A categorical action of the shifted $q = 0$ affine algebras

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NCTS

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Table of Contents

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Remark

In this talk, we work over the field $\mathbb{C}$ of complex numbers.

Representation of $\mathfrak{sl}_2$

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We can consider a more general case, which is the representation of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. The third condition is replaced by $(ef - fe)|_{V_\lambda} = [\lambda]_q Id_{V_\lambda}$, where $[\lambda]_q := q^{\lambda-1} + q^{\lambda-3} + ... + q^{-\lambda+1}$ is the quantum integer.

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- Geometry is a good resource for producing categories.
- It can be decategorified to recover the original vector space.

$$\begin{tikzcd}\text{vector spaces} \arrow[r, dotted, "categorify"] & \text{categories} \arrow[l, bend left, "decategorify"]\end{tikzcd}$$

The categorical $\mathfrak{sl}_2$ (or $\mathcal{U}_q(\mathfrak{sl}_2)$) action

We would apply the philosophy of categorification to representations of $\mathfrak{sl}_2(\mathbb{C})$ (or $\mathcal{U}_q(\mathfrak{sl}_2)$) and we call it the categorical $\mathfrak{sl}_2$ (or $\mathcal{U}_q(\mathfrak{sl}_2)$) action.

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Remark

Similarly, there is also a lift of the commutator relation $(ef - fe)|_{V_\lambda} = [\lambda]_q Id_{V_\lambda}$ for $\mathcal{U}_q(\mathfrak{sl}_2)$ to categorical level.

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Let $\mathcal{D}^bCon(\mathbb{G}(k,N))$ to be the bounded derived categories of constructible sheaves on $\mathbb{G}(k,N)$. These will be our weight categories $\mathcal{K}(\lambda) = \mathcal{D}^bCon(\mathbb{G}(k,N))$, where $\lambda = N-2k$.

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Remark

In this talk, all functors between derived categories are assumed to be derived. For example, we will use f^*, f_* instead of Lf^*, Rf_* respectively.

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Theorem 1 (Beilinson-Lusztig-MacPherson, Chuang-Rouquier)

The categories and functors defined above gives a categorical $\mathfrak{sl}_2$ *(or* $\mathcal{U}_q(\mathfrak{sl}_2)$*) action. This means that the functors defined above satisfy*

$$\mathsf{EF}|_{\mathcal{K}(\lambda)} \cong \mathsf{FE}|_{\mathcal{K}(\lambda)} \bigoplus \mathsf{Id}_{\mathcal{K}(\lambda)}^{\oplus\lambda} \text{ if } \lambda \geq 0$$

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That means that our weight categories $\mathcal{K}(\lambda)$ are bounded derived categories of coherent sheaves on $\mathbb{G}(k, N)$, which is denoted by $\mathcal{D}^b Coh(\mathbb{G}(k, N))$, where $\lambda = N - 2k$.

Our functors

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Denoting $\mathcal{V}, \mathcal{V}'$ to be the tautological bundles on $Fl(k-1,k)$ of rank k, $k-1$ respectively, then there is a natural line bundle $\mathcal{V}/\mathcal{V}'$ on $Fl(k-1,k)$.

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Instead of just pullback and pushforward, we have more functors

$$\mathsf{E}_r := p_{2*}(p_1^* \otimes (\mathcal{V}/\mathcal{V}')^r) : \mathcal{D}^b Coh(\mathbb{G}(k,N)) \to \mathcal{D}^b Coh(\mathbb{G}(k-1,N))$$
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where $r \in \mathbb{Z}$.

The main problem

Problem.
We want to understand how this $L\mathfrak{sl}_2 := \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$-like algebra acting on $\bigoplus_k \mathcal{D}^b Coh(\mathbb{G}(k, N))$, where $e \otimes t^r$ and $f \otimes t^s$ acting via the functors E_r and F_s respectively for $r,\ s \in \mathbb{Z}$.

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1. What are the categorical commutator relations between $\mathsf{E}_r\mathsf{F}_s$ and $\mathsf{F}_s\mathsf{E}_r$?
2. What is the algebra that we obtain after decategorifying?
3. If we define the algebra, can we give a definition of its categorical action like $\mathfrak{sl}_2$ in the introduction?

Fourier-Mukai (FM) transforms

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Definition 2

Let X, Y be two smooth projective varieties. A Fourier-Mukai (FM) kernel is any object $\mathcal{P} \in \mathcal{D}^b Coh(X \times Y)$. For such $\mathcal{P}$ we define the associated Fourier-Mukai (FM) transform, which is the functor

$$\Phi_{\mathcal{P}} : \mathcal{D}^b Coh(X) \to \mathcal{D}^b Coh(Y)$$

$$\mathcal{F} \mapsto \pi_{2*}(\pi_1^*(\mathcal{F}) \otimes \mathcal{P})$$

where π_1, π_2 are natural projections.

FM kernels for E_r and F_s

Then the functor $\mathsf{E}_r : \mathcal{D}^b Coh(\mathbb{G}(k,N)) \to \mathcal{D}^b Coh(\mathbb{G}(k-1,N))$ isomorphic to a FM transform with the kernel

$$\mathcal{E}_r \mathbf{1}_{(k,N-k)} := \iota_*(\mathcal{V}/\mathcal{V}')^r \in \mathcal{D}^b Coh(\mathbb{G}(k,N) \times \mathbb{G}(k-1,N))$$

where $\iota : Fl(k-1,k) \to \mathbb{G}(k,N) \times \mathbb{G}(k-1,N)$ is the natural inclusion, i.e., $\mathsf{E}_r \cong \Phi_{\mathcal{E}_r \mathbf{1}_{(k,N-k)}}$.

FM kernels for E_r and F_s

Then the functor $\mathsf{E}_r : \mathcal{D}^b Coh(\mathbb{G}(k,N)) \to \mathcal{D}^b Coh(\mathbb{G}(k-1,N))$ isomorphic to a FM transform with the kernel

$$\mathcal{E}_r \mathbf{1}_{(k,N-k)} := \iota_*(\mathcal{V}/\mathcal{V}')^r \in \mathcal{D}^b Coh(\mathbb{G}(k,N) \times \mathbb{G}(k-1,N))$$

where $\iota : Fl(k-1,k) \to \mathbb{G}(k,N) \times \mathbb{G}(k-1,N)$ is the natural inclusion, i.e., $\mathsf{E}_r \cong \Phi_{\mathcal{E}_r \mathbf{1}_{(k,N-k)}}$. Similarly, we denote

$$\mathcal{F}_s \mathbf{1}_{(k,N-k)} \in \mathcal{D}^b Coh(\mathbb{G}(k,N) \times \mathbb{G}(k+1,N))$$

to be the FM kernel for F_s, i.e., $\mathsf{F}_s \cong \Phi_{\mathcal{F}_r \mathbf{1}_{(k,N-k)}}$.

