{1/2}} : \Theta(\hbar)^{-\mathrm{rank}(T^{1/2}_{>0})} \otimes \mathscr{U}' \to \Theta(T^{1/2}) \otimes \mathscr{U}$$

satisfying a support and normalization condition. Here $\mathscr{U}'$ is a certain shift of $\mathscr{U}|_{E_{\mathsf{T}}(X^{\mathsf{A}})}$.

- (Aganagic and Okounkov 2016) proved that elliptic stable envelopes exist for hypertoric varieties and quiver varieties.
- The construction of elliptic stable envelopes was generalized in (Okounkov 2020). It depends on the existence of an *attractive line bundle*, which is given by $\Theta(T^{1/2})$ above.

Elliptic stable envelopes

When X^{A} is finite, we have

$$E_{\mathsf{T}}(X^{\mathsf{A}}) \cong \bigsqcup_{p \in X^{\mathsf{A}}} B_p, \quad B_p = B_X$$

and

$$E_{\mathsf{T}}(X) \cong \Big(\bigsqcup_{p \in X^{\mathsf{A}}} B_p \Big) / \text{Gluing data}$$

And it is equivalent to construct a section $\mathrm{Stab}_{\mathfrak{C}, T^{1/2}}(p)$ of the bundle

$$\Theta(\hbar)^{\mathrm{rank}(T^{1/2}_{>0,p})} \otimes \mathscr{U}'^{-1}_{B_p} \otimes \Theta(T^{1/2}) \otimes \mathscr{U}$$

over $E_{\mathsf{T}}(X)$ for each $p \in X^{\mathsf{A}}$. The two conditions mean that

- $\mathrm{Stab}_{\mathfrak{C}, T^{1/2}}(p)|_{B_{p'}} = 0$ if $p < p'$.
- $\mathrm{Stab}_{\mathfrak{C}, T^{1/2}}(p)|_{B_p} = \prod_{w \in \mathrm{char}_{\mathsf{T}}(N_p^-)} \vartheta(w)$

Elliptic stable envelopes

- Recall
$$ch : \mathsf{E}_\mathsf{T}(X) \to E^{\mathsf{rank}(\mathsf{T})} \times E^{|I|} \times \prod_{i \in I} S^{\mathsf{v}_i} E$$
- $\mathrm{Stab}_{\mathfrak{C}, T^{1/2}}(p)$ is a section of a line bundle pulled-back from the right side.
- We will describe a section of the line bundle on the right side that pulls back to $\mathrm{Stab}_{\mathfrak{C}, T^{1/2}}(p)$ under *ch*.

This is called the “off-shell” description of the stable envelope.

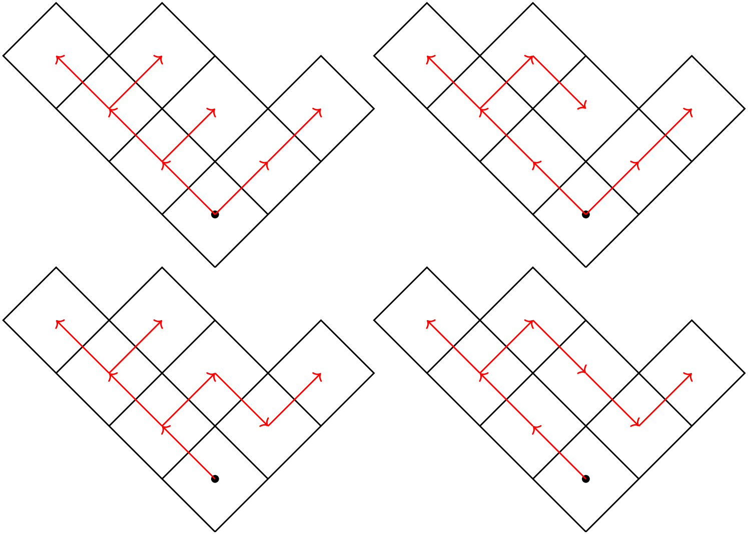
Trees in partitions

Recall: Fixed points on $X(\mathrm{v}, \mathrm{w}) \leftrightarrow \sum_{i \in I} \mathrm{w}_i$-tuples of partitions that respect v.

Consider oriented trees in a partition λ. We forbid trees with certain types of edges.

