Diagram automorphisms and canonical bases for quantum groups

Toshiaki Shoji

(joint work with Y. Ma and Z. Zhou)

Tongji University

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Canonical bases $\mathbf{B}$ and $\underline{\mathbf{B}}$

$\mathfrak{g}$: Kac-Moody algebra assoc. to symmetric Cartan Datum X

$\mathbf{U}_q(\mathfrak{g})$: quantum group assoc. to $\mathfrak{g}$ over $\mathbf{Q}(q)$
$\mathbf{U}_q^-$: negative part of $\mathbf{U}_q(\mathfrak{g})$

$\mathfrak{g}^\sigma$: orbit algebra obtained from $\mathfrak{g}$ by admissible autom. $\sigma : X \to X$
$\underline{\mathbf{U}}_q^-$: negative part of $\mathbf{U}_q(\mathfrak{g}^\sigma)$

By using the geometry of quivers, Lusztig proved;

Theorem (Lusztig)

1. There exists the canonical basis $\mathbf{B}$ for $\mathbf{U}_q^-$. σ acts on $\mathbf{B}$ as a permutation. Let $\mathbf{B}^\sigma = \{b \in \mathbf{B} \mid \sigma(b) = b\}$.
2. There exists the canonical signed basis $\widetilde{\underline{\mathbf{B}}}$ of $\underline{\mathbf{U}}_q^-$, and the natural bijection $\widetilde{\mathbf{B}}^\sigma \simeq \widetilde{\underline{\mathbf{B}}}$, where $\widetilde{\mathbf{B}}^\sigma = \mathbf{B}^\sigma \cup -\mathbf{B}^\sigma$.

Geometric construction of canonical bases

Assume, for simplicity, X : finite type, simply-laced
$Q = (I, \Omega)$: a quiver assoc. to X, I : vertex set, Ω : oriented edges

$V = \bigoplus_{i \in I} V_i$: representation space of V.
$G_V = \prod_{i \in I} GL(V_i)$ acts naturally on V.
$\sharp$ of G_V-orbits on V : finite (since X : finite type)

$\mathscr{P}_V = \{\mathrm{IC}(\overline{\mathscr{O}}, \bar{\mathbb{Q}}_l)[\dim \mathscr{O}] \mid \mathscr{O} : G_V\text{-orbit in } V\}$,
the set of G_V-equivariant simple perverse sheaves on V
Put $\mathscr{P}_Q = \bigsqcup_V \mathscr{P}_V$

$\mathscr{Q}_V$: fullsubacategory of $D^b_c(V)$, objects : complexes of the form

$$\bigoplus_{L,i} L[i], \qquad L \in \mathscr{P}_V, \quad i \in \mathbf{Z} \quad \text{(finite direct sum)}$$

$\mathbf{K}(\mathscr{Q}_V)$; Grothendieck group of $\mathscr{Q}_V$, set $\mathbf{K}(\mathscr{Q}) = \bigoplus_V \mathbf{K}(\mathscr{Q}_V)$
$\mathscr{P}_Q$ gives a basis of $\mathbf{K}(\mathscr{Q})$

${}_{\mathbf{A}}\mathbf{U}_q^-$: Lusztig's integral form of $\mathbf{U}_q^-$, $\mathbf{A}$-subalg. of $\mathbf{U}_q^-$, where $\mathbf{A} = \mathbf{Z}[q, q^{-1}]$

Lusztig proved;

- Grothendieck group $\mathbf{K}(\mathscr{Q}_Q)$ has a structure of $\mathbf{A}$-algebra, and there exists an isomorphism of $\mathbf{A}$-algebras

$$\varphi : \mathbf{K}(\mathscr{Q}_Q) \simeq {}_{\mathbf{A}}\mathbf{U}_q^-. \tag{1}$$

  Canonical basis $\mathbf{B}$ is defined by $\mathbf{B} = \varphi(\mathscr{P}_Q)$.

Remark. For X: **symmetric** Cartan datum, the Grothendieck group $\mathbf{K}(\mathscr{Q}) = \bigoplus_V \mathbf{K}(\mathscr{Q}_V)$ can be defined, and (1) holds.
But the construction of the category $\mathscr{Q}_V$ is more complicated.

σ-setup for the category $\mathscr{Q}_V$

Fix an orientation of Q : compatible with $\sigma : X \to X$
σ induces a functor $\sigma^* : \mathscr{Q}_Q \to \mathscr{Q}_Q$

$\widetilde{\mathscr{Q}}_Q$: category with autom, objects : (C, ϕ),
where $C \in \mathscr{Q}_Q$ such that $\phi : \sigma^* C \xrightarrow{\sim} C$ with certain conditions

- "Modified" Grothendieck group $\mathbf{K}(\widetilde{\mathscr{Q}}_Q)$ has a structure of $\mathbf{A}$-algebra, and there exists an $\mathbf{A}$-algebra isomorphism

$$\mathbf{K}(\widetilde{\mathscr{Q}}_Q) \simeq {}_{\mathbf{A}}\underline{\mathbf{U}}_q^- \qquad (2)$$

In (1), simple object : $A \in \mathscr{P}_Q \Longleftrightarrow$ **canonical base** in $\mathbf{B}$,

In (2), simple object : (A, ϕ) with $A \in \mathscr{P}_Q \Longleftrightarrow$ **canonical signed base** in $\widetilde{\underline{\mathbf{B}}} = \underline{\mathcal{B}} \sqcup -\underline{\mathcal{B}}$ (here $\mathcal{B}$: a basis of $\underline{\mathbf{U}}_q^-$, but not unique)

The forgetful functor $(A, \phi) \mapsto A$ gives a map $\widetilde{\underline{\mathbf{B}}} \to \widetilde{\mathbf{B}} = \mathbf{B} \sqcup -\mathbf{B}$,
induces a bijection $\quad \underline{\widetilde{\mathbf{B}}} \xrightarrow{\sim} \widetilde{\mathbf{B}}^\sigma$

Kashiwara's theory of crystal bases

- Lusztig obtained the canonical basis $\underline{\mathbf{B}}$ of $\underline{\mathbf{U}}_q^-$ from $\underline{\widetilde{\mathbf{B}}}$, by using Kashiwara's theory of crystals, and proved the bijection $\underline{\mathbf{B}} \xrightarrow{\sim} \mathbf{B}^\sigma$.