Categorical commutator relations between E_r and F_s

First, we study the relation between the two functors

$$\mathsf{E}_r \circ \mathsf{F}_s,\ \mathsf{F}_s \circ \mathsf{E}_r : \mathcal{D}^b Coh(\mathbb{G}(k,N)) \to \mathcal{D}^b Coh(\mathbb{G}(k,N)).$$

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Then $(\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)},\ (\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)} \in \mathcal{D}^b Coh(\mathbb{G}(k,N) \times \mathbb{G}(k,N))$ are FM kernels for the functors $\mathsf{E}_r \circ \mathsf{F}_s,\ \mathsf{F}_s \circ \mathsf{E}_r$, respectively.

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Categorical commutator relations between E_r and F_s

Theorem 3 (Hsu)

We have the following exact triangles in $\mathcal{D}^b Coh(\mathbb{G}(k,N) \times \mathbb{G}(k,N))$*.*

$$(\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)} \to (\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)} \to (\Psi^+ * \mathcal{H}_1)\mathbf{1}_{(k,N-k)}, \text{ if } r+s = N-k+1$$
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Here $\Psi^+\mathbf{1}_{(k,N-k)} = \Delta_* \det(\mathbb{C}^N/\mathcal{V})$, $\Psi^-\mathbf{1}_{(k,N-k)} = \Delta_* \det(\mathcal{V})^{-1}$ up to some homological shift, where $\Delta : \mathbb{G}(k,N) \to \mathbb{G}(k,N) \times \mathbb{G}(k,N)$ is the diagonal map. $\mathcal{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$ are more complicated kernels.

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Remark

The exact triangles are non-split in general.

The idea behind the proof

To compare $(\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)}$ and $(\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)}$, the geometries is exactly the same to the setting of constructible derived category for categorical $\mathfrak{sl}_2$ action.

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$$\begin{tikzcd}
Z' = \{V''' \overset{1}{\subset} V,\ V''\} \arrow[dr, "\pi'"] & & Z = \{V,\ V'' \overset{1}{\subset} V'\} \arrow[dl, "\pi"'] \\
& Y = \{(V, V'') \in \mathbb{G}(k,N) \times \mathbb{G}(k,N) \mid \dim(V \cap V'') \geq k-1\} &
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Remark

The variety Y is singular with 2 resolutions π, π'.

A comparison between constructible and coherent pictures

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In this setting π' is a small resolution and π is not a small resolution. Using the theory about perverse sheaves (IC sheaf), we have $\mathsf{EF}|_{\mathcal{K}(\lambda)} \cong \mathsf{FE}|_{\mathcal{K}(\lambda)} \bigoplus Id_{\mathcal{K}(\lambda)}^{\oplus \lambda}$.

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The non-zero extra term can be written as

$$Id_{\mathcal{K}(\lambda)}^{\oplus\lambda} \rightsquigarrow Id_{\mathcal{K}(\lambda)} \otimes H^*_{sing}(\mathbb{P}^{\lambda-1}, \mathbb{C})$$

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In this setting, the last isomorphism in Theorem 7 is reflected by the vanishing of coherent cohomology.

$$\begin{tikzcd}
(\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)} \cong (\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)},\ -k+1 \leq r+s \leq N-k-1 \arrow[d, leftrightsquigarrow] \\
H^*(\mathbb{P}^{N-1}, \mathcal{O}_{\mathbb{P}^{N-1}}(-r-s-k)) = 0,\ -N+1 \leq -r-s-k \leq -1.
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This tells us that $\mathcal{H}_1 \mathbf{1}_{(k,N-k)}$ is neither isomorphic to $\Delta_*(\mathcal{V} \oplus \mathbb{C}^N/\mathcal{V})$ nor to $\Delta_* \mathbb{C}^N$.

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Remark

We prove that $(\mathcal{H}_1 \mathbf{1}_{(k,N-k)})_R \cong \mathcal{H}_{-1} \mathbf{1}_{(k,N-k)} \cong (\mathcal{H}_1 \mathbf{1}_{(k,N-k)})_L$.

Other relations

We define the functors $\mathsf{H}_{\pm 1} : \mathcal{D}^b Coh(\mathbb{G}(k,N)) \to \mathcal{D}^b Coh(\mathbb{G}(k,N))$ to be the FM transforms with the kernels given by $\mathcal{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$.

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Similarly for $\mathsf{H}_{\pm 1}$ and F_s.
Again, we use the tools from FM transforms to study the categorical relations.
For example,

$$\text{Comparing } \mathsf{E}_r \circ \mathsf{H}_1,\ \mathsf{H}_1 \circ \mathsf{E}_r \rightsquigarrow \text{Comparing } (\mathcal{E}_r * \mathcal{H}_1)\mathbf{1}_{(k,N-k)},\ (\mathcal{H}_1 * \mathcal{E}_r)\mathbf{1}_{(k,N-k)}$$

Other relations

Theorem 4 (Hsu)

We have the following exact triangles

$$(\mathcal{H}_1 * \mathcal{E}_r)\mathbf{1}_{(k,N-k)} \to (\mathcal{E}_r * \mathcal{H}_1)\mathbf{1}_{(k,N-k)} \to (\mathcal{E}_{r+1} \bigoplus \mathcal{E}_{r+1}[1])\mathbf{1}_{(k,N-k)}$$

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Remark

Note that at the level of K-theory, this gives that the commutator relations between $h_{\pm 1}$ and e_r are trivial, i.e. $[h_{\pm 1}, e_r] = 0$. Similarly, $[h_{\pm 1}, f_s] = 0$.

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Note that at the level of K-theory, this gives that the commutator relations between $h_{\pm 1}$ and e_r are trivial, i.e. $[h_{\pm 1}, e_r] = 0$. Similarly, $[h_{\pm 1}, f_s] = 0$. In contrast, we have $[h_1, e_r] = [2]_q e_{r+1}$ for quantum loop algebra $\mathcal{U}_q(L\mathfrak{sl}_2)$.

Other relations

Theorem 4 (Hsu)

We have the following exact triangles

$$(\mathcal{H}_1 * \mathcal{E}_r)\mathbf{1}_{(k,N-k)} \to (\mathcal{E}_r * \mathcal{H}_1)\mathbf{1}_{(k,N-k)} \to (\mathcal{E}_{r+1} \bigoplus \mathcal{E}_{r+1}[1])\mathbf{1}_{(k,N-k)}$$

$$(\mathcal{E}_r * \mathcal{H}_{-1})\mathbf{1}_{(k,N-k)} \to (\mathcal{H}_{-1} * \mathcal{E}_r)\mathbf{1}_{(k,N-k)} \to (\mathcal{E}_{r-1} \bigoplus \mathcal{E}_{r-1}[1])\mathbf{1}_{(k,N-k)}$$

$$(\mathcal{F}_r * \mathcal{H}_1)\mathbf{1}_{(k,N-k)} \to (\mathcal{H}_1 * \mathcal{F}_r)\mathbf{1}_{(k,N-k)} \to (\mathcal{F}_{r+1} \bigoplus \mathcal{F}_{r+1}[1])\mathbf{1}_{(k,N-k)}$$

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Remark

Those exact triangles are non-split (in general).