Example

There are four admissible trees in $(4,3,1)$.



Tree weights

Fix $X := X(\mathsf{v}, \mathsf{w})$, $\mathfrak{C}$, $T^{1/2}$, and $\vec{\lambda} \in X^{\mathsf{A}}$. We define

- $S_{\vec{\lambda}}$, an elliptic function of the equivariant parameters, Kähler parameters, and elliptic Chern roots, depending on $\vec{\lambda}$.
- $\mathsf{W}_{\vec{\mathsf{t}}}$, an elliptic function of the same variables, for each tuple of trees in $\vec{\lambda}$.

Theorem (Dinkins 2021)

The elliptic stable envelope $\mathrm{Stab}_{\mathfrak{C}, T^{1/2}}(\vec{\lambda})$ is the pullback of the section

$$\mathsf{Sym}_1 \mathsf{Sym}_2 \dots \mathsf{Sym}_{|I|} \left(S_{\vec{\lambda}} \sum_{\vec{\mathsf{t}}} W_{\vec{\mathsf{t}}} \right)$$

under the map *ch*, where the sum is taken over all tuples of admissible trees $\vec{\mathsf{t}}$ in $\vec{\lambda}$ and Sym_i is the symmetrization over the elliptic Chern roots of each $\mathscr{V}_i$.

Proof sketch

The proof uses abelianization techniques, which relates properties of $Y/\!/G$ to properties of $Y/\!/S$, where $S \subset G$ is a maximal torus. In our case, there is a diagram:

$$\begin{array}{ccccc} \mu_{U,\mathbb{R}}^{-1}(\theta) \bigcap \mu_{U,\mathbb{C}}^{-1}(0)/(U \cap S) & \xrightarrow{\mathrm{j}_+} & \mu_G^{-1}(\mathfrak{b}^{\perp})^{\theta_S-ss}/S & \xrightarrow{\mathrm{j}_-} & AX \\ \big\downarrow \pi & & & & \\ X & & & & \end{array}$$

where $U \subset G$ is a maximal compact subgroup and $B \subset G$ is a Borel subgroup.

There is a similar diagram for the fixed locus.

Proof sketch

(Aganagic and Okounkov 2016) construct the elliptic stable envelope as the composition

$$\mathrm{Stab}_{\mathfrak{C},T^{1/2}} := \pi_* \circ \mathrm{j}_+^* \circ (\mathrm{j}_{-*})^{-1} \circ \mathrm{Stab}'_{\mathfrak{C},T^{1/2}_{AX}} \circ \mathrm{j}'_{-*} \circ ({\mathrm{j}'_+}^*)^{-1} \circ {\pi'_*}^{-1}$$

where

$$\begin{array}{ccc} \mathscr{U}' & \xrightarrow{\mathrm{j}'_{-*} \circ ({\mathrm{j}'_+}^*)^{-1} \circ {\pi'_*}^{-1}} & \Theta\left(T^{1/2}A\vec{\lambda}\right) \otimes \mathscr{U}' \\ \Big\downarrow \mathrm{Stab}_{\mathfrak{C},T^{1/2}} & & \Big\downarrow \mathrm{Stab}'_{\mathfrak{C},T^{1/2}_{AX}} \\ \Theta\left(T^{1/2}X\right) \otimes \mathscr{U} & \xleftarrow{\pi_* \circ \mathrm{j}_+^* \circ \mathrm{j}_{-*}^{-1}} & \Theta\left(T^{1/2}AX\right) \otimes \mathscr{U} \end{array}$$

Roughly, the sum over trees provides the top map. The vertical and bottom maps are given explicitly in (Aganagic and Okounkov 2016).

Why trees?

The abelianization $A\vec{\lambda}$ of a fixed point $\vec{\lambda}$ is a nontrvial variety. Trees index torus fixed points on $A\vec{\lambda}$. The sum over admissible trees leads to a miraculous (and necessary) cancellation.

Computer examples

I have written a package implementing the formulas described here. It is available at `https://tarheels.live/dinkins/`.

A choice of chamber is equivalent to an ordering of the equivariant parameters.
Quiver varieties have many natural polarizations, given by a choice of half the arrows in the framed-doubled quiver.

Future directions

- 3d mirror symmetry
- Can this formula be generalized to Cherkis bow varieties?

Questions?