In this talk, we give an alternate approach for the construction of the canonical signed basis $\underline{\widetilde{\mathbf{B}}}$ of $\underline{\mathbf{U}}_q^-$, and the bijection $\underline{\widetilde{\mathbf{B}}} \xrightarrow{\sim} \widetilde{\mathbf{B}}^\sigma$, **assuming the existence of canonical basis** $\mathbf{B}$ of $\mathbf{U}_q^-$. Once $\mathbf{B}$ is given, the discussion in other parts are elementary, in the sense we don't use the geometry of quivers, nor the theory of crystal bases.

Remark. Similar results were obtained by S.-Zhou if X is finite or affine type, by using PBW-bases of $\mathbf{U}_q^-$. In the general case, we use $\mathbf{B}$ instead of PBW-bases.

By Lusztig-Grojnowski, **canonical bases = global crystal bases**
An approach from crystal bases theory for the proof $\underline{\mathbf{B}} \simeq \mathbf{B}^\sigma$

- Naito-Sagaki : Use Littelmann's path model realization of crystal bases.

Diagram automorphism on the Cartan datum

$X = (I, (\ ,\ ))$: Cartan datum,

I : vertex set with $|I| < \infty$, $(\ ,\ )$: symmetric bilinear form on $\bigoplus_{i \in I} \mathbf{Q}\alpha_i$, with $(\alpha_i, \alpha_j) \in \mathbf{Z}$, satisfying the properties

- $(\alpha_i, \alpha_i) \in 2\mathbf{Z}_{>0}$ for any $i \in I$,
- $\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \in \mathbf{Z}_{\leq 0}$ for any $i \neq j \in I$.

The matrix $(a_{ij})_{i,j \in I}$: Cartan matrix, $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$.

X : called **symmetric** if $(\alpha_i, \alpha_i) = 2$, and **simply-laced** if symmetric and $(\alpha_i, \alpha_j) \in \{0, -1\}$ for an y $i \neq j \in I$

X : Cartan datum of arbitrary type

$\sigma : I \to I$: diagram automorphism, i.e.,
permutation such that $(\alpha_{\sigma(i)}, \alpha_{\sigma(j)}) = (\alpha_i, \alpha_j)$ for any $i, j \in I$.

$\underline{I}$: the set of orbits of σ in I.

Define a symmetric bilinear form $(\ ,\ )_1$ on $\bigoplus_{\eta \in \underline{I}} \mathbf{Q}\alpha_\eta$ by

$$(\alpha_\eta, \alpha_{\eta'})_1 = \begin{cases} (\alpha_i, \alpha_i)|\eta|, \quad (i \in \eta) & \text{if } \eta = \eta', \\ \sum_{i \in \eta, j \in \eta'} (\alpha_i, \alpha_j) & \text{if } \eta \neq \eta'. \end{cases}$$

Then $\underline{X} = (\underline{I}, (\ ,\ )_1)$ defines a Cartan datum,
called the Cartan datum induced from (X, σ).

Assumption : σ is **admissible**,
i.e, for each orbit $\eta \in \underline{I}$, $(\alpha_i, \alpha_j) = 0$ for any $i \neq j \in \eta$.

Quantum groups $\mathbf{U}_q^-$ and $\underline{\mathbf{U}}_q^-$

Let q : indeterminate. Put, for $n, m \in \mathbf{Z}$ with $m > 0$,

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [m]^! = \prod_{i=1}^{m} [i], \qquad [0]^! = 1.$$

$\mathbf{U}_q = \mathbf{U}_q(X)$: quantum group assoc. to X.
$\mathbf{U}_q^-$: negative part of $\mathbf{U}_q$.

$\mathbf{U}_q^-$: assoc. algebra over $\mathbf{Q}(q)$, with generators $\{f_i \mid i \in I\}$ and relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)} = 0, \qquad (i \neq j \in I),$$

where for $i \in I$, $n \in \mathbf{N}$, $\quad f_i^{(n)} = \dfrac{f_i^n}{[n]_{d_i}^!}, \qquad d_i = (\alpha_i, \alpha_i)/2.$

(Here for $d \in \mathbf{N}$, $[n]_d$ denotes the substitution $q \mapsto q^d$ for $[n]$.)

$\sigma : I \to I$ induces an isomorphism

$$\sigma : \mathbf{U}_q^- \xrightarrow{\sim} \mathbf{U}_q^-, \qquad f_i \mapsto f_{\sigma(i)}.$$

$\mathbf{U}_q^{-,\sigma} = \{x \in \mathbf{U}_q^- \mid \sigma(x) = x\}$: subalgebra of $\mathbf{U}_q^-$

${}_{\mathbf{A}}\mathbf{U}_q^-$: Lusztig's integral form of $\mathbf{U}_q^-$, where $\mathbf{A} = \mathbf{Z}[q, q^{-1}]$
: $\mathbf{A}$-subalgebra of $\mathbf{U}_q^-$ generated by $\{f_i^{(n)} \mid i \in I, n \in \mathbf{N}\}$.

σ acts on ${}_{\mathbf{A}}\mathbf{U}_q^-$, ${}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma}$: subalgebra of σ-fixed elements

$\underline{\mathbf{U}}_q^-$: negative part of $\underline{\mathbf{U}}_q = \mathbf{U}_q(\underline{X})$ assoc. to $\underline{X}$
: $\mathbf{Q}(q)$-algebra generated by $\{\underline{f}_\eta \mid \eta \in \underline{I}\}$.

${}_{\mathbf{A}}\underline{\mathbf{U}}_q^-$: $\mathbf{A}$-subalgebra generated by $\{\underline{f}_\eta^{(a)} \mid \eta \in \underline{I}, a \in \mathbf{N}\}$.

Remark. We want to compare the algebra structure of $\mathbf{U}_q^{-,\sigma}$ and $\underline{\mathbf{U}}_q^-$.
But no direct relations exist between them.

The algebra $\mathbf{V}_q$

Assume the order of σ : a power of a prime number p.
$\mathbf{F} = \mathbf{Z}/p\mathbf{Z}$: finite field of p elements

Put $\mathbf{A}' = \mathbf{F}[q, q^{-1}]$, and consider the $\mathbf{A}'$-algebra

$$ {}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma} = {}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma} \otimes_{\mathbf{A}} \mathbf{A}' \simeq {}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma}/p({}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma}) $$

For each $x \in \mathbf{U}_q^-$, let $O(x)$ be the oribt sum $\sum_{0 \le i < k} \sigma^i(x)$, where k: smallest integer such that $\sigma^k(x) = x$.