Keep tracking the element under the process of convolution

Here we roughly explain the technical details behind the proof.

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$$\begin{tikzcd}
\Delta_*\mathcal{V} \arrow[r] & \mathcal{H}_1\mathbf{1}_{(k,N-k)} \arrow[r] \arrow[d, squiggly, "\pi_{12}^*"] & \Delta_*\mathbb{C}^N/\mathcal{V} & (0, id) \in \mathrm{Ext}^1 \arrow[d, mapsto] \\
\pi_{12}^*\Delta_*\mathcal{V} \arrow[r] & \pi_{12}^*\mathcal{H}_1\mathbf{1}_{(k,N-k)} \arrow[r] & \pi_{12}^*\Delta_*\mathbb{C}^N/\mathcal{V} & .... \in \mathrm{Ext}^1
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\iota_*\mathcal{V} \arrow[r] & (\mathcal{E} * \mathcal{H}_1)\mathbf{1}_{(k,N-k)} \arrow[r] & \iota_*\mathbb{C}^N/\mathcal{V} & x \in \operatorname{Ext}^1
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Similarly, we obtain the exact triangle

$$\iota_*\mathcal{V}' \to (\mathcal{H}_1 * \mathcal{E})\mathbf{1}_{(k,N-k)} \to \iota_*\mathbb{C}^N/\mathcal{V}' \text{ determined by } y \in \mathrm{Ext}^1$$

Check commutativity and Cone

$$\begin{tikzcd}
\iota_*\mathcal{V}' \arrow[r] \arrow[d] & (\mathcal{H}_1 * \mathcal{E})\mathbf{1}_{(k,N-k)} \arrow[r] & \iota_*\mathbb{C}^N/\mathcal{V}' \arrow[r, "y"] \arrow[d] & \iota_*\mathcal{V}'[1] \arrow[d] \\
\iota_*\mathcal{V} \arrow[r] & (\mathcal{E} * \mathcal{H}_1)\mathbf{1}_{(k,N-k)} \arrow[r] & \iota_*\mathbb{C}^N/\mathcal{V} \arrow[r, "x"] & \iota_*\mathcal{V}[1]
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Check commutativity and Cone

Check the commutativity of the right square induces a morphism $(\mathcal{H}_1 * \mathcal{E})\mathbf{1}_{(k,N-k)} \to (\mathcal{E} * \mathcal{H}_1)\mathbf{1}_{(k,N-k)}$.

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\iota_*\mathcal{V}/\mathcal{V}' \arrow[r] & Cone \arrow[r] & \iota_*\mathcal{V}/\mathcal{V}'[1] \arrow[r] & \iota_*\mathcal{V}/\mathcal{V}'[1]
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\end{tikzcd}$$

Finally, we show that $Cone \cong \iota_*\mathcal{V}/\mathcal{V}' \oplus \iota_*\mathcal{V}/\mathcal{V}'[1] \cong \mathcal{E}_1 \oplus \mathcal{E}_1[1]$ to get the desire exact triangle.

The main result

Together with the study of other categorical relations, we obtain the following main result.

Theorem 5 (Hsu)

(1)The resulting algebra acting on $\bigoplus_k \mathcal{D}^b Coh(\mathbb{G}(k,N))$ is a new algebra, which we call it the shifted $q=0$ affine algebra. Denoted by $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$.
(2)We give a definition of the categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ action.
(3)We verify that there is a categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ action on $\bigoplus_k \mathcal{D}^b Coh(\mathbb{G}(k,N))$.

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Our main result answers these natural questions that arising from the study of the $L\mathfrak{sl}_2$-like algebra action.

Remark

More generally, we constructed a categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_n)$ action on the derived categories of coherent sheaves on n-step partial flag varieties.

Exceptional collections

Fixing a triangulated category $\mathcal{D}$, which we may assume it is $\mathbb{C}$-linear.

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An object $E \in \mathsf{Ob}(\mathcal{D})$ is called exceptional if

$$\mathrm{Hom}_{\mathcal{D}}(E, E[l]) = \begin{cases} \mathbb{C} & \text{if } l = 0 \\ 0 & \text{if } l \neq 0. \end{cases}$$

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Then we define the notion of exceptional collections.

Definition 7

An ordered collection $\{E_1, ..., E_n\}$, where $E_i \in \mathsf{Ob}(\mathcal{D})$ for all $1 \leq i \leq n$, is called an exceptional collection if each E_i is exceptional and moreover $\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[l]) = 0$ for all $i > j$ and all $l \in \mathbb{Z}$.

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A semiorthogonal decomposition (SOD for short) of $\mathcal{D}$ is a sequence of full triangulated subcategories $\mathcal{A}_1, ..., \mathcal{A}_n$ such that

1. there is no non-zero Homs from right to left, i.e. $\mathrm{Hom}_{\mathcal{D}}(A_i, A_j) = 0$ for all $A_i \in \mathsf{Ob}(\mathcal{A}_i)$, $A_j \in \mathsf{Ob}(\mathcal{A}_j)$ where $1 \leq j < i \leq n$.
2. $\mathcal{D}$ is generated by $\mathcal{A}_1, ..., \mathcal{A}_n$, i.e. the smallest full triangulated subcategory containing $\mathcal{A}_1, ..., \mathcal{A}_n$ equal to $\mathcal{D}$.

We will use the notation $\mathcal{D} = \langle \mathcal{A}_1, ..., \mathcal{A}_n \rangle$ for a semiorthogonal decomposition of $\mathcal{D}$ with components $\mathcal{A}_1, ..., \mathcal{A}_n$.

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Remark

Constructing SOD when $\mathcal{D} = \mathcal{D}^b Coh(X)$ is an active research area in algebraic geometry. There has been many developments due to Bondal-Orlov, Kuznetsov...etc.

SOD given by exceptional collection

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where $\mathcal{A} = \langle E_1, ..., E_n \rangle^{\perp}$ and E_i denote the full triangulated subcategory generated by the object E_i.

