Integrability in Quantum Field Theory

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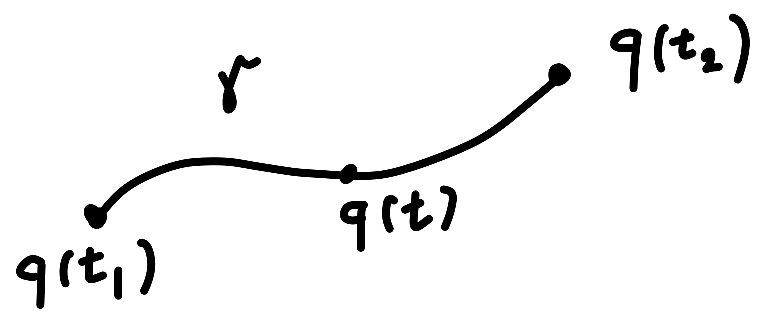
Outline

1. Classical integrable systems
2. KdV and Dirac Fermions
3. Kac-Wakimoto hierarchies
4. Tau functions in Gromov-Witten theory

Classical integrable systems

Mechanical system with coordinates $q_1, \ldots, q_n$ satisfying the Lagrangian principle:

The trajectories of the system are critical points for an action functional



$$S(\gamma) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t))dt$$

Euler-Lagrange equations $$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

Momentum $$p_i := \frac{\partial L}{\partial \dot{q}_i}$$

Energy $$H(q,p) := \sum_{i=1}^{n} p_i \dot{q}_i - L(q,\dot{q})$$

L is a function on a space with coordinates $(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$

Equivalently, after changing the coordinates, a function on the so called **phase space**

$$M := \{(q_1, \ldots, q_n, p_1, \ldots, p_n)\}$$

Euler-Lagrange equations turn into

Hamilton equations

$$\frac{\partial q_i}{\partial t} = \frac{\partial H}{\partial p_i}$$

$$\frac{\partial p_i}{\partial t} = -\frac{\partial H}{\partial q_i}$$

Poisson bracket: $\{H_1, H_2\} := \sum_{i=1}^{n} \left(\frac{\partial H_1}{\partial p_i} \frac{\partial H_2}{\partial q_i} - \frac{\partial H_1}{\partial q_i} \frac{\partial H_2}{\partial p_i} \right)$

Conservation laws: given a Hamiltonian system with Hamiltonian H

$$G(q(t), p(t)) = \text{const} \quad \Leftrightarrow \quad \{H, G\} = 0$$

Liouville integrable system:

(i) There are n independent conservation laws $G_1, \ldots, G_n$

(ii) The conserved quantities pairwise Poisson commute

$$\{G_i, G_j\} = 0$$

Liouville-Arnold theorem

If a Hamiltonian system is Liouville integrable with conserved quantities $G_1, \ldots, G_n$ and the Hamiltonian flows of G_i are *complete,* then the map

$$(G_1, \ldots, G_n) : \ M \to \mathbb{R}^n$$

is a smooth fibration with fiber $(S^1)^k \times \mathbb{R}^{n-k}$.