Let J : the $\mathbf{A}'$-submodule of ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}$ generated by

$$ \{ O(x) \mid \sigma(x) \ne x, x \in {}_{\mathbf{A}'}\mathbf{U}_q^- \} $$

J: two-sided ideal of ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}$. Define a quotient algebra $\mathbf{V}_q$ by

$$ \mathbf{V}_q = {}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}/J $$

Let $\pi : {}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma} \to \mathbf{V}_q$: the natural homomorphism.

Main theorems

For each $\eta \in \underline{I}$ and $a \in \mathbf{N}$, put $\widetilde{f}_\eta^{(a)} = \prod_{i\in\eta} f_i^{(a)}$.

$f_i f_j = f_j f_i$ for $i, j \in \eta \Longrightarrow \widetilde{f}_\eta^{(a)} \in {}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma}$.

Denote also by $\widetilde{f}_\eta^{(a)}$ its image in ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}$.

Define $g_\eta^{(a)} \in \mathbf{V}_q$ by $g_\eta^{(a)} = \pi(\widetilde{f}_\eta^{(a)}) \in \mathbf{V}_q$.

Note : ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$: generated by $\{\underline{f}_\eta^{(a)} \mid \eta \in \underline{I}, a \in \mathbf{N}\}$.

For any quantum group $\mathbf{U}_q^-$, we introduce a canonical basis $\mathbf{B}$ in an axiomatic way. Note that $\mathbf{B}$ is unique if it exists.

Theorem A

Assume that the canonical basis $\mathbf{B}$ (or the canonical signed basis $\widetilde{\mathbf{B}}$) exists for $\mathbf{U}_q^-$. Then the assignment $\underline{f}_\eta^{(a)} \mapsto g_\eta^{(a)}$ gives an isomorphism of $\mathbf{A}'$-algebras

$$\Phi : {}_{\mathbf{A}'}\mathbf{U}_q^- \xrightarrow{\sim} \mathbf{V}_q$$

Theorem B

1. Assume that $p \neq 2$. Assume that the canonical basis $\mathbf{B}$ exists for $\mathbf{U}_q^-$. There exists the canonical basis $\underline{\mathbf{B}}$ of $\underline{\mathbf{U}}_q^-$, and the natural bijection $\underline{\mathbf{B}} \xrightarrow{\sim} \mathbf{B}^\sigma$.
2. Assume that $p = 2$. A weaker statement holds, by replacing $\mathbf{B}$ by $\widetilde{\mathbf{B}}$, and $\underline{\mathbf{B}}$ by $\underline{\widetilde{\mathbf{B}}}$ (the canonical signed basis of $\underline{\mathbf{U}}_q^-$).

Consider X : symmetric type.
Then by Lusztig, there exists the canonical basis $\mathbf{B}$ for $\mathbf{U}_q^-$.

Let $\sigma : X \to X$: admissible diagram automophism,
with n : the order of σ.

Corollary

1. Assume n : odd. There exists the canonical basis $\underline{\mathbf{B}}$ of $\underline{\mathbf{U}}_q^-$, and the natural bijection $\underline{\mathbf{B}} \xrightarrow{\sim} \mathbf{B}^\sigma$.
2. Assume n : even. There exists the canonical signed basis $\underline{\widetilde{\mathbf{B}}}$ of $\underline{\mathbf{U}}_q^-$, and the natural bijection $\underline{\widetilde{\mathbf{B}}} \xrightarrow{\sim} \widetilde{\mathbf{B}}^\sigma$.

Proof of Corollary (n : odd)

There exists a sequence $X = X_0, X_1, \dots, X_k = \underline{X}$ of Cartan data, and a diagram autom. $\sigma_i : X_i \to X_i$ $(0 \le i \le k-1)$ such that

$$X_{i+1} \simeq \text{ Cartan datum induced from } (X_i, \sigma_i)$$

and that $\sigma = \sigma_{k-1} \cdots \sigma_1 \sigma_0$. Moreover, the order of σ_i : a prime power.

Let ${}_{(i)}\mathbf{U}_q^-$: (negative part of) the quantum group associated to X_i.

By induction on i, there exists the canonical basis ${}_{(i)}\mathbf{B}$ of ${}_{(i)}\mathbf{U}_q^-$.
By Theorem B, there exists the canonical basis ${}_{(i+1)}\mathbf{B}$ of ${}_{(i+1)}\mathbf{U}_q^-$, and the natural bijection

$$\xi_i : ({}_{(i)}\mathbf{B})^{\sigma_i} \xrightarrow{\sim} {}_{(i+1)}\mathbf{B}$$

Thus we obtain the canonical basis $\underline{\mathbf{B}} = {}_{(k)}\mathbf{B}$ of ${}_{(k)}\mathbf{U}_q^- = \underline{\mathbf{U}}_q^-$, and the natural bijection (commuting with Kashiwara operators)

$$\xi : \mathbf{B}^\sigma = (\cdots(\mathbf{B}^{\sigma_0})^{\sigma_1} \cdots)^{\sigma_{k-1}} \xrightarrow{\sim} \underline{\mathbf{B}}$$

Inner product on $\mathbf{U}_q^-$

Let $\mathbf{U}_q^-$ quantum group of arbitrary type

Let $Q = \bigoplus_{i \in I} \mathbf{Z}\alpha_i$: root lattice, $Q_- = \sum_{i \in I} \mathbf{Z}_{\leq i}\alpha_i$.

$\mathbf{U}_q^-$ has the weight space decomposition $\mathbf{U}_q^- = \bigoplus_{\nu \in Q_-} (\mathbf{U}_q^-)_\nu$,
where $(\mathbf{U}_q^-)_\nu$: the subspace of $\mathbf{U}_q^-$ spanned by $f_{i_1} \dots f_{i_N}$
such that $\alpha_{i_1} + \cdots + \alpha_{i_N} = -\nu$.

Define a multiplication on $\mathbf{U}_q^- \otimes \mathbf{U}_q^-$ by, for homogeneous x_1, x_2, x_1', x_2',

$$(x_1 \otimes x_2) \cdot (x_1' \otimes x_2') = q^{-(\mathrm{wt}\, x_2, \mathrm{wt}\, x_1')} x_1 x_1' \otimes x_2 x_2',$$

where $\mathrm{wt}\, x = \nu$ if $x \in (\mathbf{U}_q^-)_\nu$.