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For a full triangulated subcategory $\mathcal{C} \subset \mathcal{D}$, we define $\mathcal{C}^{\perp} = \{X \in \mathsf{Ob}(\mathcal{D}) \mid \mathrm{Hom}_{\mathcal{D}}(C, X) = 0 \; \forall \; C \in \mathsf{Ob}(\mathcal{C})\}$ to be the right orthogonal to $\mathcal{C}$ in $\mathcal{D}$. It is a triangulated subcategories of $\mathcal{D}$.

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Remark

An exceptional collection is called full if the subcategory $\mathcal{A}$ is zero.

The Beilinson-Kapranov exceptional collection

The simplest example is given by Beilinson for projective space $\mathbb{P}^{N-1} = \mathbb{G}(1, N)$.

Theorem 9 (Beilinson)

There is a full exceptional collection (thus a SOD)

$$\mathcal{D}^b Coh(\mathbb{P}^{N-1}) = \langle \mathcal{O}_{\mathbb{P}^{N-1}}(-N+1), \mathcal{O}_{\mathbb{P}^{N-1}}(-N+2), ..., \mathcal{O}_{\mathbb{P}^{N-1}} \rangle.$$

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Theorem 10 (M. Kapranov)

There is a full exceptional collection (thus a SOD)

$$\mathcal{D}^b Coh(\mathbb{G}(k, N)) = \langle\ \mathbb{S}_\lambda \mathcal{V}\ \rangle_{\lambda \in P(N-k,k)}.$$

Relate to the categorical action

Since we construct an action of $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ on $\bigoplus_k \mathcal{D}^b Coh(\mathbb{G}(k,N))$ via using FM kernels, we try to relate the Kapranov exceptional collection to this action.

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$$\begin{tikzcd}
 & & Fl(1,2,...,k) & & & & \\
 & & .... & & & & \\
 & & ... & & & & \\
 & Fl(k-1,k) \arrow[dl] \arrow[dr] & ... & & & Fl(0,1) \arrow[dl] \arrow[dr] & \\
\mathbb{G}(k,N) & & \mathbb{G}(k-1,N) \arrow[ll, dotted, "\mathsf{F}_{\lambda_1}\mathbf{1}_{(\mathsf{k},\mathsf{N}-\mathsf{k})}"'] & ... & \mathbb{G}(1,N) & & \mathbb{G}(0,N) = pt \arrow[ll, dotted, "\mathsf{F}_{\lambda_\mathsf{k}}\mathbf{1}_{(0,\mathsf{N})}"']
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Since we construct an action of $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ on $\bigoplus_k \mathcal{D}^b Coh(\mathbb{G}(k,N))$ via using FM kernels, we try to relate the Kapranov exceptional collection to this action.

$$\begin{tikzcd}
 & & Fl(1,2,...,k) \arrow[ddddll, bend right] \arrow[ddddrrrr, bend left] & & & & \\
 & & .... & & & & \\
 & & ... & & & & \\
 & Fl(k-1,k) \arrow[dl] \arrow[dr] & ... & & & Fl(0,1) \arrow[dl] \arrow[dr] & \\
\mathbb{G}(k,N) & & \mathbb{G}(k-1,N) \arrow[ll, dotted, "\mathsf{F}_{\lambda_1}\mathbf{1}_{(k,N-k)}"'] & ... & \mathbb{G}(1,N) & & \mathbb{G}(0,N)=pt \arrow[ll, dotted, "\mathsf{F}_{\lambda_k}\mathbf{1}_{(0,N)}"']
\end{tikzcd}$$

Relate to the categorical action

More precisely, by using the Borel-Weil-Bott theorem we get

$$\mathbb{S}_\lambda \mathcal{V} \cong \mathcal{F}_{\lambda_1} * ... * \mathcal{F}_{\lambda_k} \mathbf{1}_{(0,N)}$$

where $\lambda = (\lambda_1, ..., \lambda_k) \in P(N-k, k)$. Note that $\mathcal{F}_{\lambda_1} * ... * \mathcal{F}_{\lambda_k} \mathbf{1}_{(0,N)}$ is the FM kernel for the functor $\mathsf{F}_\lambda \mathbf{1}_{(0,N)} := \mathsf{F}_{\lambda_1} ... \mathsf{F}_{\lambda_k} \mathbf{1}_{(0,N)}$.

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We know that $\{\mathbb{S}_\lambda \mathcal{V} \mid \lambda \in P(N-k,k)\}$ is an exceptional collection, it is natural to ask the following question.

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We know that $\{\mathbb{S}_\lambda \mathcal{V} \mid \lambda \in P(N-k,k)\}$ is an exceptional collection, it is natural to ask the following question.
$\boldsymbol{Question}$ **:** Given an (abstract) categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ (or $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_n)$) action $\mathcal{K}$. Do the functors

$$\mathsf{F}_\lambda \mathbf{1}_{(0,N)} := \mathsf{F}_{\lambda_1} ... \mathsf{F}_{\lambda_k} \mathbf{1}_{(0,N)} : \mathcal{K}(0,N) \to \mathcal{K}(k, N-k), \ \lambda \in P(N-k,k)$$

behave like an exceptional collection?

Application 1: SOD of weight categories

Proposition 11 (Hsu)

The functors $\{\mathsf{F}_\lambda \mathbf{1}_{(0,N)} \mid \lambda \in P(N-k,k)\}$ *satisfy the following properties*

(1) $\mathrm{Hom}(\mathsf{F}_\lambda \mathbf{1}_{(0,N)}, \mathsf{F}_\lambda \mathbf{1}_{(0,N)}) \cong \mathrm{Hom}(\mathbf{1}_{(0,N)}, \mathbf{1}_{(0,N)})$ (*exceptional-like property*)

(2) $\mathrm{Hom}(\mathsf{F}_\lambda \mathbf{1}_{(0,N)}, \mathsf{F}_{\lambda'} \mathbf{1}_{(0,N)}) \cong 0,$ *if* $\lambda <_l \lambda'$ (*semiorthogonal property*)

where $<_l$ *is the lexicographical order.*

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Remark

When the weight categories are $\mathcal{K}(k, N-k) = \mathcal{D}^b Coh(\mathbb{G}(k,N))$, we have $\mathrm{Hom}(\mathbf{1}_{(0,N)}, \mathbf{1}_{(0,N)}) \cong \mathbb{C}$. This recovers that $\mathbb{S}_\lambda \mathcal{V}$ is exceptional.

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Remark

The first property (1) also implies that the functors $\mathsf{F}_\lambda \mathbf{1}_{(0,N)} : \mathcal{K}(0,N) \to \mathcal{K}(k, N-k)$ are fully faithful for $\lambda \in P(N-k,k)$.

Application 1: SOD of weight categories

Then we have the following result.