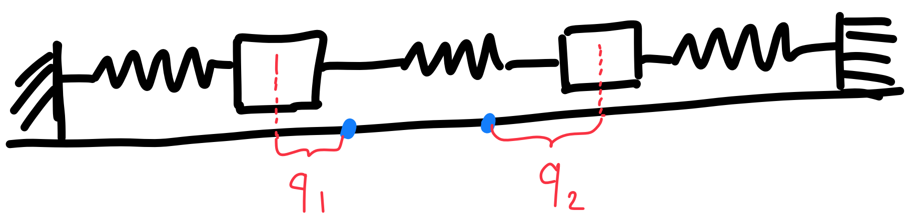
Examples

1) Free particle: $H = p_1^2 + p_2^2 + p_3^2.$

$$G_1 = p_1,\ G_2 = p_2,\ G_3 = p_3$$

2) Kepler's problem

3) Coupled oscillators

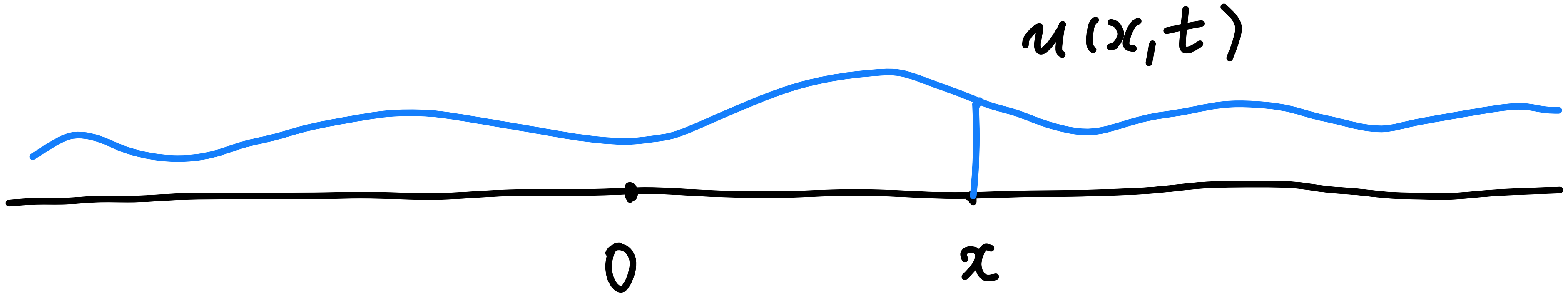


$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \sum_{i,j=1}^{2} k_{ij} q_i q_j$$

$$G_1 = H, \quad G_2 = \frac{q_1^2}{2} + \frac{q_2^2}{2} + \sum_{i,j=1}^{2} \widetilde{k}_{ij} p_i p_j$$

KdV

A model for waves in shallow water



Korteweg-de Vries (KdV) equation

$$u_t + u\,u_x + \frac{1}{12}u_{xxx} = 0$$

Introduced in the 2nd half of 19th century by Boussinesq, Kortweweg, and de Vries.

It is an Euler-Lagrange equation for a 2d field theory

$$S = \iint_{\mathbb{R}^2} L(\phi, \phi_x, \phi_t, \phi_{xx}) dxdt$$

where $\phi = \phi(x, t)$ is a scalar field $u(x, t) = \phi_x(x, t)$.

Miura, Gardner, Kruskal (1968): KdV is integrable, that is, it admits an infinite sequence of conserved quantities.

This is the beginning of a new subject: integrable systems in 2d field theories, i.e., integrable systems on loop/path spaces of manifolds.

Dirac fermions

Space time $M = \mathbb{R}^1 \times S^1$ equipped with Minkowski metric

standard flat coordinates: $x = (x_0, x_1) \mapsto (x_0, e^{\mathbf{i}x_1}) \in M$

Fields: $\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$ and $\psi^\dagger = \left(\psi_1^\dagger(x), \psi_2^\dagger(x)\right)$

Lagrangian: $L = \mathbf{i}\,\psi^\dagger \cdot \begin{pmatrix} \partial_0 - \partial_1 & 0 \\ 0 & \partial_0 + \partial_1 \end{pmatrix} \psi$

Action:

$$S = \iint_M L(\psi, \psi^\dagger) = \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \psi_n^\dagger \cdot \left(\mathbf{i}\partial_0 \psi_n + \begin{pmatrix} n & 0 \\ 0 & -n \end{pmatrix} \psi_n \right),$$

where

$$\psi(x) =: \sum_{n\in\mathbb{Z}} \psi_n(x_0)\, e^{\mathbf{i}nx_1}, \quad \psi^\dagger(x) =: \sum_{n\in\mathbb{Z}} \psi_n^\dagger(x_0)\, e^{-\mathbf{i}nx_1}\,.$$

$$H = \sum_n \left(-n\,\psi^\dagger_{n,1}\psi_{n,1} + n\,\psi^\dagger_{n,2}\psi_{n,2} \right) \qquad P = \sum_n \left(n\,\psi^\dagger_{n,1}\psi_{n,1} + n\,\psi^\dagger_{n,2}\psi_{n,2} \right)$$

Quantization:

$\psi_{n,1} \mapsto b^\dagger_{-n}$ *creating* anti-fermion with momentum $-n$ $(n > 0)$

$\psi^\dagger_{n,2} \mapsto a^\dagger_n$ *creating* fermion with momentum n $(n > 0)$

$\psi^\dagger_{n,1} \mapsto a^\dagger_n$ *creating fermion* with momentum n $(n < 0)$