There exists a unique homomorphism $r : \mathbf{U}_q^- \to \mathbf{U}_q^- \otimes \mathbf{U}_q^-$
defined by $f_i \mapsto f_i \otimes 1 + 1 \otimes f_i$ $(i \in I)$

There exists a unique bilinear form (,) on $\mathbf{U}_q^-$ satisfying the properties; $(1,1) = 1$ and

$$(f_i, f_i) = \delta_{ij}(1 - q_i)^{-1}, \quad q_i = q^{d_i} = q^{(\alpha_i,\alpha_i)/2}$$
$$(x, y'y'') = (r(x), y' \otimes y''),$$
$$(x'x'', y) = (x' \otimes x'', r(y)),$$

where the bilinear form on $\mathbf{U}_q^- \otimes \mathbf{U}_q^-$ is defined by $(x_1 \otimes x_2, x_1' \otimes x_2') = (x_1, x_1')(x_2, x_2')$.
The bilinear form (,) is symmetric, and non-degenerate.

For $i \in I$, define a $\mathbf{Q}(q)$-linear map ${}_ir : \mathbf{U}_q^- \to \mathbf{U}_q^-$ by $r(x) = f_i \otimes {}_ir(x) + \sum y \otimes z$, where y : homog. with $\operatorname{wt} y \neq -\alpha_i$.

For $i \in I$, define a $\mathbf{Q}(q)$-subspace $\mathbf{U}_q^-[i]$ of $\mathbf{U}_q^-$ by

$$\mathbf{U}_q^-[i] = \operatorname{Ker} {}_ir.$$

The following result is known by Lusztig and Kashiwara.

1. For each $i \in I$, there is a direct sum decomp. of $\mathbf{Q}(q)$-vector spaces,

$$\mathbf{U}_q^- = \bigoplus_{n \geq 0} f_i^{(n)} \mathbf{U}_q^-[i],$$

   where all the components $f_i^{(n)} \mathbf{U}_q^-[i]$ are mutually orthogonal.
2. The map $x \mapsto f_i^{(n)} x$ gives an isom. $\mathbf{U}_q^-[i] \xrightarrow{\sim} f_i^{(n)} \mathbf{U}_q^-[i]$.
3. Set ${}_{\mathbf{A}}(\mathbf{U}_q^-[i]) = \mathbf{U}_q^-[i] \cap {}_{\mathbf{A}}\mathbf{U}_q^-$. There is a decomp. as $\mathbf{A}$-submodules,

$${}_{\mathbf{A}}\mathbf{U}_q^- = \bigoplus_{n \geq 0} f_i^{(n)} {}_{\mathbf{A}}(\mathbf{U}_q^-[i])$$

4. The projection $\mathbf{U}_q^- \to f_i^{(n)} \mathbf{U}_q^-[i]$ preserves the weights.

The $\mathbf{Z}[q]$-submodule $\mathscr{L}_{\mathbf{Z}}(\infty)$

A basis $\mathcal{B}$ of $\mathbf{U}_q^-$ is called **almost orthonormal** if, for any $b, b' \in \mathcal{B}$,

$$(b, b') \in \begin{cases} 1 + q\mathbf{Z}[[q]] \cap \mathbf{Q}(q) & \text{if } b = b', \\ q\mathbf{Z}[[q]] \cap \mathbf{Q}(q) & \text{if } b \neq b'. \end{cases}$$

Recall : $\mathbf{A} = \mathbf{Z}[q, q^{-1}]$. Let $\mathbf{A}_0 = \mathbf{Q}[[q]] \cap \mathbf{Q}(q)$. Set

$$\mathscr{L}_{\mathbf{Z}}(\infty) = \{x \in {}_{\mathbf{A}}\mathbf{U}_q^- \mid (x, x) \in \mathbf{A}_0\}$$

Known : $\mathscr{L}_{\mathbf{Z}}(\infty)$: $\mathbf{Z}[q]$-submodule of ${}_{\mathbf{A}}\mathbf{U}_q^-$.
If $\mathcal{B}$ is almost orthonormal, and **integral** , i.e.,
$\mathbf{A}$-submodule generated by $\mathcal{B}$: stable by $f_i^{(n)}$ and ${}_ir$,
then $\mathcal{B}$ gives a $\mathbf{Z}[q]$-basis of $\mathscr{L}_{\mathbf{Z}}(\infty)$.

For each $i \in I$, consider the decomp. $\mathbf{U}_q^- = \bigoplus_{n \ge 0} f_i^{(n)} \mathbf{U}_q^-[i]$.
For $x \in \mathbf{U}_q^-$, write

$$x = \sum_{n \ge 0} y_n = \sum_{n \ge 0} f_i^{(n)} x_n, \qquad (x_n \in \mathbf{U}_q^-[i])$$

Lemma (Kashiwara)

Let $x = \sum_{n \ge 0} y_n$ be as above.

1. If $x \in \mathscr{L}_{\mathbf{Z}}(\infty)$, then $x_n, y_n \in \mathscr{L}_{\mathbf{Z}}(\infty)$. If, in addition, $(x, x) \in 1 + q\mathbf{A}_0$, then there exists $n_0 \ge 0$ such that $(y_{n_0}, y_{n_0}), (x_{n_0}, x_{n_0}) \in 1 + q\mathbf{A}_0$, $(y_n, y_n), (x_n, x_n) \in q\mathbf{A}_0$ for $n \ne n_0$.
2. Let $\mathcal{B}$: $\mathbf{A}$-basis of ${}_{\mathbf{A}}\mathbf{U}_q^-$, which is almost orthonormal, and integral. There exists $b \in \mathcal{B}$ such that, module $q\mathscr{L}_{\mathbf{Z}}(\infty)$,

$$y_n \equiv \begin{cases} \pm b & \text{if } n = n_0, \\ 0 & \text{if } n \ne n_0. \end{cases}$$

Canonical basis

Fix $i \in I$. Under the decom. $x = \sum_{n \geq 0} f_i^{(n)} x_n$ with $x_n \in \mathbf{U}_q^-[i]$, set

$$x_{[i;a]} = f_i^{(a)} x_a, \quad \text{(projection to } f_i^{(a)}\mathbf{U}_q^-[i])$$

Let $\mathcal{B}$: a basis of $\mathbf{U}_q^-$.