Theorem 12 (Hsu)

Given a categorical $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_2)$ *action* $\mathcal{K}$. *There is a SOD*

$$\mathcal{K}(k, N-k) = \langle \mathcal{A}, \mathcal{K}_\lambda(k, N-k) \rangle_{\lambda \in P(N-k,k)}$$

where $\mathcal{A} := \langle \mathcal{K}_\lambda(k, N-k) \rangle^\perp_{\lambda \in P(N-k,k)}$ *and* $\mathcal{K}_\lambda(k, N-k) := \langle \mathsf{F}_\lambda \mathbf{1}_{(0,N)}(\mathcal{K}(0,N)) \rangle$ *is the minimal full triangulated subcategories of* $\mathcal{K}(k, N-k)$ *generated by the class of objects which are the essential images of* $\mathsf{F}_\lambda \mathbf{1}_{(0,N)}$.

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Remark

In fact, we prove the above theorem for general $\mathfrak{sl}_n$ action.

1. Introduction
    - Categorical $\mathfrak{sl}_2$ action
    - An example from geometry
2. The motivation and the main result
3. Applications
    - Semiorthogonal decomposition
    - Demazure operators
4. Some related works

Demazure operators

Let $G = \mathrm{SL}_N(\mathbb{C})$ and $B \subset G$ be the Borel of upper triangular matrices. Consider the type A full flag variety

$$G/B = \{0 = V_0 \overset{1}{\subset} V_1 \overset{1}{\subset} ... \overset{1}{\subset} V_N = \mathbb{C}^N\}$$

and similarly the partial flag variety

$$G/P_i = \{0 \overset{1}{\subset} V_1 \overset{1}{\subset} V_2 \overset{1}{\subset} ...V_{i-1} \overset{2}{\subset} V_{i+1} \overset{1}{\subset} ...V_N = \mathbb{C}^N\}$$

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π_i induces pullback $\pi_i^* : K(G/P_i) \to K(G/B)$ and pushforward $\pi_{i*} : K(G/B) \to K(G/P_i)$ on the K-theory.

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We denote $\mathcal{V}_i$ to be the tautological bundle of rank i on G/B, and $\mathscr{L}_i = \mathcal{V}_i/\mathcal{V}_{i-1}$ the natural line bundles. Let $x_i = [\mathscr{L}_i] \in K(G/B)$ be the class in the Grothendieck group.

$q = 0$ affine Hecke algebra

The Demazure operators are defined by $\delta_i := \pi_i^* \pi_{i*} : K(G/B) \to K(G/B)$ for all i.

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$$\delta_i^2 = \delta_i \text{ (idempotent)}$$
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and thus we have an action $H_N(0)$ on $K(G/B)$.

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$\{\delta_i, x_j\}$ generate the $q = 0$ affine Hecke algebra, denoted by $\mathcal{H}_N(0)$, and the action of $H_N(0)$ can be extended to the action of $\mathcal{H}_N(0)$ on $K(G/B)$.

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Remark

Lusztig introduced an q-analogue version of δ_i, which is called the Demazure-Lusztig operator. Together with x_j, he proved that there is an action of the affine Hecke algebra on $K^{G \times \mathbb{C}^*}(G/B)$.

Lift to categorical level

The action of $\mathcal{H}_N(0)$ on $K(G/B)$ can be lifted to categorical action of $\mathcal{H}_N(0)$ on $\mathcal{D}^b Coh(G/B)$.

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There is a categorical action of the $q = 0$ affine Hecke algebra $\mathcal{H}_N(0)$ on $\mathcal{D}^b Coh(G/B)$, where the generators δ_i, x_j act by lifting to the FM transformation $\Phi_{\mathcal{T}_i}$, $\Phi_{\mathcal{X}_j}$ respectively.

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One way to prove this theorem is to verify the categorical relations directly by calculating many convolutions of the FM kernels $\mathcal{T}_i, \mathcal{X}_j$.

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Remark

One way to prove this theorem is to verify the categorical relations directly by calculating many convolutions of the FM kernels $\mathcal{T}_i, \mathcal{X}_j$. However, instead of proving this theorem by direct calculation, we prove this theorem by relating this action to the categorical action of shifted $q = 0$ affine algebra.

Relation to shifted $q = 0$ affine algebra

For any $\underline{k} = (k_1, ..., k_n) \in \mathbb{N}^n$ such that $\sum_i k_i = N$, we define

$$Fl_{\underline{k}}(\mathbb{C}^N) = \{0 \overset{k_1}{\subset} V_1 \overset{k_2}{\subset} ... \overset{k_n}{\subset} V_n = \mathbb{C}^N\}$$

to be the n-step partial flag variety. We will simply use the notation $Fl_{\underline{k}}$ if there is no ambiguity.

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In particular, when $n = N$, there is a categorical action of $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_{N})$ on $\bigoplus_{\underline{k}} \mathcal{D}^b Coh(Fl_{\underline{k}})$ and thus descend to action on $\bigoplus_{\underline{k}} K(Fl_{\underline{k}})$.

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In particular, when $n = N$, there is a categorical action of $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_{N})$ on $\bigoplus_{\underline{k}} \mathcal{D}^b Coh(Fl_{\underline{k}})$ and thus descend to action on $\bigoplus_{\underline{k}} K(Fl_{\underline{k}})$. Note that in this notation, we have $G/B = Fl_{(1,1,...,1)}$, and $K(G/B)$ is one of the direct summand. The main idea is to interpret the Demazure operators δ_i in terms of elements in $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_{N})$.

A simple observation

Recall that to construct δ_i, we need

$$G/P_i = \{0 \overset{1}{\subset} V_1 \overset{1}{\subset} V_2 \overset{1}{\subset} ...V_{i-1} \overset{2}{\subset} V_{i+1} \overset{1}{\subset} ...V_N = \mathbb{C}^N\}$$

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Observe that

$$\begin{aligned} V_{i-1} \overset{2}{\subset} V_{i+1} &= {\color{red}V_{i-1}} \overset{0}{\subset} {\color{red}V_{i-1}} \overset{2}{\subset} V_{i+1} \\ &= V_{i-1} \overset{2}{\subset} {\color{blue}V_{i+1}} \overset{0}{\subset} {\color{blue}V_{i+1}} \end{aligned}$$

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So

$$G/P_i = Fl_{(1,1,...,1)+\alpha_i} = Fl_{(1,1,...,1)-\alpha_i}$$

where $\alpha_i = (0, ..., -1, 1, ..., 0)$ is the simple root with -1 at the ith position.