$\psi_{n,2} \mapsto b^\dagger_{-n}$ *creating* anti-fermion with momentum $-n$ $(n < 0)$

All operators *anti-commute* except for

$$a_m a^\dagger_n + a^\dagger_n a_m = \delta_{m,n} \qquad b_m b^\dagger_n + b^\dagger_n b_m = \delta_{m,n}$$

Sato Grassmannian

F vector space of states created by $\psi_{n,2}$ and $\psi^{\dagger}_{n,2}$ from the vacuum state $|0\rangle$. The Lie algebra of $\mathbb{Z} \times \mathbb{Z}$ matrices with finitely many non-zero entries acts on F

$$A := (a_{ij})_{i,j\in\mathbb{Z}} \mapsto \sum_{i,j} a_{ij} : \psi^{\dagger}_{i,2}\psi_{j,2} :$$

The action can be exponentiated and we define

$$\mathrm{Gr} := \{e^{A}|0\rangle \,|\, \forall A\} \subset F^{(0)}$$ charge 0

Let us consider Dirac theory with interactions of the form

$$\int_{\mathbb{R}} \left(\int_{S^1} L(\psi, \psi^{\dagger}) - \sum_{i,j} b_{ij}\psi^{\dagger}_{i,2}\psi_{j,2} \right)$$

The ground state of the deformed theory will be a point in Gr.

Mikio Sato: The solutions of the KdV hierarchy can be obtained from the points in Gr!

Boson-Fermion isomorphism: $F^{(0)} \cong \mathbb{C}[t_1, t_2, t_3, \ldots]$

$J_k := \sum_i :\psi_i^\dagger \psi_{i+k}:$ Heisenberg Lie algebra $[J_k, J_l] = k\,\delta_{k,-l}$

$$|0\rangle \mapsto 1, \quad J_k \mapsto \frac{\partial}{\partial t_k}, \quad J_{-k} \mapsto k t_k$$

If we embed Gr in $\mathbb{C}[t_1, t_2, t_3, \ldots]$ via the Boson-Fermion isomorphism, then the Plucker relations turn into a system of PDEs known as Hirota Bilinear Equations (of the KP hierarchy).

States $|\Omega\rangle \in \mathrm{Gr}$ correspond to functions $\tau(t_1, t_2, \ldots)$ called *tau-functions*. The quantization of the Dirac field $\psi_2(x)$ corresponds to a *vertex operator*

$$\Gamma(z) = \exp\Big(\sum_{k=1}^{\infty} t_k z^k\Big)\exp\Big(\sum_{k=1}^{\infty} \partial_k \frac{z^{-k}}{-k}\Big), \quad z = e^{-\mathbf{i}x_1} \in S^1.$$

The wave function $\psi_2(x)\,|\,\Omega\rangle$ takes the form

$$\Psi(z,t) = \tau\Big(t_1 - z^{-1}, t_2 - z^{-2}/2, t_3 - z^{-3}/3, \ldots\Big)\ \exp\Big(\sum_{k=1}^{\infty} t_k z^k\Big).$$

The Plucker relations are equivalent to a system of PDEs

$$\partial_k \Psi(z,t) = B_k(\Psi, \partial_1)\Psi(z,t),$$

where $B_k = \partial_1^k + \cdots$ is a differential operator of order k whose coefficients depend on Ψ and it's t_1-derivatives.

How to recover the solutions to the KdV equation?

Suppose that $\tau(t_1, t_3, t_5, \ldots) \in \mathrm{Gr}$ is independent of the even variables. Put $L := B_2 = \partial_1^2 + u$. Then one can check that

$$B_k = (L^{k/2})_+ \quad \text{and that} \quad \partial_k L = [B_k, L].$$

This is non-trivial only if k is odd ≥ 3. If $k = 3$, after setting $t_1 := x$ and $t_3 := t$, we get precisely the KdV equation.

Kac-Wakimoto hierarchies

Main ingredients of Sato's construction of the KP hierarchy:

1) Lie algebra $\mathfrak{g} = \mathrm{gl}_\infty$.

2) Representation $V = F^{(0)}$ of $\mathfrak{g}$ that can be exponentiated.

3) Heisenberg Lie sub algebra $\mathfrak{h}$ whose Fock representation is V.

4) Bi-linear operator Ω acting on $V \otimes V$ and commuting with $\mathfrak{g}$.