Fix $i \in I$. For $b \in \mathcal{B}$, define $\varepsilon_i(b) \in \mathbf{N}$ by

$$b \in f_i^{(\varepsilon_i(b))}\mathbf{U}_q^- - f_i^{(\varepsilon_i(b)+1)}\mathbf{U}_q^-$$

Set $\mathcal{B}_{i;a} = \{b \in \mathcal{B} \mid \varepsilon_i(b) = a\}$. We have a partition

$$\mathcal{B} = \bigsqcup_{n \geq 0} \mathcal{B}_{i;n}$$

Let $^- : \mathbf{U}_q^- \to \mathbf{U}_q^-$ the **bar-involution** ; i.e., a $\mathbf{Q}$-algebra isom. defined by $q \mapsto q^{-1}, f_i \mapsto f_i$ $(i \in I)$.

We consider a basis $\mathbf{B}$ of $\mathbf{U}_q^-$ having the following properties;

(C1) $\mathbf{B}$ gives a $\mathbf{Z}[q]$-basis of $\mathscr{L}_{\mathbf{Z}}(\infty)$,

(C2) $\mathbf{B}$ is bar-invariant, i.e., $\overline{b} = b$ for $b \in \mathbf{B}$,

(C3) $\mathbf{B}$ is almost orthonormal,

(C4) For $\nu \in Q_-$, set $\mathbf{B}_\nu = \mathbf{B} \cap (\mathbf{U}_q^-)_\nu$. Then we have a partition $\mathbf{B} = \bigsqcup_{\nu \in Q_-} \mathbf{B}_\nu$, with $\mathbf{B}_\nu = \{1\}$ for $\nu = 0$, ,

(C5) If $b \in \mathbf{B}_{i;a}$ for $i \in I, a \in \mathbf{N}$, then

$$b \equiv b_{[i;a]} \mod q\mathscr{L}_{\mathbf{Z}}(\infty)$$

(C6) $\bigcap_{i \in I} \mathbf{B}_{i;0} = \{1\}$,

(C7) Let $b \in \mathbf{B}_{i;0}$, and $a > 0$. There exists a unique $b' \in \mathbf{B}_{i;a}$ such that

$$b' \equiv f_i^{(a)} b \mod f_i^{a+1}\mathbf{U}_q^-$$

The correspondence $b \mapsto b'$ gives a bijection $\pi_{i;a} : \mathbf{B}_{i;0} \xrightarrow{\sim} \mathbf{B}_{i;a}$.

Remark. If $\mathbf{B}$ exists in $\mathbf{U}_q^-$, then $\mathbf{B}$ is unique.
The basis $\mathbf{B}$ is called the **canonical basis** of $\mathbf{U}_q^-$

Theorem (Lusztig)

Assume that $\mathbf{U}_q^-$: assoc. to the symmetric Cartan datum X.
Then the canonical basis $\mathbf{B}$ exists.

We define a subset $\widetilde{\mathcal{B}}$ of $\mathbf{U}_q^-$ by

$$\widetilde{\mathcal{B}} = \{x \in \mathbf{U}_q^- \mid \overline{x} = x, (x, x) \in 1 + q\mathbf{Z}[[q]]\}$$

If there exists a basis $\mathcal{B}$ of $\mathbf{U}_q^-$ such that $\widetilde{\mathcal{B}} = \mathcal{B} \sqcup -\mathcal{B}$,
$\widetilde{\mathcal{B}}$: called **canonical signed basis**.
For the canonical signed basis $\widetilde{\mathcal{B}}$, the choice of $\mathcal{B}$ is not unique.

If $\mathbf{B}$: canonical basis, then $\mathbf{B} \sqcup -\mathbf{B}$: canonical signed basis.

Outline of the proof of Theorem A

We prove $\Phi : {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- \xrightarrow{\sim} \mathbf{V}_q$.

Step 1 : Φ is an algebra homomorphism (discussed later).

Step 2 : Φ is injective.

Let $\mathbf{F}(q)$: rational function filed, quotient field of $\mathbf{A}' = \mathbf{F}[q, q^{-1}]$

Set ${}_{\mathbf{F}(q)}\mathbf{V}_q = \mathbf{V}_q \otimes_{\mathbf{A}'} \mathbf{F}(q)$, ${}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^- = {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- \otimes_{\mathbf{A}'} \mathbf{F}(q)$.

Φ can be extended to $\Phi : {}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^- \to {}_{\mathbf{F}(q)}\mathbf{V}_q$. Step 2 follows from

Proposition

1. The bilinear forms on ${}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^-$ and on ${}_{\mathbf{F}(q)}\mathbf{V}_q$ are non-degenerate.
2. For any $x, y \in {}_{\mathbf{F}(q)}\underline{\mathbf{U}}_q^-$, $(\Phi(x), \Phi(y)) = (x, y)$.

Step 3 : Φ is surjective.

Let $\mathbf{B}$ the canonical basis of $\mathbf{U}_q^-$.
For each $i \in I, a \in \mathbf{N}$, defnie a bijection $F_i : \mathbf{B}_{i;a} \to \mathbf{B}_{i;a+1}$ by

$$F_i = \pi_{i;a+1} \circ \pi_{i;a}^{-1} : \mathbf{B}_{i;a} \longrightarrow \mathbf{B}_{i;0} \longrightarrow \mathbf{B}_{i;a+1}$$

Define $E_i : \mathbf{B}_{i;a} \to \mathbf{B}_{i;a-1}$ as the inverse of F_i if $a > 0$, and $E_i(b) = 0$ for $b \in \mathbf{B}_{i;0}$. The maps $E_i, F_i : \mathbf{B} \to \mathbf{B} \cup \{0\}$ are called **Kashiwara operators**.

Let $\sigma : \mathbf{U}_q^- \to \mathbf{U}_q^-$: alg. autom. and consider $\underline{\mathbf{U}}_q^-$.
Let $\mathbf{B}^\sigma$: the set of σ-fixed element in $\mathbf{B}$.

Let $\eta \in \underline{I}$, and $b \in \mathbf{B}^\sigma$.
$\varepsilon_i(b)$ is constant for $i \in \eta$, which we denote by $\varepsilon_\eta(b)$. Set

$$\mathbf{B}^\sigma_{\eta;a} = \{b \in \mathbf{B}^\sigma \mid \varepsilon_\eta(b) = a\}.$$

We have a partition $\mathbf{B}^\sigma = \bigsqcup_{a \geq 0} \mathbf{B}^\sigma_{\eta;a}$.