Interpretation of δ_i

We have the following picture

$$\begin{tikzcd}
K(G/P_i = Fl_{(1,1,\ldots,1)-\alpha_i}) \arrow[r, shift left, "e_{i,r}"] & K(G/B = Fl_{(1,1,\ldots,1)}) \arrow[l, shift left, "f_{i,s}"] \arrow[loop above, "\delta_i"] \arrow[r, shift left, "e_{i,r}"] & K(G/P_i = Fl_{(1,1,\ldots,1)+\alpha_i}) \arrow[l, shift left, "f_{i,s}"]
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As a result, the Demazure operators can be written as

$$\delta_i = e_i f_{i,1} (\psi_i^+)^{-1} 1_{(1,1,\dots,1)} = f_i e_{i,-1} (\psi_i^-)^{-1} 1_{(1,1,\dots,1)}.$$

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Similarly $x_j = \psi_{j-1}^+ 1_{(1,1,\ldots,1)} = (\psi_j^-)^{-1} 1_{(1,1,\ldots,1)}$.

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Similarly $x_j = \psi_{j-1}^+ 1_{(1,1,\dots,1)} = (\psi_j^-)^{-1} 1_{(1,1,\dots,1)}$.

Remark

Here we simply denote e_i for $e_{i,0}$, f_i for $f_{i,0}$. Similarly for $\mathcal{E}_i$ and $\mathcal{F}_i$.

Lifting

Lifting to the categorical level, as a result, we have the isomorphisms of FM kernels

$$\mathcal{T}_i \cong \mathcal{E}_i * \mathcal{F}_{i,1} * (\Psi_i^+)^{-1}\mathbf{1}_{(1,1,\ldots,1)} \cong \mathcal{F}_i * \mathcal{E}_{i,-1} * (\Psi_i^-)^{-1}\mathbf{1}_{(1,1,\ldots,1)}$$
$$\mathcal{X}_j \cong \Psi_{j-1}^+\mathbf{1}_{(1,1,\ldots,1)} \cong (\Psi_j^-)^{-1}\mathbf{1}_{(1,1,\ldots,1)}$$

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Thus the categorical relations that we need to verify for $\mathcal{H}_N(0)$ can be deduced from the categorical relations in $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_N)$.

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Thus the categorical relations that we need to verify for $\mathcal{H}_N(0)$ can be deduced from the categorical relations in $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_N)$.
In particular, the categorical commutator relations

$$(\mathcal{F}_{i,1} * \mathcal{E}_i)\mathbf{1}_{(1,1,\ldots,1)} \to (\mathcal{E}_i * \mathcal{F}_{i,1})\mathbf{1}_{(1,1,\ldots,1)} \to \Psi_i^+\mathbf{1}_{(1,1,\ldots,1)},$$
$$(\mathcal{E}_{i,-1} * \mathcal{F}_i)\mathbf{1}_{(1,1,\ldots,1)} \to (\mathcal{F}_i * \mathcal{E}_{i,-1})\mathbf{1}_{(1,1,\ldots,1)} \to \Psi_i^-\mathbf{1}_{(1,1,\ldots,1)},$$

are precisely the (categorical) affine Hecke relations

$$\mathcal{X}_i * \mathcal{T}_i \to \mathcal{T}_i * \mathcal{X}_{i+1} \to \mathcal{X}_{i+1},$$
$$\mathcal{T}_i * \mathcal{X}_i \to \mathcal{X}_{i+1} * \mathcal{T}_i \to \mathcal{X}_{i+1}.$$

Generalization to $Fl_{\underline{k}}$

For $\underline{k} = (k_1, ..., k_n)$ with $n < N$, the pullback induced by the natural projection $\pi : G/B = Fl_{(1,1,...,1)} \to Fl_{\underline{k}}$ makes $K(Fl_{\underline{k}})$ as a submodule of $K(G/B)$.

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One natural question is to generalize the construction of the Demazure operators δ_i from G/B to $Fl_{\underline{k}}$.
Motivating from the above interpretation of δ_i for G/B, and since there is a categorical action of $\dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_n)$ on $\bigoplus_{\underline{k}} \mathcal{D}^b Coh(Fl_{\underline{k}})$, we can try to generalize δ_i to get operators acting on $\mathcal{D}^b Coh(Fl_{\underline{k}})$ and $K(Fl_{\underline{k}})$.

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Remark

Note that the submodule $K(Fl_{\underline{k}})$ is not invariant under the action of δ_i, so $\delta_i|_{K(Fl_{\underline{k}})}$ does not work.

Generalization to $Fl_{\underline{k}}$

$$\begin{tikzcd}
K(Fl_{(1,1,\dots,1)-\alpha_i}) \arrow[r, shift left, "e_{i,r}"] & K(G/B = Fl_{(1,1,\dots,1)}) \arrow[l, shift left, "f_{i,s}"] \arrow[r, shift left, "e_{i,r}"] \arrow[loop above, "\delta_i"] & K(Fl_{(1,1,\dots,1)+\alpha_i}) \arrow[l, shift left, "f_{i,s}"] \\
K(Fl_{\underline{k}-\alpha_i}) \arrow[r, bend left, "e_{i,r}"] & K(Fl_{\underline{k}}) \arrow[l, bend left, "f_{i,s}"] \arrow[r, bend left, "e_{i,r}"] \arrow[u, hook] & K(Fl_{\underline{k}+\alpha_i}) \arrow[l, bend left, "f_{i,s}"]
\end{tikzcd}$$

Generalization to $Fl_{\underline{k}}$

$$\begin{tikzcd}
& e_i f_{i,1}(\psi_i^+)^{-1} 1_{(1,1,\dots,1)} = \delta_i \arrow[loop below] \arrow[dddl, bend right=60] & \\
K(Fl_{(1,1,\dots,1)-\alpha_i}) \arrow[r, shift left, "e_{i,r}"] & K(G/B = Fl_{(1,1,\dots,1)}) \arrow[l, shift left, "f_{i,s}"] \arrow[r, shift left, "e_{i,r}"] & K(Fl_{(1,1,\dots,1)+\alpha_i}) \arrow[l, shift left, "f_{i,s}"] \\
K(Fl_{\underline{k}-\alpha_i}) \arrow[r, bend left=10, "e_{i,r}"] & K(Fl_{\underline{k}}) \arrow[l, bend left=10, "f_{i,s}"] \arrow[u, hook] \arrow[r, bend left=10, "e_{i,r}"] & K(Fl_{\underline{k}+\alpha_i}) \arrow[l, bend left=10, "f_{i,s}"] \\
{} & &
\end{tikzcd}$$

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$$\begin{tikzcd}
& e_i f_{i,1}(\psi_i^+)^{-1} 1_{(1,1,\ldots,1)} = \delta_i \arrow[ddddl, bend right] & \\
& \circlearrowright & \\
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e_i f_{i,\textcolor{red}{k_{i+1}}}(\psi_i^+)^{-1} 1_{\textcolor{red}{\underline{k}}} =: \textcolor{blue}{\delta_i'} & &
\end{tikzcd}$$