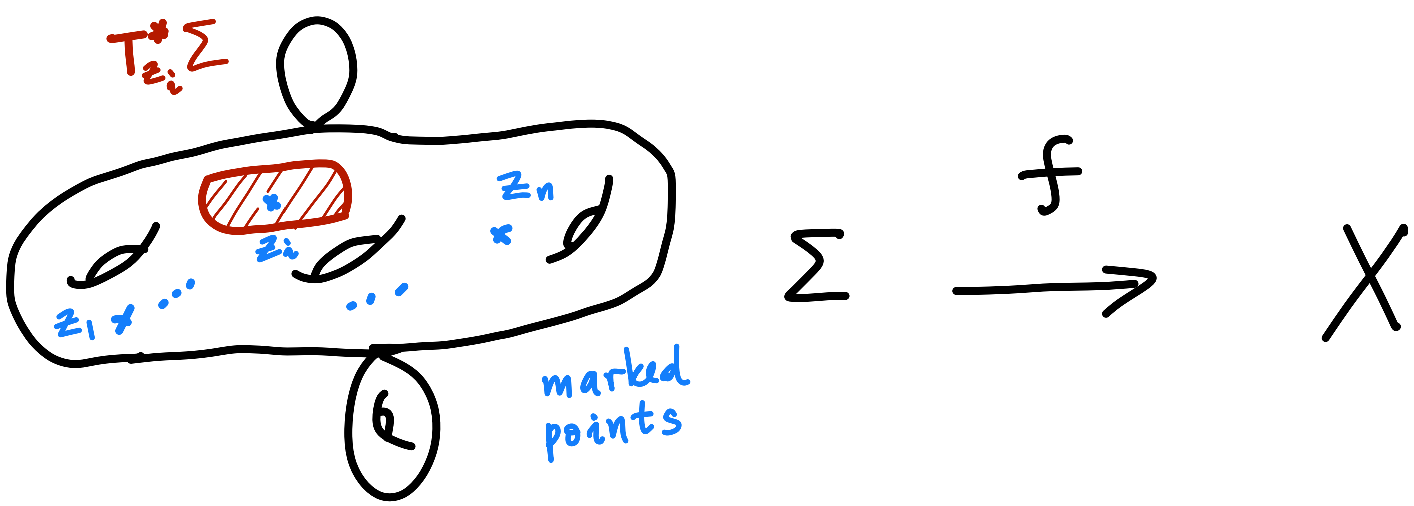
If $\mathfrak{g}$ is an affine Lie algebra of type A,D, or E, then Kac-Peterson classified all Heisenberg subalgebras: they are in one-to-one correspondence with conjugacy classes of the Weyl group.

Moreover, they proved that the Fock representation $\mathbb{C}[t]$ $t = (t_1, t_2, t_3, \ldots)$ extends to a representation $\mathbb{C}[Q_1^{\pm 1}, \ldots, Q_r^{\pm 1}, t]$ of $\mathfrak{g}$, where r is the dimension of the sub space of fixed points of an element w in the Weyl group (representing the conjugacy class).

Kac and Wakimoto proved that Sato's idea extends to all affine Lie algebras for any given conjugacy class in the Weyl group.

Type	Fock space
A	Created by Dirac fermions
D	Created by neutral fermions
E	???

Tau functions in GW theory



X : smooth projective variety or more generally orbifold

Σ : nodal Riemann surface equipped with marked points

f : holomorphic map

$(\Sigma, z_1, \ldots, z_n, f)$ is called a *stable map* if it does not have infinitesimal deformations

$\mathscr{M}_{g,n}(X,d)$:= moduli space of stable maps, such that, the Riemann surface has genus g, there are n marked points and the degree of the map f is d, that is, $f_*[\Sigma] = d \in H_2(X,\mathbb{Z})$.

Using the natural evaluation maps $\mathrm{ev}_i : \mathscr{M}_{g,n}(X,d) \to X$ and the line bundles L_i formed by the cotangent lines $T^*_{z_i}\Sigma$, we define the *total descendent potential*

GW invariants of X $\quad \psi_i = c_1(L_i)$

$$\mathscr{D}_X(\hbar,\mathbf{t}) := \exp\Big(\sum_{g,n,d} \frac{\hbar^{g-1}Q^d}{n!} \langle \phi_{i_1}\psi_1^{k_1}, \phi_{i_2}\psi_2^{k_2}, \ldots, \phi_{i_n}\psi_n^{k_n}\rangle_{g,n,d}\, t_{k_1,i_1}\cdots t_{k_n,i_n}\Big)$$

$\phi_i (1 \le i \le N)$ basis of $H^*(X)$ and $t_{k,i}$ are formal variables.