For each $\eta \in \underline{I}$, one can define a bijection $\pi_{\eta;a} : \mathbf{B}^\sigma_{\eta:0} \xrightarrow{\sim} \mathbf{B}^\sigma_{\eta;a}$, as the restriction of $\prod_{i\in\eta} \pi_{i;a}$ on $\mathbf{B}^\sigma_{\eta;0}$.

We define Kashiwara operators $\widetilde{F}_\eta, \widetilde{E}_\eta : \mathbf{B}^\sigma \to \mathbf{B}^\sigma \cup \{0\}$ by using $\pi_{\eta;a}$. $\widetilde{F}_\eta$ is the restriction of $\prod_{i\in\eta} F_i$ on $\mathbf{B}^\sigma$

Lemma

For $\eta \in \underline{I}$, set $\mathbf{U}_q^-[\eta] = \bigcap_{i\in\eta} \mathbf{U}_q^-[i]$. Then we have

$$ {}_{\mathbf{A}}\mathbf{U}_q^- = \bigoplus_{(a_i)\in\mathbf{N}^\eta} \Big(\prod_{i\in\eta} f_i^{(a_i)}\Big) {}_{\mathbf{A}}(\mathbf{U}_q^-[\eta]) $$

In particular,

$$ {}_{\mathbf{A}}\mathbf{U}_q^{-,\sigma} \equiv \bigoplus_{a\in\mathbf{N}} \widetilde{f}_\eta^{(a)} {}_{\mathbf{A}}(\mathbf{U}_q^-[\eta])^\sigma \mod J. $$

Recall $\pi : {}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma} \to \mathbf{V}_q$. $\pi(\mathbf{B}^\sigma)$ give an $\mathbf{A}'$-basis of $\mathbf{V}_q$. In order to prove the surjectivity of Φ, enough to show $\pi(b) \in \operatorname{Im}\Phi$ for $b \in \mathbf{B}^\sigma$. This is done by the Lemma, and the property of $\widetilde{F}_\eta, \widetilde{E}_\eta$.

Outline of the proof of Theorem B

Recall $\pi : {}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma} \to \mathbf{V}_q$: projection. For each $\eta \in \underline{I}$,
set $\mathbf{V}_q[\eta] = \pi({}_{\mathbf{A}'}(\mathbf{U}_q^-[\eta])^\sigma))$. By the previous lemma, we have

$$\mathbf{V}_q = \bigoplus_{a \in \mathbf{N}} g_\eta^{(a)} \mathbf{V}_q[\eta]$$

We also have a decomp.

$${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- = \bigoplus_{a \in \mathbf{N}} \underline{f}_\eta^{(a)} {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-[\eta].$$

Thus $\Phi : {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- \xrightarrow{\sim} \mathbf{V}_q$ gives an isomorphism of $\mathbf{A}'$-modules,

$$\underline{f}_\eta^{(a)} {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-[\eta] \xrightarrow{\sim} g_\eta^{(a)} \mathbf{V}_q[\eta].$$

$\pi(\mathbf{B}^\sigma)$ gives an $\mathbf{A}'$-basis of $\mathbf{V}_q$. Set $\underline{\mathbf{B}}^\bullet = \Phi^{-1}(\pi(\mathbf{B}^\sigma))$.
Then $\underline{\mathbf{B}}^\bullet$ gives an $\mathbf{A}'$-basis of ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$.

Let $\underline{\mathscr{L}}_{\mathbf{F}}(\infty)$: $\mathbf{F}[q]$-submodule of ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$ spanned by $\underline{\mathbf{B}}^\bullet$.

Lemma

$\underline{\mathbf{B}}^\bullet$ is the canonical basis of ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$, namely it satisfies similar properties as $(C_1) \sim (C7)$, by replacing $\mathscr{L}_{\mathbf{Z}}(\infty)$ by $\underline{\mathscr{L}}_{\mathbf{F}}(\infty)$, etc.
Moreover, we have a natural bijection $\mathbf{B}^\sigma \xrightarrow{\sim} \underline{\mathbf{B}}^\bullet$.

Let $\varphi : {}_{\mathbf{A}}\underline{\mathbf{U}}_q^- \to {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- = {}_{\mathbf{A}}\underline{\mathbf{U}}_q^- / p({}_{\mathbf{A}}\underline{\mathbf{U}}_q^-)$: the natural surjection.

Let $\underline{\mathscr{L}}_{\mathbf{Z}}(\infty)$: $\mathbf{Z}[q]$-submodule of ${}_{\mathbf{A}}\underline{\mathbf{U}}_q$ defined similar to $\mathscr{L}_{\mathbf{Z}}(\infty)$ for $\mathbf{U}_q^-$.

Lemma

Let $x \in \underline{\mathscr{L}}_{\mathbf{Z}}(\infty)$ be such that $\overline{x} = x$, and $(x, x) \in 1 + q\mathbf{A}_0$. Further assume that $\varphi(x) = b_\bullet \in \underline{\mathbf{B}}^\bullet$.

1. If $p \neq 2$, then x is determined uniquely by $b_\bullet$.
2. If $p = 2$, then x is unique up to sign, i.e., $\varphi^{-1}(b_\bullet) = \{\pm x\}$.

Thus cannical basis $\underline{\mathbf{B}}$ of $\underline{\mathbf{U}}_q^-$ is obtained by $\varphi : \underline{\mathbf{B}} \xrightarrow{\sim} \underline{\mathbf{B}}^\bullet$ if $p \neq 2$,
and canonical signed basis $\widetilde{\underline{\mathbf{B}}} = \varphi^{-1}(\underline{\mathbf{B}}^\bullet)$ if $p = 2$.

Homomorphism $\Phi : {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- \to \mathbf{V}_q$

We prove Φ is a homomorphism. (Step 1).

Note : ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^-$ is the $\mathbf{A}'$-algebra with generators $\underline{f}_\eta^{(a)}$, with Serre relations.

In order to prove Step 1, enough to show $\widetilde{f}_\eta^{(a)}$ satisfies similar relations, namely the relations in ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}$,

$$(A) \qquad \sum_{k=0}^{1-a_{\eta\eta'}} (-1)^k \begin{bmatrix} 1-a_{\eta\eta'} \\ k \end{bmatrix}_{d_\eta} \widetilde{f}_\eta^k \widetilde{f}_{\eta'} \widetilde{f}_\eta^{1-a_{\eta\eta'}-k} \equiv 0 \mod J \qquad (\eta \neq \eta'),$$

$$(B) \qquad [a]^!_{d_\eta} \widetilde{f}_\eta^{(a)} = \widetilde{f}_\eta^a, \qquad (a \in \mathbf{N}),$$

where $d_\eta = (\alpha_\eta, \alpha_\eta)_1/2 = |\eta| d_i$.