Generalization to $Fl_{\underline{k}}$

$$\begin{tikzcd}
 & & e_i f_{i,1}(\psi_i^+)^{-1}1_{(1,1,\ldots,1)} = \delta_i = f_i e_{i,-1}(\psi_i^-)^{-1}1_{(1,1,\ldots,1)} \arrow[loop below] \arrow[ddddll, bend right=60] \arrow[ddddrr, bend left=60] & & \\
 & & & & \\
K(Fl_{(1,1,\ldots,1)-\alpha_i}) \arrow[rr, shift left, "e_{i,r}"] & & K(G/B = Fl_{(1,1,\ldots,1)}) \arrow[ll, shift left, "f_{i,s}"] \arrow[rr, shift left, "e_{i,r}"] & & K(Fl_{(1,1,\ldots,1)+\alpha_i}) \arrow[ll, shift left, "f_{i,s}"] \\
K(Fl_{\underline{k}-\alpha_i}) \arrow[rr, bend left=10, "e_{i,r}"] & & K(Fl_{\underline{k}}) \arrow[ll, bend left=10, "f_{i,s}"] \arrow[rr, bend left=10, "e_{i,r}"] \arrow[u, hook] & & K(Fl_{\underline{k}+\alpha_i}) \arrow[ll, bend left=10, "f_{i,s}"] \\
e_i f_{i,\textcolor{red}{k_{i+1}}}(\psi_i^+)^{-1}1_{\textcolor{red}{\underline{k}}} =: \textcolor{blue}{\delta'_i} & & & & {}
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 & & e_i f_{i,1}(\psi_i^+)^{-1}1_{(1,1,\dots,1)} = \delta_i = f_i e_{i,-1}(\psi_i^-)^{-1}1_{(1,1,\dots,1)} \arrow[loop below] \arrow[dddll, bend right] \arrow[dddrr, bend left] & & \\
K(Fl_{(1,1,\dots,1)-\alpha_i}) \arrow[rr, shift left, "e_{i,r}"] & & K(G/B = Fl_{(1,1,\dots,1)}) \arrow[ll, shift left, "f_{i,s}"] \arrow[rr, shift left, "e_{i,r}"] & & K(Fl_{(1,1,\dots,1)+\alpha_i}) \arrow[ll, shift left, "f_{i,s}"] \\
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e_i f_{i,\textcolor{red}{k_{i+1}}}(\psi_i^+)^{-1}1_{\textcolor{red}{\underline{k}}} =: \textcolor{blue}{\delta_i'} & & & & \textcolor{blue}{\delta_i''} := f_i e_{i,\textcolor{red}{-k_i}}(\psi_i^-)^{-1}1_{\textcolor{red}{\underline{k}}}
\end{tikzcd}$$

Generalization to $Fl_{\underline{k}}$

$$\begin{tikzcd}
e_if_{i,1}(\psi_i^+)^{-1}1_{(1,1,\ldots,1)} = \arrow[ddd, bend right] & \delta_i \arrow[loop below] & = f_ie_{i,-1}(\psi_i^-)^{-1}1_{(1,1,\ldots,1)} \arrow[ddd, bend left] \\
K(Fl_{(1,1,\ldots,1)-\alpha_i}) \arrow[r, shift left, "e_{i,r}"] & K(G/B = Fl_{(1,1,\ldots,1)}) \arrow[l, shift left, "f_{i,s}"] \arrow[r, shift left, "e_{i,r}"] & K(Fl_{(1,1,\ldots,1)+\alpha_i}) \arrow[l, shift left, "f_{i,s}"] \\
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e_if_{i,k_{i+1}}(\psi_i^+)^{-1}1_{\underline{k}} =: \delta_i' & \in \mathrm{End}(K(Fl_{\underline{k}})) \ni & \delta_i'' := f_ie_{i,-k_i}(\psi_i^-)^{-1}1_{\underline{k}}
\end{tikzcd}$$

Demazure operators for $Fl_{\underline{k}}$

Now we have the operators

$$\delta_i' = e_i f_{i,k_{i+1}} (\psi_i^+)^{-1} 1_{\underline{k}},\ \delta_i'' = f_i e_{i,-k_i} (\psi_i^-)^{-1} 1_{\underline{k}} : K(Fl_{\underline{k}}) \to K(Fl_{\underline{k}}).$$

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Lifting to categorical level by using FM transforms, we denote

$$\mathcal{T}_i' := \mathcal{E}_i * \mathcal{F}_{i,k_{i+1}} * (\Psi_i^+)^{-1} \mathbf{1}_{\underline{k}}$$
$$\mathcal{T}_i'' := \mathcal{F}_i * \mathcal{E}_{i,-k_i} * (\Psi_i^-)^{-1} \mathbf{1}_{\underline{k}}$$

to be the kernels in $\mathcal{D}^b Coh(Fl_{\underline{k}} \times Fl_{\underline{k}})$ for the lifting of δ_i', δ_i'' respectively.

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to be the kernels in $\mathcal{D}^b Coh(Fl_{\underline{k}} \times Fl_{\underline{k}})$ for the lifting of δ_i', δ_i'' respectively.

Remark

In general, we have $\delta_i' \neq \delta_i''$ and thus $\mathcal{T}_i' \ncong \mathcal{T}_i''$. They are equal/isomorphic only when $\underline{k} = (1, 1, ..., 1)$.

Demazure operators for $Fl_{\underline{k}}$

From the categorical commutator relations,

$$(\mathcal{F}_{i,k_{i+1}} * \mathcal{E}_i)\mathbf{1}_{\underline{k}} \to (\mathcal{E}_i * \mathcal{F}_{i,k_{i+1}})\mathbf{1}_{\underline{k}} \to \Psi_i^+ \mathbf{1}_{\underline{k}},$$
$$(\mathcal{E}_{i,-k_i} * \mathcal{F}_i)\mathbf{1}_{\underline{k}} \to (\mathcal{F}_i * \mathcal{E}_{i,-k_i})\mathbf{1}_{\underline{k}} \to \Psi_i^- \mathbf{1}_{\underline{k}},$$

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since $\Psi_i^+ \mathbf{1}_{\underline{k}}$, $\Psi_i^- \mathbf{1}_{\underline{k}}$ are certain line bundles, we can convolution their inverse to get the following exact triangle

$$\mathcal{F}_{i,k_{i+1}} * \mathcal{E}_i * (\Psi_i^+)^{-1} \mathbf{1}_{\underline{k}} \to \mathcal{T}_i' = \mathcal{E}_i * \mathcal{F}_{i,k_{i+1}} * (\Psi_i^+)^{-1} \mathbf{1}_{\underline{k}} \to \mathcal{O}_\Delta,$$
$$\mathcal{E}_{i,-k_i} * \mathcal{F}_i * (\Psi_i^-)^{-1} \mathbf{1}_{\underline{k}} \to \mathcal{T}_i'' = \mathcal{F}_i * \mathcal{E}_{i,-k_i} * (\Psi_i^-)^{-1} \mathbf{1}_{\underline{k}} \to \mathcal{O}_\Delta.$$

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We denote $\mathcal{S}_i' := \mathcal{F}_{i,k_{i+1}} * \mathcal{E}_i * (\Psi_i^+)^{-1} \mathbf{1}_{\underline{k}}$ and $\mathcal{S}_i'' := \mathcal{E}_{i,-k_i} * \mathcal{F}_i * (\Psi_i^-)^{-1} \mathbf{1}_{\underline{k}}$.