GW invariants were introduced by Witten. They coincide with the correlation functions of a QFT known as the *topological string*.

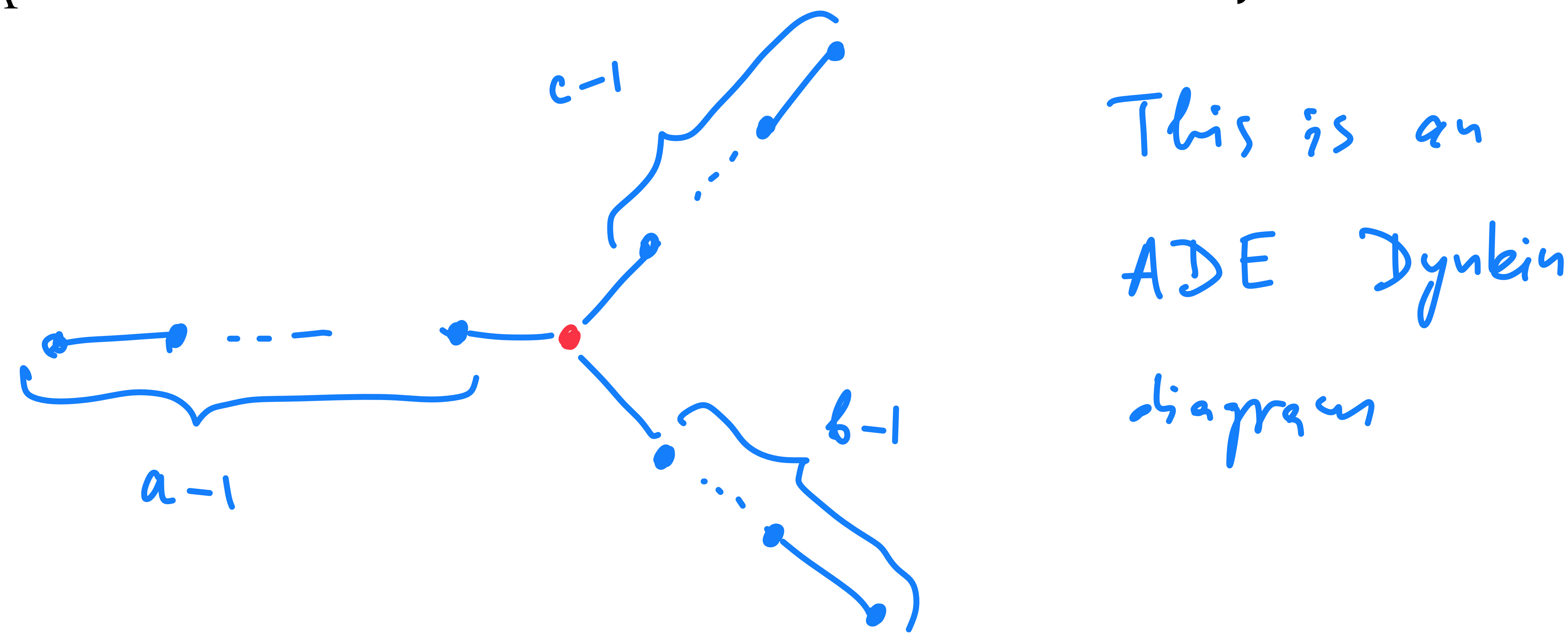
1) Witten conjectured and Kontsevich proved that $\mathscr{D}_{pt}$ is a tau function of the KdV hierarchy.

2) Egouchi-Hori-Yang conjectured that $\mathscr{D}_X$ is a highest weight vector for the Virasoro algebra. This is the main open problem in GW theory.

3) Okounkov-Pandharipande computed the GW invariants when X is a Riemann surface. If $X = \mathbb{P}^1$, they proved the so called Toda conjecture: $\mathscr{D}_{\mathbb{P}^1}$ is a tau function of the *extended* Toda hierarchy.