(B) is shown as follows. Since $|\eta|$ is a power of p, we have $([a]^!_{d_i})^{|\eta|} = [a]^!_{|\eta| d_i} = [a]^!_{d_\eta}$ in $\mathbf{A}' = \mathbf{F}[q, q^{-1}]$. Hence

$$\widetilde{f}_\eta^{(a)} = \prod_{i \in \eta} f_i^{(a)} = ([a]^!_{d_i})^{-|\eta|} \prod_{i \in \eta} f_i^a = ([a]^!_{d_\eta})^{-1} \widetilde{f}_\eta^a.$$

For the proof of (A), we consider the simplest situation;
$\mathbf{U}_q^-$: simply-laced, fix $\eta, \eta' \in \underline{I}$ such that $|\eta| = 1, |\eta'| = n-1$,
and any element in η' is joined to the element in η
(Here give **no assumption** on n)

Since $a_{ij} = -1$ for $i \in \eta, j \in \eta'$, we have

$$a_{\eta,\eta'} = -|\eta'|, \qquad a_{\eta',\eta} = -1.$$

Write $\eta = \{1\}, \eta' = \{2_1, \dots, 2_{n-1}\}$. We have $\widetilde{f}_\eta = f_1, \widetilde{f}_{\eta'} = f_{2_1} \cdots f_{2_{n-1}}$.

In order to prove (A), we need to copmpute, for various $0 \le k \le n$,
$\widetilde{f}_\eta^k \widetilde{f}_{\eta'} \widetilde{f}_\eta^{n-k} = f_1^k f_{2_1} \cdots f_{2_{n-1}} f_1^{n-k}$ (here $1 - a_{\eta\eta'} = n$).

More genrally, for $(a_1, \dots, a_n) \in \mathbf{N}^n$ such that $\sum_i a_i = n$, consider the corresp.

$$(a_1, \dots, a_n) \longleftrightarrow f_1^{a_1} f_{2_1} f_1^{a_2} f_{2_2} \cdots f_{2_{n-2}} f_1^{a_{n-1}} f_{2_{n-1}} f_1^{a_n} \in \mathbf{U}_q^-$$

The commuting relations are given by $f_1^2 f_{2_k} = f_1 f_{2_k} f_1 - f_{2_k} f_1^2$.

Combinatorial setting

Let V_n be a $\mathbf{Q}(q)$-vector space spanned by $\{\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{N}^n\}$ satisfying the relations;
For any $1 \le i \le n-1$, if $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{N}^n$ with $a_i \ge 2$, then $\mathbf{a}$ is written as

$$\mathbf{a} = [2]\mathbf{b} - \mathbf{c},$$

where $\mathbf{b}, \mathbf{c} \in \mathbf{N}^n$ are given by

$$\mathbf{b} = (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1} + 1, a_{i+2}, \dots, a_n),$$
$$\mathbf{c} = (a_1, \dots, a_{i-1}, a_i - 2, a_{i+1} + 2, a_{i+1}, \dots, a_n).$$

For each $m \ge 1$, denote by $E_n(m)$ the subspace of V_n spanned by

$$\mathscr{E}_n(m) = \{\mathbf{a} = (a_1, \dots, a_n) \mid \sum_{1 \le i \le n} a_i = m, a_i \in \{0, 1\} \text{ for } 1 \le i \le n-1 \ \}$$

If $\mathbf{a} \in V_n$ is such that $\sum_i a_i = m$, then $\mathbf{a} \in E_n(m)$.

In the case where $m = n$, set $E_n = E_n(m)$ and $\mathscr{E}_n(m) = \mathscr{E}_n$.

Example.

$$\begin{aligned}
\mathscr{E}_2 &= \{(1,1),(0,2)\}, \\
\mathscr{E}_3 &= \{(1,1,1),(1,0,2),(0,1,2),(0,0,3)\}, \\
\mathscr{E}_4 &= \{(1,1,1,1),(1,1,0,2),(1,0,1,2),(1,0,0,3), \\
&\qquad (0,1,1,2),(0,1,0,3),(0,0,1,3),(0,0,0,4)\}.
\end{aligned}$$

Lemma

Assume that $(k, 0, \dots, 0, \ell) \in V_n$. Then we have

$$(k, 0, \dots, 0, \ell) = \sum_{\substack{a_1+\cdots+a_n = k+\ell \\ a_1,\dots,a_{n-1} \in \{0,1\}}} (-1)^{a_1+\cdots+a_{n-1}+(n-1)} \Big(\prod_{1 \le i \le n-1} [k - x_i] \Big) \mathbf{a}$$

where $x_i = a_1 + \cdots + a_{i-1} + (1 - a_i)$ for each i, and $\mathbf{a} = (a_1, \dots, a_n) \in \mathscr{E}_n(k+\ell)$.

In the case where $n = 2$, the following formula holds.
For any $k \geq 0, \ell \geq 0$, we have

$$(k, \ell) = [k](1, k + \ell - 1) - [k-1](0, k + \ell). \tag{1}$$

(1) is proved by induction on k. The lemma is proved by induction on n.

Proposition

The following equality holds in E_n.

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (k, 0, \dots, 0, n-k) = 0. \tag{2}$$

Proof. By applying the lemma for $m = n$, we have

$$(k, 0, \dots, 0, n-k) = \sum_{(a_1, \dots, a_n) \in \mathscr{E}_n} (-1)^{a_n - 1} \Bigg(\prod_{1 \leq i \leq n-1} [k - x_i] \Bigg) (a_1, a_2, \dots, a_n),$$

where $x_i = a_1 + \cdots + a_{i-1} + (1 - a_i)$ for $1 \leq i < n$.

In order to prove (2), enough to see, for a fixed $\mathbf{a} = (a_1, \dots, a_n) \in \mathscr{E}_n$,

$$(3) \qquad \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{a_n - 1} \Bigg(\prod_{1 \le i \le n-1} [k - x_i] \Bigg) = 0.$$

Note : The product factor can be written as

$$\prod_{1 \le i \le n-1} [k - x_i] = \sum_{j=1}^{n} F_j(q) q^{(n-2j+1)k},$$

where $F_j(q) \in \mathbf{Q}(q)$ is independent from k.