Demazure operators for $Fl_{\underline{k}}$

Theorem 14 (Hsu)

(1) Idempotent

$$\mathcal{T}_i^{'} * \mathcal{T}_i^{'} \cong \mathcal{T}_i^{'},\ \mathcal{T}_i^{''} * \mathcal{T}_i^{''} \cong \mathcal{T}_i^{''}.$$

(2) There exist exact triangles in $\mathcal{D}^b Coh(Fl_{\underline{k}} \times Fl_{\underline{k}})$

$$\mathcal{T}_{i+1}^{'} * \mathcal{T}_i^{'} * \mathcal{T}_{i+1}^{'} * \mathcal{S}_i^{'} \to \mathcal{T}_i^{'} * \mathcal{T}_{i+1}^{'} * \mathcal{T}_i^{'} \to \mathcal{T}_{i+1}^{'} * \mathcal{T}_i^{'} * \mathcal{T}_{i+1}^{'},$$
$$\mathcal{T}_i^{''} * \mathcal{T}_{i+1}^{''} * \mathcal{T}_i^{''} * \mathcal{S}_{i+1}^{''} \to \mathcal{T}_{i+1}^{''} * \mathcal{T}_i^{''} * \mathcal{T}_{i+1}^{''} \to \mathcal{T}_i^{''} * \mathcal{T}_{i+1}^{''} * \mathcal{T}_i^{''}.$$

(3) Vanishing

$$\mathcal{T}_i^{'} * \mathcal{T}_{i+1}^{'} * \mathcal{T}_i^{'} * \mathcal{S}_{i+1}^{'} \cong \mathcal{T}_{i+1}^{''} * \mathcal{T}_i^{''} * \mathcal{T}_{i+1}^{''} * \mathcal{S}_i^{''} \cong 0$$

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(2) There exist exact triangles in $\mathcal{D}^b Coh(Fl_{\underline{k}} \times Fl_{\underline{k}})$

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Remark

In particular, when $\underline{k} = (1, 1, ..., 1)$ (which is G/B), we have $\mathcal{T}_i' \cong \mathcal{T}_i''$ and $\mathcal{S}_i' \cong \mathcal{S}_i''$. So (2) and (3) imply the categorical braid relations. Thus the categorical action of $H_N(0)$ on $\mathcal{D}^b Coh(G/B)$ is a direct consequence of this theorem.

Calculations of the actions on a basis

When $n = 2$, we obtain the action of $H_2(0)$ on $K(\mathbb{G}(k, N))$, where the generator acts by $\delta' = ef_{N-k}(\psi^+)^{-1}1_{(k,N-k)}$.

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Theorem 15 (Hsu)

$$\delta^{'}([\mathbb{S}_\lambda\mathcal{V}]) = \begin{cases} 0 & \textit{if } \lambda_1 = N-k \\ [\mathbb{S}_\lambda\mathcal{V}] & \textit{if } 0 \le \lambda_1 \le N-k-1, \end{cases}$$

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Remark

Since $(\delta')^2 = \delta'$, we have $\{[\mathbb{S}_\lambda \mathcal{V}] \mid \lambda \in P(N-k, k)\}$ is an eigenbasis for δ'.

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Remark

In fact, we calculate the action of δ'_i on the basis given by the Kapranov exceptional collection for the n-step partial flag variety $Fl_{\underline{k}}$.

Relate to the action of (quantum) loop algebra

Consider the K-theory of cotangent bundle of n-step partial flag varieties ($\mathfrak{sl}_n$ Nakajima quiver varieties) $K(T^*Fl_{\underline{k}})$.

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Consider the K-theory of cotangent bundle of n-step partial flag varieties ($\mathfrak{sl}_n$ Nakajima quiver varieties) $K(T^*Fl_{\underline{k}})$. There is an action of $\mathcal{U}(L\mathfrak{sl}_n)$ (or $\mathcal{U}_q(L\mathfrak{sl}_n)$ if we work $\mathbb{C}^*$-equivariantly) on it. The action is constructed by Hecke correspondence with the loop structure also comes from twisting of line bundles. Since there is the Thom isomorphism, it would be interesting to see the relation between this action and our action of shifted $q = 0$ affine algebra.

$$\begin{tikzcd}
\bigoplus_{\underline{k}} K(T^*Fl_{\underline{k}}) \arrow[r, "Thom", "\cong"'] & \bigoplus_{\underline{k}} K(Fl_{\underline{k}}) \\
\mathcal{U}(L\mathfrak{sl}_n) \arrow[loop above] \arrow[r, dotted, leftrightarrow, "?"] & \dot{\mathcal{U}}_{0,N}(L\mathfrak{sl}_n) \arrow[loop above]
\end{tikzcd}$$

Relate to shifted quantum affine algebra

Consider the affine Grassmannain $Gr_{GL_N} := GL_N(\mathcal{K})/GL_N(\mathcal{O})$ where $\mathcal{O} = \mathbb{C}[[t]]$ is the formal power series ring and $\mathcal{K} = \mathbb{C}((t))$ its fraction field.

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$$\mathfrak{U}_{q,-2N\alpha^\vee} \twoheadrightarrow \underset{\text{(quantized K-theoretic Coulomb branch)}}{K_{loc}^{\widetilde{GL_N}(\mathcal{O})\rtimes\widetilde{\mathbb{C}^*}}(Gr_{GL_N})}$$

where $\mathfrak{U}_{q,-2N\alpha^\vee}$ are certain truncated shifted quantum affine algebra.

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On the other hand, for $\mathfrak{g} = \mathfrak{sl}_n$ they also construct action of

$$\mathcal{U}_{q,\mu} \circlearrowright \underset{\text{(K-theory of Laumon based parabolic quasiflag spaces)}}{\bigoplus_{\underline{k}} K_{loc}^{\widetilde{T}\times\widetilde{\mathbb{C}^*}}(\mathfrak{Q}_{\underline{k}})}$$

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It would also be interesting to see the relations to these works, e.g. lifting the above results from K-theory to derived categories by using our categorical action.

Thank you for your attention.