Remark: the Toda hierarchy is a Kac-Wakimoto hierarchy of type A_1 with conjugacy class $w = 1$.

4) In collaboration with H.-H. Tseng and Y. Shen we proved that if $X = \mathbb{P}^1_{a,b,c}$ sphere with 3 orbifold points, such that, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$ (positive orbifold Euler characteristic), then $\mathcal{D}_X$ is a tau function of a Kac-Wakimoto hierarchy.



$w =$ composition of all reflections different from the red node

5) Dubrovin-Zhang proved that if the **quantum cohomology of X is semi-simple**, then the GW invariants define an integrable hierarchy. The DZ construction resembles the situation on slide 14 — the integrable system there was defined using the Plucker relations. Dubrovin and Zhang used the Topological recursion relations in GW theory and Virasoro constraints.

a) DZ hierarchy can not be used to compute GW invariants! It is defined using GW invariants and $\mathscr{D}_X$ is tautologically a solution.

b) Nevertheless the result of Dubrovin and Zhang is very important for the theory of integrable systems. If we exclude the examples of integrable systems coming from the theory of simple or affine Lie algebras, then we will be left with very few examples. GW theory provides a huge class of examples that go beyond the theory of simple or affine Lie algebras.

Hirota Quadratic Equations?

The DZ hierarchies admit the notion of a tau-function. Is the set of tau functions a manifold similar to Sato's Grassmannian?

My approach is the following:

1) Construct HQEs for $\mathscr{D}_X$.

2) Prove that the HQEs define an integrable system.

3) Prove that the integrable system coincides with the DZ hierarchy.

The most difficult step is 1). Nevertheless, it seems to be going in the right direction.

Lattice vertex algebras

Let X be a smooth projective variety, such that, $D^b(X)$ has a full exceptional collection.

Lattice $\Lambda := K^0(X)$ — the topological K-ring of X

Euler pairing $\chi(E,F) := \sum_i (-1)^i \dim(\mathrm{Ext}^i(E,F))$

Intersection pairing (E|F):=$\chi(E,F) + \chi(F,E)$

Mukai lattice

Let $\alpha_1, \ldots, \alpha_N$ be the K-theoretic classes of a full exceptional collection. Each α_i determines a reflection $r_i(x) := x - (\alpha_i \,|\, x)\alpha_i$

Let W be the reflection group generated by all r_i . The smallest W-invariant subset R of Λ containing all α_i is called the set of reflection vectors. Let us recall the lattice vertex algebra

$$V_\Lambda = \bigoplus_{\alpha \in \Lambda} \mathbb{C}[\![\alpha_i, \partial\alpha_i, \partial^2\alpha_i, \ldots]\!] e^\alpha$$

It is equipped with products $v_{(n)}w$ (one for each integer n), satisfying generalized Jacobi identities.

Bakalov-M: If we can find a state $\Omega \in V_\Lambda \otimes V_\Lambda$, such that, $(e^\alpha_{(0)} \otimes 1 + 1 \otimes e^\alpha_{(0)})\Omega = 0$ for all reflection vectors α, then there is a general method to produce HQEs for $\mathscr{D}_X$.

The general method is the following: quantum cohomology gives a family of Fuchsian connections whose local mondoromies are precisely the reflections r_i. Using the solutions to the Fuchsian connection we construct vertex operators. The vertex operators define a twisted representation of V_Λ. Therefore, the state Ω can be represented by a differential operator acting on $\mathscr{D}_X \otimes \mathscr{D}_X$. The coefficients of the differential operator are meromorphic functions in a parameter λ. The screening condition in the definition of Ω implies analyticity at $\lambda = 0 \ \Rightarrow$ we get a system of quadratic equations for the coefficients of $\mathscr{D}_X$.

There is a progress in understanding the kernel of the screening operators. If Q is a quiver, then we have a natural lattice, Euler pairing, and reflection vectors. The problem of finding the kernel of the screening operators makes sense.

Kimura-Pestun: the qq-characters of Nekrasov define states in the kernel of the screening operators.

Unfortunately, lattices associated with quivers and K-rings rarely coincide. Nevertheless, if $X = \mathbb{P}^1_{a,b,c}$ with arbitrary a, b, c, then the K-ring can be identified with a quiver lattice. The conjecture of Kimura and Pestun should give many interesting results in this case.

Thank you!