Thus (3) follows from the following statement.

$$(4) \qquad \sum_{k=0}^{n} (-1)^k q^{(n-2j+1)k} \begin{bmatrix} n \\ k \end{bmatrix} = 0 \qquad \text{for } j = 1, \dots, n.$$

By the quantum binomial formula,

$$\prod_{\ell=0}^{n-1}(1+q^{2\ell}z) = \sum_{k=0}^{n} q^{k(n-1)} \begin{bmatrix} n \\ k \end{bmatrix} z^k,$$

where z : another indeterminate. If we put $z = -q^{-2j+2}$ for $j = 1, \dots, n$, (4) holds. The proposition is proved.

The proposition can be translated to the following.

Corollary

Assume that $\eta = \{1\}$ and $\eta' = \{2_1, \dots, 2_{n-1}\}$. Assume that 1 is joined to $2_1, \dots, 2_{n-1}$ (by single edge). Then $1 - a_{\eta\eta'} = n$, and

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} f_1^k (f_{2_1} \cdots f_{2_{n-1}}) f_1^{n-k} = 0. \tag{1}$$

Remark. (1) holds for any $n \in \mathbf{N}$, without using modulo J.

Assume that $|\eta| = n$ and $|\eta'| = 1$ with $\eta' = \{j\}$,
and any $i \in \eta$ is joined to $j \in \eta'$ (with single edge).

Thus $a_{\eta\eta'} = -1, 1 - a_{\eta\eta'} = 2$ and $d_\eta = (\alpha_\eta, \alpha_\eta)_1/2 = n$.

Recall : $\widetilde{f}_\eta = \prod_{i \in \eta} f_i$, $\widetilde{f}_{\eta'} = f_j$.

For any subset $X \subset \eta$, let $\widetilde{f}_X = \prod_{i \in X} f_i$. Set $X' = \eta - X$.

The following formula is proved by a similar argument as before.

$$[2]^n \widetilde{f}_\eta \widetilde{f}_{\eta'} \widetilde{f}_\eta = \sum_{X \subset \eta} \widetilde{f}_X^2 \widetilde{f}_{\eta'} \widetilde{f}_{X'}^2.$$

Consider $\sigma : \mathbf{U}_q^- \to \mathbf{U}_q^-$. We have $\sigma(\widetilde{f}_X \widetilde{f}_{\eta'} \widetilde{f}_{X'}) = \widetilde{f}_{\sigma(X)} \widetilde{f}_{\eta'} \widetilde{f}_{\sigma(X')}$.

Note : $\sigma(X) = X$ if and only if $X = \emptyset$ or η.
If n is a prime power, $[2]^n = [2]_n = [2]_{d_\eta}$.

Propopsition

Assume that n is a prime power. Then we have, in ${}_{\mathbf{A}'}\mathbf{U}_q^{-,\sigma}$,

$$\widetilde{f}_\eta^2 \widetilde{f}_{\eta'} - [2]_{d_\eta} \widetilde{f}_\eta \widetilde{f}_{\eta'} \widetilde{f}_\eta + \widetilde{f}_{\eta'} \widetilde{f}_\eta^2 \equiv 0 \mod J.$$

The case σ : not admissible

- Lusztig showed that $\mathbf{B}^\sigma \simeq \underline{\mathbf{B}}$ for X : finite type, σ: non-admissible.

Theorem

Assume σ : non-admissible, and X : finite or affine type.
Except the cases $(X, \underline{X}) = (A_2^{(1)},\ A_2^{(2)}),\ (A_3^{(1)}, A_1^{(1)}),\ (A_{n-1}^{(1)}, A_1)$,
there exists an isom. $\Phi : {}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- \xrightarrow{\sim} \mathbf{V}_q$, and a bijection $\mathbf{B}^\sigma \xrightarrow{\sim} \underline{\mathbf{B}}$.

Remark. The definition of the map Φ must be modified.
$\prod_{i\in\eta} f_i^{(a)}$: not necessarly σ-stable.

Example 1 : $X : A_2,\ \underline{X} : C_1$. Here $I = \{1, 2\} = \eta$, $\sigma : 1 \leftrightarrow 2$.

$$\mathbf{B} = \{f_1^{(\ell)} f_2^{(m)} f_1^{(n)} \mid m \ge \ell + n\} \cup \{f_2^{(\ell)} f_1^{(m)} f_2^{(n)} \mid m \ge \ell + n\},$$

$$\mathbf{B}^\sigma = \{f_1^{(a)} f_2^{(2a)} f_1^{(a)} = f_2^{(a)} f_1^{(2a)} f_2^{(a)} \mid a \in \mathbf{N}\} \simeq \underline{\mathbf{B}} = \{\underline{f}_\eta^{(a)} \mid a \in \mathbf{N}\}.$$

Set $g_\eta^{(a)} = \pi(f_1^{(a)} f_2^{(2a)} f_1^{(a)})$.
Then $\Phi : \underline{f}_\eta \mapsto g_\eta^{(a)}$ gives isom. ${}_{\mathbf{A}'}\underline{\mathbf{U}}_q^- \xrightarrow{\sim} \mathbf{V}_q$.

Note : σ-stable PBW-basis does not exist for $\mathbf{U}_q^-$.

Example 2. $X = A_{n-1}^{(1)}$, $\underline{X} = A_1$, order of $\sigma = n$.
The canonical basis of $\mathbf{U}_q^-$ was classified by Luszitg.
Let

$$\mathscr{P}^{(n)} = \{\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)}) \mid \lambda^{(i)} : \text{partition } \}$$

$\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$: called **aperiodic** if $\lambda^{(1)}, \dots, \lambda^{(n)}$ have no common parts $c = \lambda_j^{(i)}$.

$$\mathbf{B} \simeq \{\boldsymbol{\lambda} \in \mathscr{P}^{(n)} \mid \boldsymbol{\lambda} : \text{aperiodic } \}.$$

σ acts on $\mathbf{B}$ as a cyclic permutation of $\boldsymbol{\lambda} \in \mathscr{P}^{(n)}$. Hence $\mathbf{B}^\sigma = \emptyset$.

Note : σ-stable canonical basis does not exist for $\mathbf{U}_q^-$.
Thus the theorem does not hold for $\mathbf{U}_q^-$.